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Inverse problems of determining an unknown depending on time coefficient for a parabolic equation with involution and periodicity conditions

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The solution of the investigated problem with an unknown coefficient in the equation was constructed by using the method of separation of variables. The properties of the induced spectral problem for the second-order differential equation with involution are studied. The dependence on the equation involutive part of the spectrum and its multiplicity as well as the structure of the system of root functions and partial solutions of the problem were investigated. The conditions for the existence and uniqueness of the solution of the inverse problem have been established. To determine the required coefficient, Volterra's integral equation of the second kind was found and solved.

Key words and phrases: inverse problem, heat conduction equation, method of separation of variables, nonlocal condition, involution, Riesz basis.

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1 Introduction

Problems of determining the coefficients or the right-hand side of a differential equation simultaneously with its solution are called inverse problems of mathematical physics. Such problems appear, for example, in the simulation of hyperthermia, thrombosis and sclerosis of vessels, optical tomography.

Inverse heat conduction problems arise in various branches of applied heat engineering. In particular, the problem of modeling the thermodiffusion process is described in the paper [9]. The authors analyzed a problem that describes a mathematical model of the process of heat diffusion in a closed metal rod, the insulation of which is slightly permeable. Therefore, the temperature at the point of the rod on one side of the insulation affects the diffusion process in the rod on the other side of the insulation. The authors proposed to consider the following heat conduction equation with involution for modeling the process:

$$\frac{\partial u(x,t)}{\partial t} = \alpha \frac{\partial^2 u(x,t)}{\partial x^2} + \beta \frac{\partial^2 u(-x,t)}{\partial x^2}, \quad \alpha, \beta \in \mathbb{R}. \tag{1}$$

In the paper [1], for the equation

$$\frac{\partial u(x,t)}{\partial t} - \frac{\partial^2 u(x,t)}{\partial x^2} + \frac{\partial^2 u(-x,t)}{\partial x^2} = f(x), \quad (x,t) \in \Omega := \{-\pi < x < \pi,\ 0 < t < T\}, \tag{2}$$

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inverse problems of determining a pair of unknown functions $\{u(x,t),\ f(x)\}$ with boundary conditions

$$\begin{aligned} u(-\pi,t)=u(\pi,t)=0, &\qquad \frac{\partial u(-\pi,t)}{\partial x}=\frac{\partial u(\pi,t)}{\partial x}=0,\\ u(-\pi,t)-u(\pi,t)=0, &\quad \frac{\partial u(-\pi,t)}{\partial x}-\frac{\partial u(\pi,t)}{\partial x}=0,\\ u(-\pi,t)+u(\pi,t)=0, &\quad \frac{\partial u(-\pi,t)}{\partial x}+\frac{\partial u(\pi,t)}{\partial x}=0, \end{aligned} \tag{3}$$

are investigated. In the paper [22], for equation (1), the inverse problem with nonlocal conditions

$$\frac{\partial u(-\pi,t)}{\partial x}-\frac{\partial u(\pi,t)}{\partial x}-\alpha u(\pi,t)=0, \quad u(-\pi,t)-u(\pi,t)=0,$$

which are weak perturbations of conditions (3), was considered. In [23], for the equation (2) the inverse problem of finding $\{u(x,t),\ f(x)\}$ with the initial condition $u(x,0)=\varphi(x)$, condition of redefinition $u(x,E)=\psi(x)$ and Ionkin-type conditions

$$\frac{\partial u(-\pi,t)}{\partial x}+\alpha\frac{\partial u(\pi,t)}{\partial x}=0, \quad u(-\pi,t)-u(\pi,t)=0,$$

are investigated.

The inverse problem of mathematical biology is considered in [10], namely, the problem of finding a time-dependent source function for a population model with nonlocal boundary conditions of the population density.

So, in $\Omega=\{0<x<1,\ 0<t<T\}$, for the equation

$$\frac{\partial u(x,t)}{\partial t}=\frac{\partial^2 u(x,t)}{\partial x^2}+r(t)u(x,t)+u(x,t)$$

the inverse problem of finding $\{u(x,t),\ r(t)\}$ with the initial condition $u(x,0)=\varphi(x)$, condition of redefinition

$$\int_0^1 u(x,t)dt=E(t)$$

and perturbed antiperiodicity conditions

$$\frac{\partial u(0,t)}{\partial x}+\frac{\partial u(1,t)}{\partial x}=0, \quad u(0,t)+bu(1,t)=0,$$

was considered.

In the papers [11–13], inverse problems of determining $\{u(x,t),\ r(t)\}$ with nonlocal boundary conditions used in models of population age description were investigated.

Mixed and boundary value problems for equations with partial derivatives, which contain involution, were studied in [2,3,16,20,23,25]. For ordinary differential operators with involution boundary value problems were studied in the papers [4,5,18,19,26,27].

2 Notations and main results

Denote

$$W_2^2(-1,1):=\Big\{y\in L_2(-1,1):\ y^{(m)}\in C[-1,1],\ y^{(2)}\in L_2(-1,1),\ m=0,1\Big\},$$

$$(y;u)_{W_2^2(-1,1)}:=\sum_{k=0}^{2}(y^{(k)};u^{(k)})_{L_2(-1,1)}, \quad \|y\|^2_{W_2^2(-1,1)}:=(y;y)_{W_2^2(-1;1)}.$$

Let E be the identity map in $L_2(-1,1)$, $I : L_2(-1,1) \ni y(x) \mapsto y(-x) \in L_2(-1,1)$ be the involution operator. Denote $p_j := \frac{1}{2}(E + (-1)^j I)$ the orthoprojectors of space $L_2(-1,1)$ and

$$L_{j,2}(-1,1) := \{y \in L_2(-1,1) : \; y = p_j y\}, \quad j = 0, 1.$$

In the domain $D_T = \{-1 < x < 1, 0 < t \le T\}$, let us consider the heat conduction equation with involution

$$\begin{aligned}\frac{\partial u(x,t)}{\partial t} = \frac{\partial^2 u(x,t)}{\partial x^2} &+ \alpha_1(1+\gamma x)\Big(\frac{\partial^2 u(x,t)}{\partial x^2} - \frac{\partial^2 u(-x,t)}{\partial x^2}\Big) \\ &+ \alpha_2\Big(\frac{\partial u(x,t)}{\partial x} + \frac{\partial u(-x,t)}{\partial x}\Big) - r(t)u(x,t) + f(x,t), \quad (x,t) \in D_T,\end{aligned} \tag{4}$$

with boundary conditions

$$\begin{cases} u(-1,t) - u(1,t) = 0, \\ \beta_1 \dfrac{\partial u(-1,t)}{\partial x} + \beta_2 \dfrac{\partial u(1,t)}{\partial x} = 0, \end{cases} \qquad 0 \le t \le T, \quad \beta_1, \beta_2 \in \mathbb{R}, \quad \beta_1 \ne \beta_2, \tag{5}$$

initial condition

$$u(x,0) = \eta(x), \quad -1 \le x \le 1, \tag{6}$$

and redefinition condition

$$\int_{-1}^{1} u(x,t)dx = E(t). \tag{7}$$

Definition 1. *A pair of functions $\{r(t), u(x,t)\}$ from the set $C[-1,1] \times \big(C^{2,1}(D_T) \cap C^{1,0}(\overline{D_T})\big)$ is called a classical solution of the inverse problem* (4)–(7).

Let $\boldsymbol{L} : L_2(-1,1) \to L_2(-1,1)$ be an operator of the problem

$$\begin{aligned}-v''(x) + \alpha_1(1+\gamma x)(v''(x) - v''(-x)) + \alpha_2(v'(x) - v'(-x)) = f(x),& \\ \alpha_1, \alpha_2 \in \mathbb{R}, \quad -1 < x < 1,&\end{aligned} \tag{8}$$

$$\begin{cases} \ell_1 v := v(-1) - v(1) = 0, \\ \ell_2 v := \beta_1 v'(-1) + \beta_2 v'(1) = 0, \end{cases} \qquad \beta_1 \ne \beta_2, \tag{9}$$

$$D(\boldsymbol{L}) = \{v \in W_2^2(-1,1) : \; \ell_1 v = \ell_2 v = 0\}.$$

Theorem 1.

A. *For any $\beta_1, \beta_2 \in \mathbb{R}$, if $\beta_1 \ne \beta_2$ then the operator $\boldsymbol{L}$ has the system of root functions*

$$\begin{aligned}V_h := \Big\{v_{s,m}(x) \in L_2(-1,1) : v_{0,0}(x) = \frac{1}{\sqrt{2}}, v_{1,m}(x) = (1+hx)\sin m\pi x,& \\ v_{0,m} = \cos m\pi x, m \in \mathbb{N}\Big\},&\end{aligned} \tag{10}$$

which is the Riesz basis of the space $L_2(-1,1)$, $h = \dfrac{\beta_1 + \beta_2}{\beta_1 - \beta_2}$.

In this case, there is a biorthogonal system

$$\begin{aligned}W_h := \Big\{w_{s,m}(x) \in L_2(-1,1) : w_{0,0}(x) = \frac{1}{\sqrt{2}}(1-hx), w_{1,m}(x) = \sin m\pi x,& \\ w_{0,m} = (1-hx)\cos m\pi x, m \in \mathbb{N}\Big\}.&\end{aligned}$$

A.1. *Let* $\gamma = \alpha_2 = \dfrac{\beta_1 + \beta_2}{\beta_1 - \beta_2}$. *Then the operator* $\mathbf{L}$ *has the set of eigenvalues* $\sigma \cup \sigma_1$, *where* $\sigma := \{\lambda_k \in \mathbb{R},\ \lambda_k = \pi^2 k^2,\ k \in \mathbb{N}\}$, $\sigma_1 := \{\lambda_{1,k} \in \mathbb{R},\ \lambda_{1,k} = (1 - 2\alpha_1)\lambda_k,\ \lambda_k \in \sigma,\ k \in \mathbb{N}\}$, *and the system of eigenfunctions* V_h.

A.2. *Let* $\alpha_1 = 0$, $\gamma \neq \dfrac{\beta_1 + \beta_2}{\beta_1 - \beta_2}$. *Then the operator* $\mathbf{L}$ *has the set of double eigenvalues* σ *and the system of eigenfunctions* V_h.

A.3. *Let* $\alpha_1 = 0$, $\gamma = \dfrac{\beta_1 + \beta_2}{\beta_1 - \beta_2}$. *Then the operator* $\mathbf{L}$ *has the set of double eigenvalues* σ *and the system of eigenfunctions* V_h.

Let

$$f(t,x) = f_{0,0}(t)\nu_{0,0}(x) + \sum_{k=1}^{\infty} \big(f_{0,k}(t)\nu_{0,k}(x) + f_{1,k}(t)\nu_{1,k}(x)\big),$$
$$\eta(x) = \eta_{0,0}\nu_{0,0}(x) + \sum_{k=1}^{\infty} \big(\eta_{0,k}\nu_{0,k}(x) + \eta_{1,k}\nu_{1,k}(x)\big),$$

where $f_{s,k} = (f; w_{r,k})_{L_2(-1,1)}$, $\eta_{s,k} = (\eta; w_{r,k})_{L_2(-1,1)}$ for $s = 0, 1$ and $k \in \mathbb{N}$.

Theorem 2.
A.1. *Let* $\gamma = \alpha_2 = \dfrac{\beta_1 + \beta_2}{\beta_1 - \beta_2}$ *and the following assumptions hold:*

B1) $\eta \in C^4[-1,1]$, $\eta(-1) - \eta(1) = 0$, $\beta_1\eta'(-1) + \beta_2\eta'(1) = 0$, $\int_{-1}^{1} \eta(x)dx = E(0)$;

B2) $E(t) \in C^1[-1,1]$;

B3) $f(x,t) \in C(\overline{D_T}) \cap C^4(D_T)$, $f(-1,t) - f(1,t) = 0$, $\beta_1 \dfrac{\partial f(-1,t)}{\partial x} + \beta_2 \dfrac{\partial f(1,t)}{\partial x} = 0$, $\int_{-1}^{1} f(x,t)dx \neq 0$;

B4) $\mu_k = (h - \alpha_2)(2k\pi - 1)$, $k \in \mathbb{N}$.

Then there is a unique solution of the problem (4)–(6) *of the following form*

$$\begin{aligned} u(x,t) =& \Big(\eta_{0,0} + \int_0^t r(\tau) f_{0,0}(\tau)d\tau\Big)\nu_{0,0}(x) \\ &+ \sum_{k=1}^{\infty} \bigg(\Big(\eta_{1,k}e^{-\lambda_{1,k}t} + \int_0^t r(\tau) f_{1,k}(\tau) e^{-\lambda_{1,k}(t-\tau)}d\tau\Big)\nu_{1,k}(x) \\ &\quad + \Big(\eta_{0,k}e^{-\lambda_k t} + \int_0^t r(\tau) f_{0,k}(\tau) e^{-\lambda_k(t-\tau)}d\tau\Big)\nu_{0,k}(x)\bigg), \end{aligned} \tag{11}$$

and the pair of functions $\{r(t), u(x,t)\}$ *is the unique solution of the inverse problem* (4)–(7).

A.2. *Let* $\alpha_1 = 0$, $\gamma \neq \dfrac{\beta_1 - \beta_2}{\beta_2 + \beta_1}$ *and the assumptions B1)–B3) hold. Then there is a unique solution of the problem* (4)–(6) *of the following form*

$$\begin{aligned} u(x,t) =& \Big(\eta_{0,0} + \int_0^t r(\tau) f_{0,0}(\tau) d\tau\Big) \nu_{0,0}(x) \\ &+ \sum_{k=1}^{\infty} \bigg(\Big(\eta_{0,k} e^{-\lambda_k t} + \int_0^t r(\tau) f_{0,k}(\tau) e^{-\lambda_k (t-\tau)} d\tau\Big) \nu_{0,k}(x) \\ &\qquad + \Big(\eta_{1,k} e^{-\lambda_k t} + \int_0^t r(\tau) f_{1,k}(\tau) e^{-\lambda_k (t-\tau)} d\tau\Big) \nu_{1,k}(x) \bigg), \end{aligned} \tag{12}$$

and the pair of functions $\{r(t), u(x,t)\}$ *is the unique solution of the inverse problem* (4)–(7).

A.3. *Let* $\alpha_1 = 0$, $\gamma = \dfrac{\beta_1 - \beta_2}{\beta_2 + \beta_1}$ *and the assumptions B1)–B3) hold. Then there is a unique solution of the problem* (4)–(6) *of the following form*

$$\begin{aligned} u(x,t) =& \Big(\eta_{0,0} + \int_0^t r(\tau) f_{0,0}(\tau) d\tau\Big) \nu_{0,0}(x) \\ &+ \sum_{k=1}^{\infty} \bigg(\Big(\eta_{0,k} e^{-\lambda_k t} + \int_0^t r(\tau) f_{0,k}(\tau) e^{-\lambda_k (t-\tau)} d\tau\Big) \nu_{0,k}(x) \\ &\qquad - \mu_k \int_0^t \Big(\int_0^\tau r(\varrho) f_{1,k}(\varrho) e^{-\lambda_k(\tau-\varrho)} d\varrho \Big) e^{-\lambda_k(t-\tau)} d\tau \\ &\qquad \times \Big(\eta_{1,k} e^{-\lambda_k t} \nu_{0,k}(x) + \int_0^t r(\tau) f_{1,k}(\tau) e^{-\lambda_k (t-\tau)} d\tau\Big) \nu_{1,k}(x) \bigg), \end{aligned} \tag{13}$$

and the pair of functions $\{r(t), u(x,t)\}$ *is the unique solution of the inverse problem* (4)–(7).

3 Proof of the Theorem 1

Let us consider the eigenvalue problem for the equation

$$-\nu''(x) = \lambda \nu(x), \quad \lambda \in \mathbb{C}, \quad -1 \le x \le 1, \tag{14}$$

with boundary conditions (9).

Determine the fundamental system of solutions for the equation (14)

$$\begin{cases} \nu_0(x,\varrho) = e^{\varrho x} + e^{-\varrho x}, \\ \nu_1(x,\varrho) = e^{\varrho x} - e^{-\varrho x}, \end{cases} \quad \operatorname{Re} \varrho \le 0, \quad \lambda = \varrho^2,$$

and substitute the general solution $\nu(x,\varrho) = C_0 \nu_0(x,\varrho) + C_1 \nu_1(x,\varrho)$, $C_0, C_1 \in \mathbb{R}$, of the equation (14) into boundary conditions (9).

To determine the parameters C_0, C_1 we obtain the system of linear algebraic equations with the matrix of coefficients

$$\Omega(\varrho) = \begin{pmatrix} 0 & \omega_2(\varrho) \\ \omega_1(\varrho) & \omega_3(\varrho) \end{pmatrix},$$

where $\omega_2(\varrho) = 2(e^{\varrho} - e^{-\varrho})$, $\omega_1(\varrho) = 2\varrho(\beta_1 - \beta_2)(e^{-\varrho} - e^{\varrho})$, $\omega_3(\varrho) = 2\varrho(\beta_1 + \beta_2)(e^{-\varrho} + e^{\varrho})$.

To determine the eigenvalues of problem (14), (9) we obtain the characteristic equation $\det \Omega(\varrho) = 4\varrho(\beta_1 - \beta_2)(e^{-\varrho} - e^{\varrho})^2$, which has the roots $0, \pi k, k = \pm 1, \pm 2, \ldots$.

Therefore, problem (13), (9) has eigenvalues $\lambda_k = \pi^2 k^2$, $k = 0, 1, \ldots$, and corresponding eigenfunctions $v_{0,0}(x) := \frac{1}{\sqrt{2}}$, $v_{1,k}(x) := \cos \pi k x$, $k \in \mathbb{N}$.

The attached functions of the problem are defined by relations $v_{1,k} := (1 + hx)\sin \pi x$, $k \in \mathbb{N}$. Substituting these expressions into the boundary conditions (9), we obtain $h = \frac{\beta_1 - \beta_2}{\beta_2 + \beta_1}$.

Therefore, the operator of the problem (14), (9) has a spectrum σ and the system of functions V_h, which are the root functions in the sense of ratios [17]:

$$\begin{cases} -v''_{0,k}(x) = \lambda_k v_{0,k}(x), \\ -v''_{1,k}(x) = \lambda_k v_{1,k}(x) + \mu_k v_{0,k}(x), \end{cases} \quad k \in \mathbb{N},$$

where $\mu_k = 2k\pi h$, $k \in \mathbb{N}$.

Note, that in the case $\beta_2 = -\beta_1$ the boundary conditions (9) coincide with the periodicity conditions, $\mu_k = 0$, $k \in \mathbb{N}$, and the system of functions (10) is an orthonormal basis in the space $L_2(-1, 1)$:

$$V_0 = \left\{ \tau_{s,k}(x) \in L_2(-1,1) : \ \tau_{0,0}(x) = \frac{1}{\sqrt{2}}, \ \tau_{0,k}(x) = \cos \pi k x, \ \tau_{1,k}(x) = \sin \pi k x, \ k \in \mathbb{N} \right\}.$$

If $\beta_2 = -\beta_1$, then the boundary conditions (9) are singular and $\det \Omega(\varrho) \equiv 0$ (see [19]).

The operator of the conjugate problem to (14), (9) (see [3, 19])

$$-w''(x) = \overline{\lambda} w(x), \quad \lambda \in \mathbb{C}, \ -1 \le x \le 1,$$

$$\begin{cases} \beta_2 w(-1) + \beta_1 w(1) = 0, \\ w'(-1) - w'(1) = 0, \end{cases}$$

has the system (10) of root functions that is orthogonal in the sense of equalities

$$(v_{r,k}; w_{s,m})_{L_2(-1,1)} = \delta_{r,s}\delta_{k,m}, \quad r, s = 0, 1, \ k, m \in \mathbb{N}.$$

Lemma 1. *For arbitrary numbers β_1, $\beta_2 \in \mathbb{R}$, $\beta_1 \neq -\beta_2$, the system of functions V_h is the Riesz basis of the space $L_2(-1, 1)$.*

Proof. The boundary conditions (9) are regular by Birkhoff [19]. Therefore systems of functions V_h, W_h are complete and minimal in space $L_2(-1, 1)$.

From the definition of these systems for an arbitrary function $\varphi \in L_2(-1, 1)$ we obtain Bessel inequalities [14]:

$$\begin{cases} (\varphi, v_{0,0})^2_{L_2(-1,1)} + \sum_{k=1}^{\infty} \sum_{r=0}^{1} (\varphi, v_{r,k})^2_{L_2(-1,1)} \le M_0 \|\varphi\|^2_{L_2(-1,1)}, \\ (\varphi, w_{0,0})^2_{L_2(-1,1)} + \sum_{m=1}^{\infty} \sum_{s=0}^{1} (\varphi, w_{s,m})^2_{L_2(-1,1)} \le M_0 \|\varphi\|^2_{L_2(-1,1)}, \end{cases} \quad \text{where } M_0 = 2(1 + h^2).$$

Therefore, applying theorem of N.K. Bari (see [4]), we obtain the statement of lemma. □

Thus, the statement **A.1** of Theorem 1 is proved.

Let $O(V_h, \sigma)$ be the set of operators $\boldsymbol{L}: L_2(-1,1) \to L_2(-1,1)$, which have the point spectrum σ and the system of root functions V_h in the sense of ratios

$$\begin{cases} \boldsymbol{L}v_{0,k}(x) = \lambda_k v_{0,k}(x), \ k = 0, 1 \ldots, \\ \boldsymbol{L}v_{1,k}(x) = \lambda_k v_{1,k}(x) + \mu_k v_{0,k}(x), \ k = 1, 2, \ldots, \end{cases} \tag{15}$$

for some real numbers μ_k, $k \in \mathbb{N}$.

Let us consider the operator $\boldsymbol{L}: L_2(-1,1) \to L_2(-1,1)$, generated by the equation

$$\boldsymbol{L}v := -v''(x) + \alpha\big(v'(x) + v'(-x)\big) = \lambda v(x) = 0, \quad \lambda \in \mathbb{C}, \ \gamma \in \mathbb{R}, \ -1 < x < 1, \tag{16}$$

and boundary conditions (9).

By substituting functions (10) into the equation (16), we obtain the relations (15), where

$$\mu_k = (\alpha - h)2k\pi, \quad k \in \mathbb{N}.$$

Therefore, $\boldsymbol{L} \in O(V_h, \sigma)$. Thus, the statement **A.2** of Theorem 1 is proved.

If equality $\alpha = h$ holds, then $\mu_k = 0$. In this case the elements of system V_h are eigenfunctions of operator $\boldsymbol{L}$. Therefore, **A.3** of Theorem 1 is proved.

Let $\sigma_1 := \{\lambda_{1,k} \in \mathbb{R}, k \in \mathbb{N}\}$, and $O(V_h, \sigma, \sigma_1)$ be the set of operators $\boldsymbol{L}: L_2(-1,1) \to L_2(-1,1)$ with the point spectrum $\sigma \cup \sigma_1$ and system of eigenfunctions V_h

$$\begin{cases} \boldsymbol{L}v_{0,k}(x) = \lambda_k v_{0,k}(x), \quad \lambda_k \in \sigma, \ k = 0, 1, \ldots, \\ \boldsymbol{L}v_{1,k}(x) = \lambda_{1,k} v_{1,k}(x), \quad \lambda_{1,k} \in \sigma_1. \end{cases}$$

Let us consider the operator $\boldsymbol{L}$ of the problem (8)–(9). By substituting functions (10) into equation (8), we obtain

$$\begin{cases} \boldsymbol{L}v_{0,k}(x) = \lambda_k v_{0,k}(x), \\ \boldsymbol{L}v_{1,k}(x) = \lambda_k v_{1,k}(x) - 2\alpha_1\lambda_k(1+\gamma x)\tau_{1,k}(x) + \mu_k v_{0,k}(x), \end{cases}$$

$$\mu_k = (h - \alpha_2)(2k\pi - 1),$$

$$\boldsymbol{L}v_{1,k}(x) = \lambda_{1,k} v_{1,k}(x) - 2\alpha_1(\gamma - h)x\tau_{1,k}(x) + \mu_k v_{0,k}(x), \ \lambda_{1,k} := (1 - 2\alpha_1)\lambda_k,$$

for $k \in \mathbb{N}$. Therefore, $\boldsymbol{L} \notin O(V_h, \sigma)$.

If $\gamma = \alpha_2 = h = \dfrac{\beta_1 - \beta_2}{\beta_2 + \beta_1}$, then $\mu_k = 0$ for $k \in \mathbb{N}$.

Therefore,

$$\begin{cases} \boldsymbol{L}v_{0,k}(x) = \lambda_k v_{0,k}(x), \\ \boldsymbol{L}v_{1,k}(x) = \lambda_{1,k} v_{1,k}(x), \end{cases} \quad k \in \mathbb{N}. \tag{17}$$

Thus, V_h is the system of eigenfunctions of operator $\boldsymbol{L}$, for which the equalities (17) hold, where $\sigma_1 := \{\lambda_{1,k} \in \mathbb{R}, \ \lambda_{1,k} = (1 - 2\alpha_1)\lambda_k, \ k \in \mathbb{N}\}$.

Then, $\boldsymbol{L} \in O(V_h, \sigma, \sigma_1)$. Therefore, taking into account Lemma 1, we obtain the following statement.

Lemma 2. *For any numbers $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{R}$, where $2\alpha_1 \neq 1$, $\beta_1 \neq -\beta_2$, the system of eigenfunctions V_h of operator* **L** *is Riesz basis of the space $L_2(-1,1)$.*

Consequently, the statement **A.1** of Theorem 1 holds.

Note, that for the case $\alpha_2 = \gamma = 0$ the spectral properties of operator $\boldsymbol{L}$ are investigated in the papers [17, 18].

4 Existence and uniqueness of solution to the problem (4)–(7)

4.1. Let the conditions $\gamma = \alpha_2 = \dfrac{\beta_1 - \beta_2}{\beta_2 + \beta_1}$ and assumptions *B1)–B3)* hold. Partial solutions of the problem (4)–(6) are determined by relations

$$\begin{cases} u_{0,0}(x,t) = T_{0,0}(t)v_{0,0}(x), \\ u_{0,k}(x,t) = T_{0,k}(t)v_{0,k}(x), \quad k \in \mathbb{N}. \\ u_{1,k}(x,t) = T_{1,k}(t)v_{1,k}(x), \end{cases}$$

To determine functions $T_{r,k}(t)$, we obtain problems that are solved sequentially

$$\begin{cases} T'_{0,0}(t) = r(t)f_{0,0}(t),\ T_{0,0}(0) = \eta_{0,0}, \\ T'_{0,k}(t) = \lambda_k T_{0,k}(t) + r(t)f_{0,k},\ T_{0,k}(0) = \eta_{0,k}, \qquad k \in \mathbb{N}. \\ T'_{1,k}(t) = \lambda_{1,k}T_{1,k}(t) + r(t)f_{1,k}(t),\ T_{1,k}(0) = \eta_{1,k}, \end{cases}$$

Therefore,

$$\begin{cases} T_{0,0}(t) = \eta_{0,0}t + \displaystyle\int_0^t r(\tau)f_{0,0}(\tau)d\tau, \\ T_{1,k}(t) = \eta_{1,k}e^{-\lambda_{1,k}t} + \displaystyle\int_0^1 r(\tau)f_{1,k}(\tau)e^{-\lambda_{1,k}(t-\tau)}d\tau, \quad k \in \mathbb{N}. \\ T_{0,k}(t) = \eta_{0,k}e^{-\lambda_k t} + \displaystyle\int_0^1 r(\tau)f_{0,k}(\tau)e^{-\lambda_k(t-\tau)}d\tau, \end{cases}$$

$$\begin{cases} u_{0,0}(x,t) = \left(\eta_{0,0}t + \displaystyle\int_0^t r(\tau)f_{0,0}(\tau)d\tau\right)v_{0,0}(x), \\ u_{0,k}(x,t) = \left(\eta_{0,k}e^{-\lambda_{1,k}t} + \displaystyle\int_0^t r(\tau)f_{0,k}(\tau)e^{-\lambda_{1,k}(t-\tau)}d\tau\right)v_{0,k}(x), \quad k \in \mathbb{N}. \\ u_{1,k}(x,t) = \left(\eta_{1,k}e^{-\lambda_k t} + \displaystyle\int_0^t r(\tau)f_{0,k}(\tau)e^{-\lambda_k(t-\tau)}d\tau\right)v_{1,k}(x), \end{cases}$$

From the continuity of $\eta(x)$ and the boundedness of functions (10) we obtain inequalities $|\eta_{r,k}| \leq M$, $r = 0, 1$, $k \in \mathbb{N}$. Taking into account these inequalities, we have the estimates

$$|u_{0,0}(x,t)| \leq |\eta_{0,0}| + \max\left(|r(t)| \cdot |f_{0,0}(t)|\right) \leq M_0,$$

$$|u_{1,k}(x,t)| \leq |\eta_{1,k}| + \max\left(|r(t)| \cdot |f_{1,k}(t)|\right)e^{-\lambda_{1,k}\varepsilon} \leq M_1 e^{-\lambda_{1,k}\varepsilon},$$

$$|u_{0,k}(x,t)| \leq |\eta_{0,k}| + \max\left(|r(t)| \cdot |f_{0,k}(t)|\right)e^{-\lambda_k\varepsilon} \leq M_2 e^{-\lambda_k\varepsilon}.$$

So, the functional series

$$u_{0,0}(x,t) + \sum_{k=1}^{\infty}\sum_{s=0}^{1} u_{s,k}(x,t) \tag{18}$$

is estimated by an absolutely convergent numerical series

$$M_0 + \sum_{k=1}^{\infty}\left(M_1 e^{-\lambda_{1,k}\varepsilon} + M_2 e^{-\lambda_k\varepsilon}\right).$$

Therefore, according to the Weierstrass M-Test, the series (18) is uniformly convergent to a continuous function for $t \geq \varepsilon$. Thus, the sum of the series (18) determines the continuous function $u(x,t)$, which satisfies the initial condition (6).

Differentiating the series (18) element-by-element by the variable t, we obtain

$$r(t)f_{0,0}(t)v_{0,0}(x)+\sum_{k=1}^{\infty}\Big(\Big(-\lambda_{1,k}\eta_{1,k}e^{-\lambda_{1,k}t}-r(t)f_{1,k}(t)+\int_0^t r(\tau)f_{1,k}(\tau)e^{-\lambda_{1,k}(t-\tau)}d\tau\Big)v_{1,k}(x)$$
$$+\Big(-\lambda_k\eta_{0,k}e^{-\lambda_k t}-r(t)f_{0,k}(t)+\int_0^t r(\tau)f_{0,k}(\tau)e^{-\lambda_k(t-\tau)}d\tau\Big)v_{0,k}(x)\Big).$$

Let us consider

$$\Big|\frac{\partial u_{0,0}(x,t)}{\partial t}\Big|\le|\eta_{0,0}|+\max\big(|r(t)|\cdot|f_{0,0}(t)|\big),$$

$$\Big|\frac{\partial u_{1,k}(x,t)}{\partial t}\Big|\le\max\big(|r(t)|\cdot|f_{1,k}(t)|\big)+\big(|\lambda_{1,k}||\eta_{1,k}|+\max|r(t)|T|f_{1,k}(t)|\big)e^{-\lambda_{1,k}\varepsilon},$$

$$\Big|\frac{\partial u_{0,k}(x,t)}{\partial t}\Big|\le\max\big(|r(t)|\cdot|f_{0,k}(t)|\big)+\big(|\lambda_k||\eta_{0,k}|+\max|r(t)|T|f_{0,k}(t)|\big)e^{-\lambda_k\varepsilon}.$$

From the assumption *B3)* and the embedding theorems, we obtain uniform convergence and continuity of the sum of series $\max|r(t)|\cdot\Big(|f_{0,0}(t)|+\sum_{k=1}^{\infty}\big(|f_{0,k}(t)|+|f_{1,k}(t)|\big)\Big)$ in the domain D_T.

The series

$$\sum_{k=1}^{\infty}\Big(\lambda_{1,k}|\eta_{1,k}|+\Big|\int_0^t r(\tau)f_{1,k}(\tau)e^{-\lambda_{1,k}(t-\tau)}d\tau\Big|+\lambda_k|\eta_{0,k}|+\Big|\int_0^t r(\tau)f_{0,k}(\tau)e^{-\lambda_k(t-\tau)}d\tau\Big|\Big)$$

is estimated for $t\ge\varepsilon$ by series $M_3\sum_{k=1}^{\infty}\big(\lambda_{1,k}e^{-\lambda_{1,k}\varepsilon}+\lambda_k e^{-\lambda_k\varepsilon}\big)$.

Therefore, the sum of the series $\Big|\frac{\partial u_{0,0}(x,t)}{\partial t}\Big|+\sum_{k=1}^{\infty}\sum_{s=0}^{1}\frac{\partial u_{s,k}(x,t)}{\partial t}$ is a continuous function in D_T and coincides with $\frac{\partial u(x,t)}{\partial t}$.

Differentiating the series (18) element-by-element twice by the variable x, we obtain

$$\sum_{k=1}^{\infty}\Big(-\lambda_{1,k}\Big(\eta_{1,k}e^{-\lambda_{1,k}t}+\int_0^t r(\tau)f_{1,k}(\tau)e^{-\lambda_{1,k}(t-\tau)}d\tau\Big)v_{1,k}(x)$$
$$-\lambda_k\Big(\eta_{0,k}e^{-\lambda_k t}+\int_0^t r(\tau)f_{0,k}(\tau)e^{-\lambda_k(t-\tau)}d\tau\Big)v_{0,k}(x)\Big).$$

The obtained series is estimated for $t\ge\varepsilon$ by series $M_4\sum_{k=1}^{\infty}\big(\lambda_{1,k}e^{-\lambda_{1,k}\varepsilon}+\lambda_k e^{-\lambda_k\varepsilon}\big)$. Therefore, sum of this series is a continuous function in D_T and coincides with $\frac{\partial^2 u(x,t)}{\partial x^2}$.

Similarly, the smoothness of the function $\frac{\partial^2 u(-x,t)}{\partial x^2}$ is investigated. Further, by the embedding theorems we obtain the continuity of functions $\frac{\partial u(x,t)}{\partial x}$, $\frac{\partial u(-x,t)}{\partial x}$ in D_T.

Therefore, the sum of the series (12) is a classical solution of the problem (4)–(6).

Let us consider the equation to define the function $r(t)$:

$$\int_0^1 \frac{\partial u(x,t)}{\partial t}dx = E'(t) = \sqrt{2}r(t)f_{0,0}(t) + 4h\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{2\pi k}\Big(r(t)f_{1,k}(t) - \lambda_{1,k}\eta_{1,k}e^{-\lambda_{1,k}t} - \lambda_{1,k}\int_0^t r(\tau)f_{1,k}(\tau)e^{-\lambda_{1,k}(t-\tau)}d\tau\Big).$$

Then we construct the equation to determine the function $r(t)$:

$$r(t)\Big(\sqrt{2}f_{0,0}(t) + 2h\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{\pi k}f_{1,k}(t)\Big) = E'(t) + h\sum_{k=1}^{\infty}\frac{2(-1)^{k-1}}{\pi k}\eta_{1,k}\lambda_{1,k}e^{-\lambda_{1,k}t} + h\sum_{k=1}^{\infty}\frac{2(-1)^{k-1}}{\pi k}\lambda_{1,k}\int_0^t r(\tau)f_{1,k}(\tau)e^{-\lambda_{1,k}(t-\tau)}d\tau,$$

hence

$$r(t) = \frac{E'(t)+2(1-2\alpha_1)h\sum\limits_{k=1}^{\infty}\frac{(-1)^{k-1}}{\pi k}\eta_{1,k}e^{-\lambda_{1,k}t}}{\sqrt{2}f_{0,0}(t)+2h\sum\limits_{k=1}^{\infty}\frac{(-1)^{k-1}}{\pi k}f_{1,k}(t)} + \frac{2h(1-2\alpha_1)\sum\limits_{k=1}^{\infty}\frac{(-1)^{k-1}}{\pi k}\int\limits_0^t r(\tau)f_{1,k}(\tau)e^{-\lambda_{1,k}(t-\tau)}d\tau}{\sqrt{2}f_{0,0}(t)+2h\sum\limits_{k=1}^{\infty}\frac{(-1)^{k-1}}{\pi k}f_{1,k}(t)}.$$

So, to determine the function $r(t)$ the Volterra integral equation of the second kind is obtained:

$$r(t) = F(t) + \int_0^t K(t,\tau)r(\tau)d\tau, \tag{19}$$

where

$$F(t) = \frac{E'(t)+2(1-2\alpha_1)h\sum\limits_{k=1}^{\infty}\frac{(-1)^{k-1}}{\pi k}\eta_{1,k}e^{-\lambda_{1,k}t}}{\sqrt{2}f_{0,0}(t)+2h\sum\limits_{k=1}^{\infty}\frac{(-1)^{k-1}}{\pi k}f_{1,k}(t)}, \tag{20}$$

$$K(t,\tau) = \frac{2h(1-2\alpha_1)\sum\limits_{k=1}^{\infty}\frac{(-1)^{k-1}}{\pi k}f_{1,k}(\tau)e^{-\lambda_{1,k}(t-\tau)}}{\sqrt{2}f_{0,0}(t)+2h\sum\limits_{k=1}^{\infty}\frac{(-1)^{k-1}}{\pi k}f_{1,k}(t)}. \tag{21}$$

The denominator of fractions (20), (21) is not equal to zero, because the assumption *B3)* is obtained

$$\int_{-1}^1 f(x,t)dx = \sqrt{2}f_{0,0}(t) + 2h\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{\pi k}f_{1,k}(t) \neq 0.$$

According to assumptions *B1)–B3)*, the function $F(t)$ and the kernel $K(t,\tau)$ are continuous functions on $[0,T]$ and $[0,1]\times[0,T]$, respectively.

Therefore, equation (19) has a unique solution. This solution is a continuous function $r(t)$ on $[0,T]$, which forms a unique solution $\{r(t), u(x,t)\}$ of the inverse problem (4)–(7) together with the given Fourier series (11) as a solution $u(x,t)$ of the direct problem (4)–(6).

The statement **A.1** of Theorem 2 is proved.

4.2. Let $\alpha_1 = 0$. Then $\lambda_k = \lambda_{1,k}$ for $k \in \mathbb{N}$.

In the case of $\gamma \neq \dfrac{\beta_1 - \beta_2}{\beta_2 + \beta_1}$ the elements of system V_h are the root functions of the operator $\boldsymbol{L}$, for which the equalities (15) hold.

*Proof of statement **A.2** of Theorem 2.* The partial solutions of the problem (4)–(6) are determined by relations (17). To find the functions $T_{r,k}(t)$ we obtain the following problems

$$\begin{cases} T'_{0,0}(t) = r(t) f_{0,0}, \; T_{0,0}(0) = \eta_{0,0}, \\ T'_{0,k}(t) = \lambda_k T_{0,k}(t) + r(t) f_{0,k}(t), \; T_{0,k}(0) = \eta_{0,k}, \quad k \in \mathbb{N}, \\ T'_{1,k}(t) = \lambda_k T_{1,k}(t) + r(t) f_{0,k}(t), \; T_{1,k}(0) = \eta_{1,k}, \end{cases}$$

which are solved sequentially.

Therefore,

$$\begin{cases} T_{0,0}(t) = \eta_{0,0} t + \int_0^t r(\tau) f_{0,0}(\tau) d\tau, \\ T_{0,k}(t) = \eta_{0,k} e^{-\lambda_k t} + \int_0^t r(\tau) f_{0,k}(\tau) e^{-\lambda_k (t-\tau)} d\tau, \quad k \in \mathbb{N}, \\ T_{1,k}(t) = \eta_{1,k} e^{-\lambda_k t} + \int_0^t r(\tau) f_{1,k}(\tau) e^{-\lambda_k (t-\tau)} d\tau, \end{cases}$$

$$\begin{cases} u_{0,0}(x,t) = \Big(\eta_{0,0} t + \int_0^t r(\tau) f_{0,0}(\tau) d\tau\Big) v_{0,0}(x), \\ u_{0,k}(x,t) = \Big(\eta_{0,k} e^{-\lambda_k t} + \int_0^t r(\tau) f_{0,k}(\tau) e^{-\lambda_k (t-\tau)} d\tau\Big) v_{0,k}(x), \quad k \in \mathbb{N}. \\ u_{1,k}(x,t) = \Big(\eta_{1,k} e^{-\lambda_k t} + \int_0^t r(\tau) f_{1,k}(\tau) e^{-\lambda_k (t-\tau)} d\tau\Big) v_{1,k}(x), \end{cases}$$

Taking into account the assumptions *B1)–B3)* of the theorem, we obtain the estimates

$$|u_{0,0}(x,t)| \leq |\eta_{0,0}| + \max\big(|r(t)| \cdot |f_{0,0}(t)|\big) \leq M_0,$$

$$|u_{1,k}(x,t)| \leq \max|v_{1,k}(t)|\big(|\eta_{1,k}| + \max|r(t)| \cdot |f_{1,k}(t)|\big) e^{-\lambda_k \varepsilon} \leq M_5 e^{-\lambda_k \varepsilon}, \tag{22}$$

$$|u_{0,k}(x,t)| \leq \big(|\eta_{0,k}| + \max|r(t)| \cdot |f_{0,k}(t)|\big) e^{-\lambda_k \varepsilon} \leq M_6 e^{-\lambda_k \varepsilon}. \tag{23}$$

Therefore, the functional series (18) is majorized by an absolutely convergent numerical series $M_7 \sum_{k=1}^{\infty} e^{-\lambda_k \varepsilon}$ for $t \geq \varepsilon$. Then, according to the Weierstrass M-Test, the series (18) is uniformly convergent to a continuous function for $t \geq \varepsilon$. Thus, the sum of the series (18) defines a continuous function that satisfies the initial condition (6).

By direct substitution, we make sure that

$$\begin{cases} \dfrac{\partial u_{0,k}(x,t)}{\partial t} = \Big(-r(t) f_{0,k}(t) - \lambda_k \big(\eta_{0,k} e^{-\lambda_k t} + \int_0^t r(\tau) f_{0,k}(\tau) e^{-\lambda_k (t-\tau)} d\tau\big)\Big) v_{0,k}(x), \\ \dfrac{\partial u_{1,k}(x,t)}{\partial t} = \Big(-r(t) f_{1,k}(t) - \lambda_k \big(\eta_{1,k} e^{-\lambda_k t} + \int_0^t r(\tau) f_{1,k}(\tau) e^{-\lambda_k (t-\tau)} d\tau\big)\Big) v_{1,k}(x), \end{cases}$$

for $k \in \mathbb{N}$.

According to the scheme of the proof of **A.1**, we differentiate the series (18) element-by-element by the variable t and twice by the variable x. Then we define majorant number series for $t \geq \varepsilon$ and obtain uniform continuity of the sum of this series by the Weierstrass M-Test for $t \geq \varepsilon$.

We differentiate the series (18) element-by-element twice by the variable x:

$$\sum_{k=1}^{\infty}\sum_{r=0}^{1}\Big|\frac{\partial^2 u_{r,k}(x,t)}{\partial x^2}\Big| = \sum_{k=1}^{\infty}\bigg(-\lambda_k\Big(\eta_{0,k}e^{-\lambda_k t} + \int_0^t r(\tau)f_{0,k}(\tau)e^{-\lambda_k(t-\tau)}d\tau\Big)v_{0,k}(x)$$
$$-\lambda_k\Big((\eta_{1,k}+\mu_k\eta_{0,k})e^{-\lambda_k t} + \int_0^t r(\tau)\big(f_{1,k}(\tau)+\mu_k f_{0,k}(\tau)e^{-\lambda_k(t-\tau)}d\tau\big)v_{1,k}(x)\Big)\bigg).$$

Taking into account the estimates (22), (23), we obtain

$$\Big|\frac{\partial^2 u_{0,k}(x,t)}{\partial x^2}\Big| \leq 2\lambda_k M_5 e^{-\lambda_k\varepsilon}, \quad \Big|\frac{\partial^2 u_{1,k}(x,t)}{\partial x^2}\Big| \leq 2\lambda_k M_6 e^{-\lambda_{1,k}\varepsilon}$$

for $k \in \mathbb{N}$.

Then, the obtained series is majorized by series $M_7\sum\limits_{k=1}^{\infty}\lambda_k e^{-\lambda_k\varepsilon}$ for $t \geq \varepsilon > 0$ and $M_7 > 0$. Therefore, the sum of this series is a continuous function in the domain D_T and coincides with $\dfrac{\partial^2 u(x,t)}{\partial x^2}$. Similarly, the smoothness of the function $\dfrac{\partial^2 u(-x,t)}{\partial x^2}$ is obtained.

By embedding theorems, we obtain continuity of functions $\dfrac{\partial u(x,t)}{\partial x}$, $\dfrac{\partial u(-x,t)}{\partial x}$ in D_T. Thus, the function $u(x,t)$, defined by the series (12), is a classical solution of the problem (4)–(6).

Let us consider the following equation to define the function $r(t)$:

$$\int_0^1 \frac{\partial u(x,t)}{\partial t}dx = E'(t) = \sqrt{2}r(t)f_{0,0}(t)$$
$$+ 2h\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{\pi k}\Big(r(t)f_{1,k}(t) - \lambda_k\eta_{1,k}e^{-\lambda_k t}$$
$$- \lambda_k\int_0^t r(\tau)f_{1,k}(\tau)e^{-\lambda_k(t-\tau)}d\tau\Big).$$

Then we construct the equation to determine the function $r(t)$:

$$r(t)\Big(\sqrt{2}f_{0,0}(t) + 2h\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{\pi k}f_{1,k}(t)\Big) = E'(t) + 2h\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{\pi k}\eta_{1,k}e^{-\lambda_k t}$$
$$+ 2h\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{\pi k}\int_0^t r(\tau)f_{1,k}(\tau)e^{-\lambda_k(t-\tau)}d\tau,$$

hence

$$r(t) = \frac{E'(t) + 2h\sum\limits_{k=1}^{\infty}\frac{(-1)^{k-1}}{\pi k}\eta_{1,k}e^{-\lambda_k t}}{\sqrt{2}f_{0,0}(t) + 2h\sum\limits_{k=1}^{\infty}\frac{(-1)^{k-1}}{\pi k}f_{1,k}(t)} + \frac{2h\sum\limits_{k=1}^{\infty}\frac{(-1)^{k-1}}{\pi k}\int\limits_0^t r(\tau)f_{1,k}(\tau)e^{-\lambda_k(t-\tau)}d\tau}{\sqrt{2}f_{0,0}(t) + 2h\sum\limits_{k=1}^{\infty}\frac{(-1)^{k-1}}{\pi k}f_{1,k}(t)}.$$

Therefore, to determine the function $r(t)$ the Volterra integral equation of the second kind is obtained:

$$r(t) = F(t) + \int_0^t K(t,\tau) r(\tau) d\tau, \tag{24}$$

where

$$F(t) = \frac{E'(t) + 2h \sum\limits_{k=1}^{\infty} \frac{(-1)^{k-1}}{\pi k} \eta_{1,k} e^{-\lambda_k t}}{\sqrt{2} f_{0,0}(t) + 2h \sum\limits_{k=1}^{\infty} \frac{(-1)^{k-1}}{\pi k} f_{1,k}(t)}, \tag{25}$$

$$K(t,\tau) = \frac{2h \sum\limits_{k=1}^{\infty} \frac{(-1)^{k-1}}{\pi k} f_{1,k}(\tau) e^{-\lambda_k (t-\tau)}}{\sqrt{2} f_{0,0}(t) + 2h \sum\limits_{k=1}^{\infty} \frac{(-1)^{k-1}}{\pi k} f_{1,k}(t)}. \tag{26}$$

The denominator of fractions (25), (26) is not equal to zero, because the assumption *B3)* is obtained

$$\int_{-1}^{1} f(x,t) dx = \sqrt{2} f_{0,0}(t) + 2h \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{\pi k} f_{1,k}(t) \neq 0.$$

According to assumptions *B1)–B3)*, the function $F(t)$ and the kernel $K(t,\tau)$ are continuous functions on $[0,T]$ and $[0,1] \times [0,T]$, respectively.

Therefore, equation (24) has a unique solution. This solution is a continuous function $r(t)$ on $[0,T]$, which forms a unique solution $\{r(t), u(x,t)\}$ of the inverse problem (4)–(7) together with solution $u(x,t)$ of the direct problem (4)–(6).

The statement **A.2** of Theorem 2 is proved. $\square$

4.3. Let $\alpha_2 = h = \dfrac{\beta_2 - \beta_1}{\beta_2 + \beta_1}$. In this case the elements of system V_h are the eigenvalues of operator $\boldsymbol{L}$, for which the following equalities hold

$$\begin{cases} \boldsymbol{L} v_{0,k}(x) = \lambda_k v_{0,k}(x), \\ \boldsymbol{L} v_{1,k}(x) = \lambda_k v_{1,k}(x), \end{cases} \quad k \in \mathbb{N}.$$

The proof of statement **A.3** of Theorem 2 repeats the proof of **A.1** of this theorem.

We note that inverse problem is investigated in [21] for $\beta_1 = 1$, $\beta_2 = b$, $\gamma = 0$, $\alpha_1 = -\varepsilon$, $\alpha_2 = 0$. In addition, if $1 - 2\alpha_1 \leq 0$, then more research is needed.

References

[1] Al-Salti N., Kirane M., Torebek B.T. *On a class of inverse problems for a heat equation with involution perturbation*. Hacet. J. Math. Stat. 2019, **48** (3), 669–681. doi:10.15672/HJMS.2017.538

[2] Ashyralyev A., Sarsenbi A. *Well-posedness of a parabolic equation with involution*. Numer. Funct. Anal. Optim. 2017, **38** (10), 1295–1304. doi:10.1080/01630563.2017.1316997

[3] Ashyralyev A., Sarsenbi A. *Well–posedness of an elliptic equations with involution*. Electron. J. Differential Equations 2015, **2015** (284), 1–8.

[4] Baranetskij Ya.O., Kalenyuk P.I., Kolyasa L.I., Kopach M.I. *Nonlocal multipoint problem for ordinary differential equations of even order involution*. Mat. Stud. 2018, **49** (1), 80–94. doi:10.15330/ms.49.1.80-94

[5] Baranetskij Ya.O, Kalenyuk P.I., Kopach M.I., Solomko A.V. *The nonlocal multipoint problem with Dirichlet-type conditions for an ordinary differential equation of even order with involution*. Mat. Stud. 2020, **54** (1), 64–78. doi:10.30970/ms.54.1.64-78

[6] Baranetskij Ya.O., Kalenyuk P.I. *Boundary-value problems with Birkhoff regular but not strongly regular conditions for a second-order differential operator*. J. Math. Sci. 2019, **238** (1), 1–21. doi:10.1007/S10958-019-04214-Z

[7] Baranetskij Ya.O., Kalenyuk P.I., Kolyasa L.I. *Spectral properties of nonself-adjoint nonlocal boundary-value problems for the operator of differentiation of even order*. Ukrainian Math. J. 2018, **70** (6), 851–865.

[8] Baranetskij Ya.O., Kalenyuk P.I. *Nonlocal multipoint problem with multiple spectrum for an ordinary 2n–th order differential equation*. J. Math. Sci. 2020, **246** (2), 152–169. doi:10.1007/s10958-020-04727-y

[9] Cabada A., Tojo A.F. *Equations with Involutions*. Workshop on Differential Equations, Malla Moravka, Czech Republic, 2014, 240.

[10] Hazanee A., Lesnic D., Ismailov M.I., Kerimov N.B. *Inverse time-dependent source problems for the heat equation with nonlocal boundary conditions*. Appl. Math. Comput. 2019, **346**, 800–815. doi:10.1016/j.amc.2018.10.059

[11] Hazanee A., Lesnic D., Ismailov M.I., Kerimov N.B. *An inverse time-dependent source problem for the heat equation with a non-classical boundary condition*. Appl. Math. Model. 2015, **39** (20), 6258–6272. doi: 10.1016/j.apm.2015.01.058

[12] Hazanee A., Lesnic D. *Determination of a time-dependent heat source from nonlocal boundary conditions*. Eng. Anal. Bound. Elem. 2013, **37** (6), 936–956. doi:10.1016/j.enganabound.2013.03.003

[13] Hazanee A., Lesnic D. *Determination of a time-dependent coefficient in the bioheat equation*. Int. J. Mech. Sci. 2014, **88**, 259–266. doi:10.1016/j.ijmecsci.2014.05.017

[14] Gohberg I.C., Krein M.G. Introduction to the theory of linear nonself-adjoint operators. Providence: Amer. Math. Soc. 1969.

[15] Il'in V.A. *Existence of a reduced system of eigen- and associated functions for a nonself-adjoint ordinary differential operator*. Proc. Steklov Inst. Math. 1979, **142**, 157–164.

[16] Kirane M., Al-Salti N. *Inverse problems for a nonlocal wave equation with an involution perturbation*. J. Nonlinear Sci. Appl. 2016, **9** (3), 1243–1251. doi:10.22436/jnsa.009.03.49

[17] Kritskov L.V., Sarsenbi A.M. *Basicity in L_p of root functions for differential equations with involution*. Electron. J. Differ. Equ. 2015, **2015** (278), 1–9.

[18] Kritskov L.V., Sadybekov M.A., Sarsenbi A.M. *Properties in L_p of root functions for a nonlocal problem with involution*. Turkish J. Math. 2019, **43** (1), 393–401. doi:10.3906/mat-1809-12

[19] Naimark M.A. Linear differential operators. Frederick Ungar Publ. Co., New York, 1967.

[20] Orazov I., Sadybekov M.A. *On a class of problems of determining the temperature and density of heat sources given initial and final temperature*. Sib. Math. J. 2012, **53** (1), 146–151.

[21] Sadybekov M., Dildabek G., Ivanova M.B. *Direct and inverse problems for nonlocal heat equation with boundary conditions of periodic type*. Bound. Value Probl. 2022, **53**. doi:10.1186/s13661-022-01632-y

[22] Sadybekov M.A., Dildabek G., Ivanova M.B. *On an inverse problem of reconstructing a heat conduction process from nonlocal data*. Adv. Math. Phys. 2018, **2018**, 1–8. doi:10.1155/2018/8301656

[23] Sarsenbi A. *Solvability of a mixed problem for a heat equation with an involution perturbation*. AIP Conference Proceedings 2019, **2183** (1). doi:10.1063/1.5136187

[24] Sarsenbi A. *The ill-posed problem for the heat transfer equation with involution*. Zhurnal SVMO. 2019, **21** (1), 48–59. doi:10.15507/2079-6900.21.201901.48-59

[25] Sarsenbi A. *On a class of inverse problems for a parabolic equation with involution*. AIP Conference Proceedings 2017, **1880** (1). doi:10.1063/1.5000637

[26] Vladykina V.E., Shkalikov A.A. *Regular Ordinary Differential Operators with Involution*. Math. Notes 2019, **106** (5), 674–687. doi:10.1134/S0001434619110026

[27] Vladykina V.E., Shkalikov A.A. *Spectral properties of ordinary differential operators with involution*. Dokl. Math. 2019, **99** (1), 5–10.

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Методом відокремлення змінних побудовано розв'язок досліджуваної задачі з невідомим коефіцієнтом у рівнянні. Вивчено властивості індукованої спектральної задачі для диференціального рівняння другого порядку з інволюцією. Досліджено залежність спектра та його кратності, а також структури системи кореневих функцій і частинних розв'язків задачі від інволютивної частини цього рівняння. Встановлено умови існування та єдиності розв'язку оберненої задачі. Для визначення шуканого коефіцієнта знайдено та розв'язано інтегральне рівняння Вольтера другого роду.

Ключові слова і фрази: обернена задача, рівняння теплопровідності, метод відокремлення змінних, нелокальна умова, інволюція, базис Рісса.