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Sub-Gaussian random variables and Wiman's inequality for analytic functions

Kuryliak A.O., Skaskiv O.B.

Let f be an analytic function in $\{z : |z| < R\}$ of the form $f(z) = \sum\limits_{n=0}^{+\infty} a_n z^n$. In the paper, we consider the Wiman-type inequality for random analytic functions of the form $f(z,\omega) = \sum\limits_{n=0}^{+\infty} Z_n(\omega) a_n z^n$, where (Z_n) is a sequence on the Steinhaus probability space of real independent centered sub-Gaussian random variables, i.e. $(\exists D > 0)$ $(\forall k \in \mathbb{N})$ $(\forall \lambda \in \mathbb{R})$: $\mathbf{E}(e^{\lambda Z_k}) \le e^{D\lambda^2}$, and such that $(\exists \beta > 0)$ $(\exists n_0 \in \mathbb{N})$: $\inf\limits_{n \ge n_0} \mathbf{E}|Z_n|^{-\beta} < +\infty$.

It is proved that for every $\delta > 0$ there exists a set $E(\delta) \subset [0, R)$ of finite h-logarithmic measure $\big($i.e. $\int_E h(r) d\ln r < +\infty\big)$ such that almost surely for all $r \in (r_0(\omega), R)\backslash E$ we have

$$M_f(r,\omega) := \max\left\{|f(z,\omega)| \colon |z| = r\right\} \le \sqrt{h(r)}\mu_f(r)\Big(\ln^3 h(r) \ln\{h(r)\mu_f(r)\}\Big)^{1/4+\delta},$$

where $h(r)$ is any fixed continuous non-decreasing function on $[0; R)$ such that $h(r) \ge 2$ for all $r \in (0, R)$ and $\int_{r_0}^R h(r) d\ln r = +\infty$ for some $r_0 \in (0, R)$.

Key words and phrases: analytic function, Levy's phenomenon, Wiman's inequality, sub-Gaussian random variables.

Ivan Franko Lviv National University, 1 Universytetska str., 79000, Lviv, Ukraine
E-mail: andriykuryliak@gmail.com (Kuryliak A.O.), olskask@gmail.com (Skaskiv O.B.)

1 Introduction

Let $\mathcal{E}_R$ be the class of analytic functions f represented by power series of the form

$$f(z) = \sum_{n=0}^{+\infty} a_n z^n \tag{1}$$

with the radius of convergence $R \in (0; +\infty]$. For $r \in [0, R)$, the maximum modulus and maximal term of series are denoted by $M_f(r) = \max\{|f(z)| \colon |z| = r\}$ and $\mu_f(r) = \max\{|a_n| r^n \colon n \ge 0\}$, respectively. We also denote by $\mathcal{H}_R$ the class of continuous non-decreasing on $[0; R)$, $R \le +\infty$, functions such that $h(r) \ge 2$ for all $r \in (0, R)$ and $\int_{r_0}^R h(r) d\ln r = +\infty$ for some $r_0 \in (0, R)$. In [1], it is proved that if $h \in \mathcal{H}_R$ and $f \in \mathcal{E}_R$, then for any $\delta > 0$ there exist $E(\delta, f, h) := E \subset (0, R)$, $r_0 \in (0, R)$ such that for all $r \in (r_0, R)\backslash E$ we have

$$M_f(r) \le h(r)\mu_f(r)\big\{\ln h(r) \ln\big(h(r)\mu_f(r)\big)\big\}^{1/2+\delta} \tag{2}$$

and

$$h\text{-meas}\, E = \int_E h(r) d\ln r < +\infty.$$

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From this statement, for $h(r) \equiv 2$, the classical Wiman-Valiron theorem (see [2–6]) for entire functions follows: for any $\delta > 0$ and for every non-constant entire function, the Wiman inequality $\ln M_f(r) \le \mu_f(r) \ln^{1/2+\delta} \mu_f(r)$ holds for all $r \in (r_0, +\infty)\backslash E$. For $h(r) \equiv 2/(1-r)$, the statement implies the theorem (see [7–9]) on the Kovari-type inequality for analytic functions in the unit disk $\mathbb{D}$. In the Wiman inequality (see [6, 10–12]) on the Steinhaus probability space $(\Omega, \mathcal{A}, P)$ exponent $1/2$ almost surely (a.s.) can be replaced by $1/4$ (*Levy's phenomenon*), where $\Omega = [0;1]$, $\mathcal{A}$ is the σ-algebra of Borel's subsets of $[0;1]$ and P is the Lebesque measure.

Let $f \in \mathcal{E}_{+\infty}$ be an entire function and $\big(X_n(\omega)\big)$ be a *multiplicative system* (MS), i.e. the sequence of real random variables on Steinhaus probability space such that

$$\mathbf{E}(X_{i_1} X_{i_2} \cdots X_{i_k}) = 0$$

for any $i_1 < i_2 < \ldots < i_k$, $k \ge 1$, where $\mathbf{E}\xi$ is the expectation of random variable ξ, i.e. $\mathbf{E}\xi = \int_\Omega \xi(\omega) P(d\omega)$. We denote

$$\mathcal{K}(f, \mathcal{Z}) = \left\{ f(z,t) = \sum_{k=0}^{+\infty} a_k Z_k(t) z^k \colon t \in [0,1] \right\},$$

where $\mathcal{Z} = (Z_k(t))$ is a sequence complex-valued random variables.

In [12] (see also [6]), it is proved that if $f \in \mathcal{E}_{+\infty}$ is non-constant entire function, and $\mathcal{Z} = (Z_k)$ is a sequence of complex valued variables such that for any $k \ge 0$ we get $\big(\operatorname{Re} Z_k(t)\big) \in MS$, $\big(\operatorname{Im} Z_k(t)\big) \in MS$ and $|Z_k(t)| = 1$ a.s., then for every $\varepsilon > 0$ a.s. in $\mathcal{K}(f, \mathcal{Z})$ there exists a set $E := E(\varepsilon, t, f) \subset [1, +\infty)$ of finite logarithmic measure such that the inequality

$$M_f(r,t) := \max\big\{ |f(z,t)| : \ |z| = r \big\} \le \mu_f(r) \big(\ln \mu_f(r) \big)^{1/4+\varepsilon} \tag{3}$$

holds for $r \in [1; +\infty) \setminus E$.

In the cases $\mathcal{Z} = \mathcal{R}$, $\mathcal{Z} = \mathcal{H}$ and $\mathcal{Z} = \mathcal{S}$ we obtain suitable results from [10], [11] and [13], respectively (see also [14]), where $\mathcal{R} = \big(R_k(t)\big)$ is *the Rademacher sequence*, i.e. a sequence of independent random variables, such that $\mathbb{P}\{t \colon R_k(t) = -1\} = \mathbb{P}\{t \colon R_k(t) = 1\} = 0{,}5$, $k \in \mathbb{N}$, and $\mathcal{H} = \big(H_k(t)\big)$ is *the Steinhaus sequence*, i.e. a sequence independent random variables $H_k(t) = \exp\{2\pi i \eta_k(t)\}$, where $\{\eta_k(t)\}$ is a sequence independent uniformly distributed on $[0;1]$ random variables, $\mathcal{S} = \big(\exp\{2\pi i \theta_k \cdot t\}\big)$, where (θ_k) is the sequence of integer numbers such that $\theta_{k+1}/\theta_k \ge q > 2$, $k \ge 0$. We remark that $\big(\cos(2\pi\theta_k t)\big) \in MS$, $\big(\sin(2\pi\theta_k t)\big) \in MS$ in this case (see [13] for $q > 1$).

Let us denote by Θ the class of the sequences of real independent sub-Gaussian random variables (Z_n) on Steinhaus probability space, i.e. such that

$$(\exists D > 0)\ (\forall k \in \mathbb{N})\ (\forall \lambda \in \mathbb{R}) \colon \ \mathbf{E}(e^{\lambda Z_k}) \le e^{D\lambda^2}. \tag{4}$$

In general, the exponent $1/4 + \varepsilon$ in the inequality (3) cannot be replaced by the number $1/4$. From one result in [15] it follows that for any sequence of complex valued variables $\mathcal{Z} = (Z_k) \in MS$ and $|Z_k(t)| \ge 1$ a.s., $k \ge 0$, there exists an entire function $f \in \mathcal{E}_{+\infty}$ such that

$$\lim_{r \to +\infty} \frac{M_f(r,t)}{\mu_f(r) (\ln \mu_f(r))^{1/4}} = +\infty \quad \text{a.s. in } \mathcal{K}(f, \mathcal{Z}).$$

Similar results regarding to the Lévy fenomenon were also proved in the classes of random analytic functions in the unit disk (see [16, 17, 20]).

Note that in above cited results all sequences (MS, $\mathcal{R}$, $\mathcal{H}$, $\mathcal{S}$) are uniformly bounded a.s. Therefore, a natural question arises (see [19]): *is the Lévy phenomenon valid in the case of an unbounded sequence of random variables?* From result obtained in [19] it follows that for any sequence $Z \in \Theta$ there exists a set $E(\delta)$ of finite logarithmic measure such that for all $r \in (r_0(t), +\infty)\backslash E$ we have

$$M_f(r,t) \le \mu_f(r) \ln^{1/4+\delta} \mu_f(r) \quad \text{a.s. in } \mathcal{K}(f, \mathcal{Z}). \tag{5}$$

In this paper, we prove a similar statement regarding the inequality (2) for the analytic functions $f \in \mathcal{E}_R$, $R \in (0, +\infty]$, and the some sub-Gaussian sequences $Z \in \Theta$.

2 Auxiliary lemmas

We need the following elementary statement (see also [21, 22]).

Proposition 1. *If a sequence of random variables $\big(Z_n(\omega)\big)$ satisfies the conditions*

$$(\exists \alpha > 0)\ (\exists n_1 \in \mathbb{N}) \colon\ \sup\{\mathbf{E}|Z_n|^{\alpha} \colon\ n \ge n_1\} < +\infty, \tag{6}$$

$$(\exists \beta > 0)\ (\exists n_2 \in \mathbb{N}) \colon\ \inf\{\mathbf{E}|Z_n|^{-\beta} \colon\ n \ge n_1\} < +\infty, \tag{7}$$

then a.s.

$$(\exists N_1(\omega) \ge \max\{n_1, n_2\})\ (\forall n > N_1(\omega)) \colon\ \frac{1}{n^{1/\beta} \ln^{2/\beta} n} \le |Z_n(\omega)| \le n^{1/\alpha} \ln^{2/\alpha} n.$$

Indeed, by Markov's inequality and conditions (6) and (7) we get

$$\sum_{n=n_1}^{+\infty} \mathbb{P}\{\omega \colon\ |Z_n(\omega)|^{\alpha} \ge n \ln^2 n\} \le \sum_{n=n_1}^{+\infty} \frac{\mathbf{E}|Z_n(\omega)|^{\alpha}}{n \ln^2 n} < +\infty,$$

$$\sum_{n=n_2}^{+\infty} \mathbb{P}\{\omega \colon\ |Z_n(\omega)|^{-\beta} \ge n \ln^2 n\} \le \sum_{n=n_2}^{+\infty} \frac{\mathbf{E}|Z_n(\omega)|^{-\beta}}{n \ln^2 n} < +\infty.$$

Therefore, the First Lemma of Borel-Cantelli implies the statement of Proposition 1.

By conditions (6) and (7) the radius of convergence of a series

$$f(z, \omega) = \sum_{n=0}^{+\infty} Z_n(\omega) a_n z^n \tag{8}$$

is a.s. equal to the radius of convergence of series (1).

Suppose that $Z = (Z_n) \in \Theta$ is a sequence of real centered independent sub-Gaussian random variables such that condition (7) holds. The class of such random variables is denoted by $\mathcal{L}$.

For $Z \in \mathcal{L}$ we have (see [18, Exercise 7.8, p.81])

$$\sup_{k \in \mathbb{N}} \mathbf{E}(Z_k^2) = \sup_{k \in \mathbb{N}} \mathbf{D}(Z_k) \le 2D \quad \text{and} \quad \mathbf{E}(Z_k) = 0,\ \forall k \in \mathbb{N},$$

where $\mathbf{D}(Z_k) := \mathbf{E}(Z_k^2) - (\mathbf{E}Z_k)^2$ is the variance of random variable Z_k and D is positive constant from (4).

Remark that any sequence of random variables $\{Z_n\} \in \mathcal{L}$ satisfies conditions of Proposition 1 with $\alpha = 2$ and the radius of convergence R_ω of random power series of the form (8) is a.s. equal to the radius of convergence of series (1).

For $r \geq 0$ and an analytic function of the form (1) we denote

$$\nu_f(r) = \max\{n\colon |a_n|r^n = \mu_f(r)\},$$

$$\mathfrak{M}_f(r) = \sum_{n=0}^{+\infty} |a_n|r^n, \qquad M_f(r, N, \omega) = \max_{0\leq\theta<2\pi} \left| \sum_{n=0}^{N} Z_n(\omega) a_n r^n e^{in\theta} \right|,$$

$$S_f^2(r) = \sum_{n=0}^{+\infty} |a_n|^2 r^{2n}, \quad S_N = S_N(r) = \left(\sum_{n=0}^{N} |a_n|^2 r^{2n} \right)^{1/2}.$$

In what follows, $[x]$ denotes an integer part of a real $x \in \mathbb{R}$.

Similarly to [19] one can prove the following analogue of the Salem-Zygmund theorem (see [18, 23]).

Lemma 1. *Let $Z \in \mathcal{L}, h \in \mathcal{H}_R, f \in \mathcal{E}_R, N \in \mathbb{N}$. Then there exist an absolute constant $C > 0$ and a set E of finite h-logarithmic measure such that*

$$P\Big\{M_f(r, N, \omega) \geq CS_N \ln\ln S_N \sqrt{\ln N}\Big\} \leq \frac{2}{N^2}, \quad r \to R. \tag{9}$$

Lemma 2. *Let $Z \in \mathcal{L}$, $f \in \mathcal{E}_R$, $h \in \mathcal{H}_R$, $N = \left[h^5(r) \ln^5\{h(r)\mu_f(r)\}\right]$ and D is the positive constant from (4). There exist an absolute constant $C > 0$ and a set E of finite h-logarithmic measure such that for $r \to R$, $r \notin E$, we have*

$$\mathbb{P}\Big\{\omega\colon M_f(r, \omega) \geq CS_N(r) \ln\ln S_N(r)\sqrt{\ln N(r)} + \mu_f(r)\Big\} \leq \frac{2D+2}{N^2(r)}. \tag{10}$$

Proof. For $\delta > 0$ we denote

$$E_1 = \Big\{r\colon r\frac{\partial}{\partial r} \ln \mathfrak{M}_f(r) > h(r) \ln^{1+\delta} \mathfrak{M}_f(r),\ \ln \mathfrak{M}_f(r) > e\Big\}.$$

Then

$$\int_{E_1} \frac{h(r)dr}{r} < \int_{E_1} \frac{\frac{\partial}{\partial r} \ln \mathfrak{M}_f(r)dr}{\ln^{1+\delta} \mathfrak{M}_f(r)} < \int_1^{+\infty} \frac{du}{u^{1+\delta}} < +\infty.$$

Therefore for $r \notin E_1$ we get

$$\sum_{n=0}^{+\infty} n|a_n|r^n \leq h(r)\mathfrak{M}_f(r) \ln^{1+\delta} \mathfrak{M}_f(r).$$

Let

$$f_1(z) = \sum_{n=0}^{+\infty} na_n z^n, \qquad f_2(z) = \sum_{n=0}^{+\infty} n^2 a_n z^n.$$

So, there exists a set E_2 of finite h-logarithmic measure such that

$$\begin{aligned}
\sum_{n=0}^{+\infty} n^3|a_n|r^n &\leq h(r)\mathfrak{M}_{f_2}(r) \ln^{1+\delta} \mathfrak{M}_{f_2}(r) \\
&\leq h^2(r)\mathfrak{M}_{f_1}(r) \ln^{1+\delta} \mathfrak{M}_{f_1}(r) \ln^{1/2+\delta}\{h(r)\mathfrak{M}_{f_1}(r) \ln^{1+\delta} \mathfrak{M}_{f_1}(r)\} \\
&\leq h^2(r) \ln^2 h(r)\mathfrak{M}_{f_1}(r) \ln^{2+3\delta} \mathfrak{M}_{f_1}(r) \\
&\leq h^3(r) \ln^2 h(r)\mathfrak{M}_f(r) \ln^{1+\delta} \mathfrak{M}_f(r) \ln^{2+3\delta}\{h(r)\mathfrak{M}_f(r) \ln^{1+\delta} \mathfrak{M}_f(r)\} \\
&\leq h^{3+\delta}(r)\mathfrak{M}_f(r) \ln^{4+7\delta} \mathfrak{M}_f(r).
\end{aligned}$$

For $n \geq N(r)$ we consider events $B_n = \{\omega\colon |Z_n(w)| \geq n^{3/2}\}$. Using Markov's inequality we can estimate the probabilities of these events

$$\mathbb{P}(B_n) = \mathbb{P}\{\omega\colon |Z_n(\omega)|^2 \geq n^3\} \leq \frac{\mathbf{D}Z_n}{n^3} \leq \frac{2D}{n^3},$$

$$\sum_{n=N(r)}^{+\infty} \mathbb{P}(B_n) \leq 2D \sum_{n=N(r)}^{+\infty} \frac{1}{n^3} \leq \frac{2D}{N^2(r)}, \quad r \to R.$$

Let $B = \bigcup\limits_{n=N(r)}^{+\infty} B_n$. Then $\mathbb{P}(B) \leq \frac{2D}{N^2(r)}$, $r \to R$. From (2) for $\omega \notin B$ and $r \notin E_2$ we get

$$\begin{aligned}
\max_{0\leq\theta<2\pi}\Big|\sum_{n=N(r)}^{+\infty} Z_n(\omega)a_n r^n e^{in\theta}\Big| &\leq \sum_{n=N(r)}^{+\infty} |Z_n(\omega)||a_n|r^n \leq \sum_{n=N(r)}^{+\infty} n^{3/2}\frac{n}{N(r)}|a_n|r^n \\
&\leq \frac{1}{N(r)}\sum_{n=N(r)}^{+\infty} n^3|a_n|r^n \leq \frac{1}{N(r)} h^{3+\delta}(r)\mathfrak{M}_f(r)\ln^{4+7\delta}\mathfrak{M}_f(r) \\
&\leq \frac{1}{N(r)} h^{4+2\delta}(r)\mu_f(r)\big(\ln(h(r)\mu_f(r))\big)^{9/2+9\delta} \\
&\leq \frac{1}{N(r)} h^{5}(r)\mu_f(r)\ln^5(h(r)\mu_f(r)) \leq \mu_f(r), \quad r \to R.
\end{aligned}$$

Therefore,

$$\mathbb{P}\Big\{\omega\colon \max_{0\leq\theta<2\pi}\Big|\sum_{n=N(r)}^{+\infty} Z_n(\omega)a_n r^n e^{in\theta}\Big| \geq \mu_f(r)\Big\} \leq \frac{2D}{N^2(r)}.$$

Finally, from Lemma 1 for $r \notin E_2, r \to R$, we obtain

$$\mathbb{P}\Big\{\omega\colon \max_{0\leq\theta<2\pi}\Big|\sum_{n=0}^{+\infty} Z_n(\omega)a_n r^n e^{in\theta}\Big| \geq CS_N \ln\ln S_N\sqrt{\ln N(r)} + \mu_f(r)\Big\} \leq \frac{2D+2}{N^2(r)}.$$

$\square$

Similarly to [12] (see also [6, 13]) we get the following lemma.

Lemma 3. *Let $l(r)$ be a continuous increasing to $+\infty$ function on $(r_0; R)$, $E \subset (r_0; R)$ and $\sup E = R$. Then there is an infinite sequence $1 < r_1 \leq \cdots \leq r_n \to R$ as $n \to +\infty$ such that*

(i) $\forall n \in \mathbb{N}\colon\ r_n \notin E$;

(ii) $\forall n \in \mathbb{N}\colon\ \ln l(r_n) \geq \frac{n}{2}$;

(iii) if $(r_n; r_{n+1}) \cap E \neq (r_n, r_{n+1})$, *then* $l(r_{n+1}) \leq el(r_n)$;

(iv) the set of indices, for which (iii) holds, is unbounded.

3 Main result

Theorem 1. *Let $Z \in \mathcal{L}$, $f \in \mathcal{E}_R$, $h \in \mathcal{H}_R$, $\delta > 0$. Then there exists a set $E(\delta)$ of finite h-logarithmic measure such that for all $r \in (r_0(\omega), R)\backslash E$ almost surely in $\mathcal{K}(f, \mathcal{Z})$ we have*

$$M_f(r,\omega) \leq \sqrt{h(r)}\mu_f(r)\Big(\ln^3 h(r)\ln\{h(r)\mu_f(r)\}\Big)^{1/4+\delta}. \tag{11}$$

Proof. Choose $k(r) = h(r)\mu_f(r)$, a set E and a sequence $\{r_k\}$ from Lemma 2. Let

$$F_k = \Big\{\omega\colon M_f(r_k,\omega) \geq CS_{N(r_k)}(r_k)\ln\ln S_{N(r_k)}(r_k)\sqrt{\ln N(r_k)} + \mu_f(r_k)\Big\}.$$

By Lemma 3 and by the definition of $N(r)$ we get

$$\sum_{k=1}^{+\infty} P(F_k) \leq \sum_{k=1}^{+\infty}\frac{2D+2}{N^2(r_k)} \leq \sum_{k=1}^{+\infty}\frac{2D+2}{\ln^{10} h(r)\ln^{10}(h(r_k)\mu_f(r_k))} \leq \frac{D+1}{2}\sum_{k=1}^{+\infty}\frac{1}{k^{10}} < +\infty.$$

Then by Borel-Cantelli's lemma for almost all $\omega \in [0,1]$ there exists $k_0(\omega)$ such that for $k \geq k_0(\omega)$ we obtain

$$M_f(r_k,\omega) < CS_{N(r_k)}(r_k)\ln\ln S_{N(r_k)}(r_k)\sqrt{\ln N(r_k)} + \mu_f(r_k).$$

Using inequalites $S_{N(r)}(r) \leq \mathfrak{M}_f(r)\mu_f(r)$ and definition of $N(r)$ we get

$$\begin{aligned} M_f(r_k,\omega) &< C\sqrt{\mathfrak{M}_f(r_k)\mu_f(r_k)}\ln\ln\Big(\mathfrak{M}_f(r_k)\mu_f(r_k)\Big)\ln^{1/2}\Big(h^5(r_k)\ln^5\big(h(r_k)\mu_f(r_k)\big)\Big) + \mu_f(r_k) \\ &< \sqrt{h(r_k)}\mu_f(r_k)\Big(\ln^3 h(r_k)\ln\{h(r_k)\mu_f(r_k)\}\Big)^{1/4+3\delta}. \end{aligned}$$

Let $r \geq r_{k_0(\omega)}$ be an arbitrary number outside set the E, $r \in (r_p, r_{p+1})$. By Lemma 3

$$h(r_{p+1})\mu_f(r_{p+1}) \leq eh(r_p)\mu_f(r_p) \leq eh(r)\mu_f(r),$$

and $h(r_{p+1}) \leq eh(r)$, $\mu_f(r_{p+1}) \leq e\mu_f(r)$. Therefore for almost all $\omega \in [0;1]$ and $r \geq r_0(\omega)$ outside a set of finite h-logarithmic measure E we have

$$\begin{aligned} M_f(r,\omega) &\leq M_f(r_{p+1},\omega) \\ &< \sqrt{h(r_{p+1})}\mu_f(r_{p+1})(\ln^3 h(r_{p+1})\ln\{h(r_{p+1})\mu_f(r_{p+1})\})^{1/4+3\delta} \\ &\leq e\sqrt{eh(r)}\mu_f(r_{p+1})(\ln^3(eh(r))\ln\{eh(r)\mu_f(r)\})^{1/4+3\delta} \\ &\leq \sqrt{h(r)}\mu_f(r_{p+1})(\ln^3 h(r)\ln\{h(r)\mu_f(r)\})^{1/4+4\delta}. \end{aligned}$$

□

In the case of complex random variables we get such a statement.

Corollary 1. *Let $\operatorname{Re} Z \in \mathcal{L}$, $\operatorname{Im} Z \in \mathcal{L}$, $f \in \mathcal{E}_R$, $h \in \mathcal{H}_R$, $\delta > 0$. Then there exists a set $E(\delta)$ of finite h-logarithmic measure such that for all $r \in (r_0(\omega), R)\backslash E$ almost surely in $\mathcal{K}(f, \mathcal{Z})$ we have*

$$M_f(r,\omega) \leq \sqrt{h(r)}\mu_f(r)\Big(\ln^3 h(r)\ln\{h(r)\mu_f(r)\}\Big)^{1/4+\delta}.$$

4 Some examples

There exists $Z \notin \mathcal{L}$ such that $\mathbf{E}Z_n = 0$, $\sup_n \mathbf{D}Z_n = +\infty$ and inequality (11) does not hold. It follows from the following statement for entire functions $f \in \mathcal{E}_{+\infty}$.

Theorem 2 ([19]). *For any $\alpha > 0$ there exist a sequence of real independent random variables satisfying*

$$\sup_n \mathbf{D}Z_n = +\infty, \qquad \mathbf{E}Z_n = 0 \ \textit{for all} \ n \in \mathbb{Z}_+,$$

entire function $f \in \mathcal{E}_{+\infty}$ and $h \in \mathcal{H}_{+\infty}$ such that almost surely in $\mathcal{K}(f, \mathcal{Z})$

$$M_f(r,\omega) \geq \sqrt{h(r)}\mu_f(r)\Big(\ln^3 h(r) \ln\{h(r)\mu_f(r)\}\Big)^{1/4+\alpha}, \quad r \in (r_0(\omega), +\infty).$$

We prove similar statement for $f \in \mathcal{E}_1$.

Theorem 3. *For any $\alpha > 0$ there exist a sequence of real independent random variables satisfying*

$$\sup_n \mathbf{D}Z_n = +\infty, \qquad \mathbf{E}Z_n = 0 \ \textit{for all} \ n \in \mathbb{Z}_+,$$

analytic function $f \in \mathcal{E}_1$ and $h \in \mathcal{H}_1$ such that almost surely in $\mathcal{K}(f, \mathcal{Z})$

$$M_f(r,\omega) \geq \sqrt{h(r)}\mu_f(r) \ln^\alpha \mu_f(r)\Big(\ln\{h(r)\mu_f(r)\}\Big)^{1/4}, \quad r \in (r_0(\omega), 1).$$

Proof. We choose

$$f(z) = \sum_{n=1}^{+\infty} \frac{\exp(n^\varepsilon) z^n}{n^\alpha}, \qquad g(z) = \sum_{n=1}^{+\infty} \exp(n^\varepsilon) z^n$$

and a sequence of independent random variables (Z_n) such that

$$\mathbb{P}\{\omega\colon Z_n(\omega) = -n^\alpha\} = \mathbb{P}\{\omega\colon Z_n(\omega) = n^\alpha\} = \frac{1}{2}.$$

Then

$$\mathbf{E}Z_n = -n^\alpha \frac{1}{2} + n^\alpha \frac{1}{2} = 0, \qquad \mathbf{D}Z_n = n^{2\alpha}\frac{1}{2} + n^{2\alpha}\frac{1}{2} = n^{2\alpha}, \qquad \sup_n \mathbf{D}Z_n = +\infty.$$

Denote

$$f(z,\omega) = \sum_{n=1}^{+\infty} Z_n(\omega)\frac{\exp(n^\varepsilon) z^n}{n^\alpha} = \sum_{n=1}^{+\infty} R_n(\omega)\exp(n^\varepsilon) z^n = g(z,\omega), \quad M_f(r,\omega) = M_g(r,\omega),$$

where $\{R_n(\omega)\}$ is a sequence of the Rademacher random variables. In [20], for $g(z,\omega)$ there was proved that for $g(z,\omega)$, $h(r) = (1-r)^{-1}$ and some $C > 0$ we have

$$M_f(r,\omega) = M_g(r,\omega) \geq \sqrt{h(r)}\mu_g(r)\Big(\ln\{h(r)\mu_g(r)\}\Big)^{1/4}, \quad r \uparrow 1.$$

Remark that

$$\mu_g(r) = \max_{n\in\mathbb{Z}_+}\Big\{\exp(n^\varepsilon) r^n\Big\} = \max_{n\in\mathbb{Z}_+}\Big\{n^\alpha \frac{\exp(n^\varepsilon) r^n}{n^\alpha}\Big\} \geq \nu_f^\alpha(r)\mu_f(r).$$

From
$$\ln \mu_h(r) - \ln \mu_h(r_0) = \int_{r_0}^{r} \frac{\nu_h(t)dt}{t} \leq \nu_h(r)(\ln r - \ln r_0)$$
it follows that for any $r > r_2 > r_0$ there exists a constant $c > 0$ such that
$$\nu_f(r) \geq \frac{\ln \mu_f(r) - \ln \mu_f(r_0)}{\ln r - \ln r_0} \geq \frac{c \ln \mu_f(r)}{-\ln r_0}.$$

Then $\mu_g(r) \geq \nu_f^{\alpha}(r)\mu_f(r) \geq C_1 \mu_f(r) \ln^{\alpha} \mu_f(r)$. Finally, almost surely in $\mathcal{K}(f, \mathcal{Z})$ we get
$$\begin{aligned} M_f(r,\omega) &> \sqrt{h(r)}\mu_g(r)\Big(\ln \big\{h(r)\mu_g(r)\big\}\Big)^{1/4} \\ &\geq \sqrt{h(r)}C_1\mu_f(r) \ln^{\alpha} \mu_f(r)\Big(\ln \big\{h(r)C_1\mu_f(r) \ln^{\alpha} \mu_f(r)\big\}\Big)^{1/4} \\ &\geq \sqrt{h(r)}\mu_f(r) \ln^{\alpha} \mu_f(r)\Big(\ln \big\{h(r)\mu_f(r)\big\}\Big)^{1/4}, \quad r \in (r_0(\omega), 1). \end{aligned}$$

□

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Куриляк А.О., Скасків О.Б. *Субгаусові випадкові величини та нерівність Вімана для аналітичних функцій* // Карпатські матем. публ. — 2023. — Т.15, №1. — С. 306–314.

Нехай f — аналітічна функція в $\{z : |z| < R\}$ вигляду $f(z) = \sum\limits_{n=0}^{+\infty} a_n z^n$. У статті доводиться нерівність типу Вімана для випадкових аналітичних функцій вигляду $f(z,\omega) = \sum\limits_{n=0}^{+\infty} Z_n(\omega) a_n z^n$, де (Z_n) — послідовність на ймовірнісному просторі Штейнгауса дійсних незалежних центрованих субгаусових випадкових величин, тобто $(\exists D > 0)$ $(\forall k \in \mathbb{N})$ $(\forall \lambda \in \mathbb{R})\colon$ $\mathbf{E}(e^{\lambda Z_k}) \le e^{D\lambda^2}$, і таких, що $(\exists \beta > 0)$ $(\exists n_0 \in \mathbb{N})\colon$ $\inf\limits_{n\ge n_0} \mathbf{E}|Z_n|^{-\beta} < +\infty$.

Доведено, що для кожного $\delta > 0$ існує множина $E(\delta) \subset [0,R)$ скінченної логарифмічної h-міри $\big($тобто $\int_E h(r)d\ln r < +\infty\big)$ така, що майже напевно для всіх $r \in (r_0(\omega), R)\backslash E$ маємо

$$M_f(r,\omega) := \max\big\{|f(z,\omega)|\colon |z| = r\big\} \le \sqrt{h(r)}\mu_f(r)\Big(\ln^3 h(r)\ln\{h(r)\mu_f(r)\}\Big)^{1/4+\delta},$$

де $h(r)$ — довільна фіксована неперервна неспадна на $[0;R)$ функція така, що $h(r) \ge 2$ для всіх $r \in (0,R)$ і $\int_{r_0}^{R} h(r)d\ln r = +\infty$ для деякого $r_0 \in (0,R)$.

Ключові слова і фрази: аналітична функція, феномен Леві, нерівність Вімана, субгаусові випадкові величини.