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Lipschitz symmetric functions on Banach spaces with symmetric bases

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We investigate Lipschitz symmetric functions on a Banach space X with a symmetric basis. We consider power symmetric polynomials on ℓ_1 and show that they are Lipschitz on the unbounded subset consisting of vectors $x \in \ell_1$ such that $|x_n| \leq 1$. Using functions max and min and tropical polynomials of several variables, we constructed a large family of Lipschitz symmetric functions on the Banach space c_0 which can be described as a semiring of compositions of tropical polynomials over c_0.

Key words and phrases: Lipschitz symmetric function on Banach space, symmetric basis, tropical polynomial.

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Introduction

Symmetric functions of infinitely many variables play an important role in the nonlinear functional analysis and its applications [12]. Let X be a real Banach space. We recall that a Schauder basis (e_n) in X is *symmetric* if it is equivalent to the basis $(e_{\sigma(n)})$ for every permutation σ of the set of positive integers $\mathbb{N}$. Let us denote by $\mathcal{S}^\infty$ the group of all permutations (bijections) on the set of natural numbers $\mathbb{N}$. Any $\sigma \in \mathcal{S}^\infty$ acts on X by

$$\sigma(x) = (x_{\sigma(1)}, \ldots, x_{\sigma(n)}, \ldots), \quad x = \sum_{n=1}^{\infty} x_n e_n = (x_1, \ldots, x_n, \ldots) \in X.$$

A function $f \colon X \to \mathbb{R}$ is called *symmetric* if $f(\sigma(x)) = f(x)$ for all $x \in X$ and $\sigma \in \mathcal{S}^\infty$. It is naturally to study symmetric polynomials and analytic functions on X as “simple” nonlinear symmetric functions. Algebras of symmetric analytic functions and their generalizations on real and complex Banach spaces were investigated by many authors (see [1–9, 14, 16–18] and references therein). However, some Banach spaces like c_0 do not support symmetric polynomials while support a lot of symmetric Lipschitz functions. In [12] it was proposed symmetric slice polynomials for approximation of uniformly continuous symmetric functions on c_0. But the slice polynomials are not Lipschitz in the general case. In this paper we consider some classes of Lipschitz symmetric functions on a real Banach space X with a symmetric basis.

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A mapping P from a Banach space X to a Banach space Y is an *n-homogeneous polynomial* if there is an n-linear map B on the nth Cartesian power X^n to Y such that $P(x) = B(x, \ldots, x)$. A finite sum of homogeneous polynomials $P = P_0 + P_1 + \cdots + P_m$ is a polynomial of degree m if each P_n is an n-homogeneous polynomial, $0 \le n \le m$, and $P_m \not\equiv 0$.

Let us recall that a function f from a metric space (M_1, ρ_1) to a metric space (M_2, ρ_2) is *Lipschitz* if there is a constant L such that $\rho_2(f(x), f(y)) \le L\rho_1(x, y)$, $x, y \in M_1$. The infimum over all constants L satisfying the inequality above is called the *Lipschitz constant* of f and denoted by $L(f)$. We refer the reader to the N. Weaver book [19] for details about Lipschitz mappings and to J. Mujica book [13] for details on polynomials and analytic functions on Banach spaces.

In Section 1, we discuss the question about existence of Lipschitz symmetric polynomials on an unbounded domain in ℓ_1. In Section 2, we consider Lipschitz functions on X, constructed using operations max, min and their linear combinations. It leads us to so-called Tropical Mathematics [15] which proceeds with semirings involving such operations. Some connections between Lipschitz symmetric functions on X and tropical polynomials of infinitely many variables are found.

1 Lipschitz symmetric polynomials

It is clear that any polynomial of degree greater than 1 is not Lipschitz on X, even if X is finite dimensional. However polynomials are locally Lipschitz. Among of symmetric polynomials we can find Lipschitz ones on some unbounded sets.

Let c_{00} be the space of all finite sequences, that is, if $x = (x_1, \ldots, x_n, \ldots) \in c_{00}$, then only a finite number of coordinates x_n is not equal to zero. We consider power symmetric polynomials

$$F_m(x) = \sum_{i=1}^{\infty} x_i^m,$$

and elementary symmetric polynomials

$$G_m(x) = \sum_{i_1 < \cdots < i_m} x_{i_1} \cdots x_{i_m}, \quad m \in \mathbb{N}, \quad x \in c_{00}.$$

It is well known that polynomials (F_m) and (G_m), form algebraic bases in the algebra of all symmetric polynomials on c_{00} (see e.g. [11]). Both (F_m) and (G_m) can be extended to ℓ_1 for every $m \in \mathbb{N}$. We will use the same symbols for the extensions.

Lemma 1. *Symmetric polynomials $F_m^k(x) = (\sum_{i=1}^n x_i^m)^k$ on $\mathbb{R}^n$ with the ℓ_1-norm are Lipschitz with constants $n^{k-1} \le L(F_m^k) \le mkn^{k-1}$ on the domain $D_n = \{x \in \mathbb{R}^n \colon |x_i| \le 1,\ i = 1, \ldots, n\}$.*

Proof. We have

$$\begin{aligned}
|F_m^k(x) - F_m^k(y)| &= |F_m(x) - F_m(y)||F_m^{k-1}(x) + F_m^{k-2}(x)F_m(y) \\
&\quad + \ldots + F_m(x)F_m^{k-2}(y) + F_m^{k-1}(y)| \\
&\le \sum_{i=1}^n |x_i^m - y_i^m| \Bigg| \left(\sum_{i=1}^n x_i^m\right)^{k-1} + \left(\sum_{i=1}^n x_i^m\right)^{k-2} \sum_{i=1}^n y_i^m \\
&\quad + \ldots + \sum_{i=1}^n x_i^m \left(\sum_{i=1}^n y_i^m\right)^{k-2} + \left(\sum_{i=1}^n y_i^m\right)^{k-1} \Bigg|.
\end{aligned}$$

Since $|x_i| \le 1$ and $|y_i| \le 1$, it follows that

$$\begin{aligned} |F_m^k(x) - F_m^k(y)| &\le kn^{k-1} \sum_{i=1}^n |x_i^m - y_i^m| \\ &= kn^{k-1} \sum_{i=1}^n |(x_i - y_i)(x_i^{m-1} + x_i^{m-2} y_i + \cdots + x_i y_i^{m-2} + y_i^{m-1})| \\ &\le mkn^{k-1} \sum_{i=1}^n |x_i - y_i| = mkn^{k-1} \|x - y\|_{\ell_1}. \end{aligned}$$

To get a lower estimation, we set $x_0 = (\underbrace{1, \dots, 1}_{n})$ and $y_0 = 0$. Then

$$|F_m^k(x) - F_m^k(y)| \ge |F_m^k(x_0) - F_m^k(y_0)| = n^k = n^{k-1} \|x_0 - y_0\|_{\ell_1}.$$

□

Since all norms on $\mathbb{R}^n$ are equivalent, we have that F_m^k are Lipschitz functions for any norm on $\mathbb{R}^n$. But for the case of ℓ_1-norm we have estimations for the Lipschitz constant which do not depend on n if $k = 1$. Thus we can prove the following theorem.

Theorem 1. *Polynomials F_m, $m \in \mathbb{N}$ are Lipschitz functions on $D_\infty = \{x \in \ell_1 \colon |x_i| \le 1, i \in \mathbb{N}\}$ with $1 \le L(F_m) \le m$ and F_m^k are not Lipschitz on D_∞ for every $k > 1$.*

Proof. Since the estimation $1 \le L(F_m) \le m$ holds for every $(\mathbb{R}^n, \|\cdot\|_{\ell_1})$, it is still correct if $n \to \infty$. For $k > 1$ we have that $n^{k-1} \le L(F_m^k)$ and so $L(F_m^k) \to \infty$ as $n \to \infty$. □

Note that polynomials G_m are not Lipschitz on D_∞. For example, routine calculations show that for the Lipschitz constant of the restriction of G_2 to D_2 we have $(n-1)/2 \le L(G_2) \le n-1$.

2 Banach spaces and tropical semirings of Lipschitz functions

It is well-known (see, e.g., [10, p. 114]) that, on every Banach space with a symmetric basis, there is an equivalent symmetric norm $\|\cdot\|$ and

$$\|x\| = \left\| \sum_{n=1}^\infty x_n e_n \right\| = \left\| \sum_{n=1}^\infty |x_n| e_n \right\|, \quad x \in X.$$

Throughout this section we suppose that the real Banach space X has a symmetric basis $(e_{\sigma(n)})$, $n \in \mathbb{N}$, is endowed with a symmetric norm $\|\cdot\| = \|\cdot\|_X$, and the c_0-norm is continuous on X, that is there is a constant $C > 0$, such that $\|x\|_{c_0} = \sup_n |x_n| \le C\|x\|_X$ for every $x = (x_1, \dots, x_n, \dots) \in X$. Spaces c_0 and ℓ_p, for $1 \le p < \infty$ are typical examples of such spaces. Let $\mathbb{Z}_0 = \mathbb{Z} \setminus \{0\}$. We will use notation $X(\mathbb{Z}_0)$ for the "two-side X", that is $X \oplus X$ indexed by negative and positive integers.

Let $x = (x_1, x_2, \dots, x_n, \dots) \in X$. We denote by $\operatorname{supp}^+(x) := \{k \in \mathbb{N} \colon x_k > 0\}$ and by $\operatorname{supp}^-(x) := \{k \in \mathbb{N} \colon x_k < 0\}$. Clearly, $\operatorname{supp}(x) = \operatorname{supp}^+(x) \cup \operatorname{supp}^-(x)$. For every vector $x \in X$ we assign a vector $\widehat{x} = (\dots, 0, \widehat{x}_{-j}, \dots, \widehat{x}_{-1}, \widehat{x}_1, \dots, \widehat{x}_m, 0, \dots)$ in the space $X(\mathbb{Z}_0)$, ordered by the following way: $m = |\operatorname{supp}^+(x)|$, $j = |\operatorname{supp}^-(x)|$ (m and/or j may be equal to infinity) and $\widehat{x}_{-j} \le \cdots \le \widehat{x}_{-1}$, and $\widehat{x}_1 \ge \cdots \ge \widehat{x}_m$.

We denote by $\mathcal{M}_X \subset X(\mathbb{Z}_0)$ the set $\{\widehat{x} \colon x \in X\}$. The set $\mathcal{M}_X$ can be considered as the quotient of X with respect to the following equivalence: $x \sim y$ if and only if $\widehat{x} = \widehat{y}$. We suppose that $\mathcal{M}_X$ is endowed with the quotient topology, that is, the strongest topology such that the quotient map $x \mapsto \widehat{x}$ is continuous.

Proposition 1. *Let Y be a topological space and $f\colon \mathcal{M}_X \to Y$ be a continuous map. Then $\check{f}(x) := f(\widehat{x})$ is a symmetric and continuous map on X.*

Proof. The continuity of $\check{f}$ follows from the fact that the quotient map is open. Also, by the definition of f, $f(x) = f(y)$ if $x \sim y$. □

For a given $\widehat{x} = (\ldots, 0, \widehat{x}_{-j}, \ldots, \widehat{x}_m, 0, \ldots)$ and every $n \in \mathbb{Z}_0$ we define

$$g_n(x) = \begin{cases} \widehat{x}_n, & \text{if } -j \le n \le m, \\ 0, & \text{otherwise.} \end{cases}$$

Theorem 2. *Functions g_n are Lipschitz symmetric on X with $1 \le L(g_n) \le C$, where $C > 0$ is such that $\|u\|_{c_0} \le C\|u\|_X$, $u \in X$. If $\widehat{x} = (\ldots, 0, \widehat{x}_{-j}, \ldots, \widehat{x}_m, 0, \ldots)$, then*

$$g_n(x) = \max_{i_1 < \cdots < i_n} \big(\min(\widehat{x}_{i_1}, \ldots, \widehat{x}_{i_n}) \big) \quad \text{for} \quad n > 0, \tag{1}$$

and

$$g_n(x) = \min_{i_1 < \cdots < i_n} \big(\max(\widehat{x}_{-i_1}, \ldots, \widehat{x}_{-i_n}) \big) \quad \text{for} \quad n < 0. \tag{2}$$

Proof. The symmetry of g_n follows from Proposition 1. Equation (1) is correct because if $0 < n \le m$, then $\max_{i_1<\cdots<i_n} \big(\min(\widehat{x}_{i_1}, \ldots, \widehat{x}_{i_n}) \big)$ will be attended at the tuple $(\widehat{x}_1, \widehat{x}_2, \ldots, \widehat{x}_n)$ and will be equal to $\widehat{x}_n$. If $n > m$, then each tuple $(\widehat{x}_1, \widehat{x}_2, \ldots, \widehat{x}_n)$ will contain 0 and so $g_n(x) = 0$. Formula (2) can be proved in a similar way.

Let us show that g_n is a Lipschitz function for $n > 0$. Let $x, y \in X$ and $k, i \in \mathbb{N}$ are such that $x_k = \widehat{x}_n$ and $y_i = \widehat{y}_n$. Without loss of the generality, we can assume that $x_k \ge y_i$. Consider the case $n = 1$. Since $g_1(y) = y_i$, it follows that $y_i \ge y_s$ for every $s \in \mathbb{N}$. In particular, $y_i \ge y_k$. Thus, we have $x_k \ge y_i \ge y_k \ge 0$. Consequently,

$$|g_1(x) - g_1(y)| = |x_k - y_i| \le |x_k - y_k| \le \|x - y\|_{c_0} \le C\|x - y\|_X.$$

Consider the case $n > 1$. Since $g_n(x) = x_k$, it follows that there exists the set of indexes $\{s_1, \ldots, s_n\}$ such that $x_{s_1} \ge x_k, \ldots, x_{s_n} \ge x_k$. Since $g_n(y) = y_i$, it follows that there exists not more than $n-1$ indexes $m \in \mathbb{N}$ such that $y_m > y_i$. Therefore, taking into account that $|\{s_1, \ldots, s_n\}| = n > n-1$, there exists $j \in \{1, \ldots, n\}$ such that $y_{s_j} \le y_i$. Thus, we have $x_{s_j} \ge x_k \ge y_i \ge y_{s_j} \ge 0$. Consequently,

$$|g_n(x) - g_n(y)| = |x_k - y_i| \le |x_{s_j} - y_{s_j}| \le \|x - y\|_{c_0} \le C\|x - y\|_X.$$

In the case $n < 0$ the proof is analogical. □

Note that the functions g_n are nonlinear and $g_1(x) = \sup_k x_k$. It can be checked that we have the following formula for representation of any element in $\mathcal{M}_X$.

Proposition 2. *Every $\widehat{x} \in \mathcal{M}_X \subset X(\mathbb{Z}_0)$ can be represented by*

$$\widehat{x} = \sum_{n \in \mathbb{Z}_0} g_n(x) e_n.$$

Theorem 2 and Proposition 2 imply the following corollary.

Corollary 1. *The mapping $\iota\colon c_0 \ni x \mapsto \widehat{x} \in \mathcal{M}_{c_0}$ is 1-Lipschitz.*

Proof. By Proposition 2,

$$\|\widehat{x}-\widehat{y}\| = \Big\| \sum_{n\in\mathbb{Z}_0} (g_n(x)-g_n(y))e_n \Big\|_{c_0} = \sup_{n\in\mathbb{Z}_0} |g_n(x)-g_n(y)|.$$

By Theorem 2, $L(g_n)=1$ for the case $X=c_0$. Therefore $|g_n(x)-g_n(y)| \le \|x-y\|_{c_0}$. Consequently, $\|\widehat{x}-\widehat{y}\| \le \|x-y\|_{c_0}$. □

Let $\varphi \in c_0(\mathbb{Z}_0)^*$ be a linear continuous functional on $c_0(\mathbb{Z}_0)$. Then φ is completely defined by the sequence of its values on the basis vectors, $(c_n) = (\varphi(e_n)) \in \ell_1(\mathbb{Z}_0)$. In this sense, we will say that $c_0(\mathbb{Z}_0)^*$ coincides with $\ell_1(\mathbb{Z}_0)$.

Corollary 2. *For every $\varphi=(c_n)\in \ell_1(\mathbb{Z}_0)$ the function*

$$g_\varphi(x) := \sum_{n\in\mathbb{Z}_0} c_n g_n(x)$$

is a Lipschitz symmetric function and $L(g_\varphi)\le L(\varphi)$.

Proof. According to Proposition 2, we have that $g_\varphi(x)=\varphi(\widehat{x})$ and so it is well-defined. From Corollary 1 it follows that $g_\varphi = \varphi\circ\iota$ is a composition of two Lipschitz mappings and so it is Lipschitz with $L(g_\varphi)\le L(\iota)L(\varphi)=L(\varphi)$ [19, p. 4]. □

Let us estimate the norm of g_φ,

$$\|g_\varphi\| = \sup_{\|x\|\le 1} |g_\varphi(x)| = \sup_{\|x\|\le 1} |g_\varphi(x)-g_\varphi(0)| \le L(\varphi) = \|\varphi\|.$$

Theorem 3. *For every $x\in c_0$,*

$$\|x\| = \sup_{\|\varphi\|\le 1} |g_\varphi(x)|.$$

Proof. For given $x\in c_0$ and $\varepsilon>0$ let $\psi_\varepsilon \in c_0^*$, $\|\psi_\varepsilon\|=1$, be such that $|\psi_\varepsilon(x)| = \|x\|-\varepsilon$. Such a functional ψ_ε exists according to the Hahn–Banach Theorem. Let $\psi_\varepsilon(e_n)=b_n$ and $\gamma\colon \mathbb{Z}_0\to\mathbb{N}$ be a map such that $\gamma(k)=j$ if $g_k(x)=x_j$. Clearly, γ is a bijection from $\operatorname{supp}(\widehat{x})$ to $\operatorname{supp}(x)$. Let us define a functional $\varphi_\varepsilon\in c_0(\mathbb{Z}_0)^*$ so that $c_k = b_{\gamma(k)}$, $k\in\mathbb{Z}_0$. Then $\|\varphi_\varepsilon\|\le 1$ and

$$|g_{\varphi_\varepsilon}(x)| = |\varphi_\varepsilon(\widehat{x})| = |\psi_\varepsilon(x)| = \|x\|-\varepsilon.$$

Since it is true for every $\varepsilon>0$, the required equality holds. □

Note that functionals g_φ, where $\varphi\in c_0(\mathbb{Z}_0)^*$, does not cover all symmetric Lipschitz functions on c_0. It is known [19, p. 16] that if f and h are Lipschitz functions on a metric space, then both $\max(f(x),h(x))$ and $\min(f(x),h(x))$ are Lipschitz functions with Lipschitz constants bounded by $\max(L(f),L(h))$.

Example. *Let $f(x)=\max(g_1(x),2g_2(x))$. Then f can not be represented in the form g_φ. Indeed, if $f=g_\varphi$ for some $\varphi\in c_0(\mathbb{Z}_0)^*$, then, since $f(x)$ depends only on $\widehat{x}_1$ and $\widehat{x}_2$, it should be of the form $f(x)=c_1g_1(x)+c_2g_2(x)$ for some constants c_1, c_2. If x is such that $\widehat{x}_1=5$, $\widehat{x}_2=1$, then $f(x)=5$; y is such that $\widehat{y}_1=5$, $\widehat{y}_2=2$, then $f(y)=5$; and z is such that $\widehat{z}_1=5$, $\widehat{z}_2=3$, then $f(z)=6$. But there are no constants c_1, c_2 which satisfy these conditions.*

Another example of a Lipschitz symmetric function which can not be represented as g_φ is $x \mapsto \|x\|_{c_0} = \max(g_1(x), -g_{-1}(x))$.

Let us recall that a *tropical semiring* is the semiring $(\mathbb{R} \cup \{+\infty\}, \oplus, \odot)$, where the operations $\oplus$ and $\odot$ are defined by

$$a \oplus b = \min(a, b) \quad \text{and} \quad a \odot b = a + b, \quad a, b \in \mathbb{R} \cup \{+\infty\}.$$

It is known (see, e.g., [15]) that $\mathbb{R} \cup \{+\infty\}$ is actually a semiring, where $\oplus$ plays the role of addition, where $+\infty$ is the zero-element, and $\odot$ plays the role of multiplication. Note that $\max(a, b) = -\min(-a, -b) = a + b - \min(a, b)$. A *tropical polynomial* of several variables $t_1, \ldots, t_n$ in $\mathbb{R} \cup \{+\infty\}$ is a function of the form

$$\begin{aligned} p(t_1, \ldots, t_n) &= a \odot t_1^{i_1} \ldots t_n^{i_n} \oplus b \odot t_1^{j_1} \cdots t_n^{j_n} \oplus \ldots \\ &= \min(a + i_1 t_1 + \cdots + i_n t_n, b + j_1 t_1 + \cdots + j_n t_n, \ldots), \end{aligned}$$

where the coefficients $a, b, \ldots$ are real numbers and the exponents $i_1, j_1, \ldots$ are integers. We can see that any tropical polynomial can be represented as minimum of some affine functions. Hence, every tropical polynomial is a Lipschitz function on $\mathbb{R}^n$ and a finite composition of tropical polynomials is a Lipschitz function. Note that a composition of tropical polynomials is not a tropical polynomial in the general case. Thus we have the following result.

Proposition 3. *Let $g_{\varphi_1}, \ldots, g_{\varphi_n}$ be Lipschitz functions on c_0, generated by functionals*

$$\varphi_1, \ldots, \varphi_n \in c_0(\mathbb{Z}_0)^*$$

as in Corollary 2 and $q(t_1, \ldots, t_n)$ be a finite composition of tropical polynomials of variables $t_1, \ldots, t_n$. Then

$$Q(x) = q(g_1(x), \ldots, g_n(x)), \quad x \in c_0, \tag{3}$$

is a Lipschitz function on c_0.

Question. *Under which conditions on c_0 every Lipschitz symmetric function can be approximated by functions of the form* (3) *uniformly on c_0?*

Note that the norm in c_0 can be written exactly as a composition of tropical polynomials of g_1 and g_{-1}

$$\begin{aligned} \|x\|_{c_0} &= \max(g_1(x), -g_{-1}(x)) = g_1(x) - g_{-1}(x) - \min(g_1(x), -g_{-1}(x)) \\ &= g_1(x) \odot g_{-1}^{-1}(x) \odot (g_1(x) \oplus (g_{-1})^{-1})^{-1}(x). \end{aligned}$$

Thus, we have represented the Lipschitz symmetric function $x \mapsto \|x\|_{c_0}$ in the form (3).

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Досліджено ліпшицеві симетричні функції на банаховому просторі X з симетричним базисом. Розглянуто степеневі симетричні поліноми на ℓ_1 і показано, що вони є ліпшицевими у необмеженій області, яка складається з векторів $x \in \ell_1$ координати яких $|x_n| \leq 1$. Використовуючи функції max та min і тропічні поліноми від кількох змінних, побудовано широкий клас ліпшицевих симетричних функцій на банаховому просторі c_0, який можна описати як напівкільце композицій тропічних поліномів над простором c_0.

Ключові слова і фрази: Ліпшицева симетрична аналітична функція на банаховому просторі, симетричний базис, тропічний поліном.