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The crossing numbers of join products of eight graphs of order six with paths and cycles

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The crossing number $\mathrm{cr}(G)$ of a graph G is the minimum number of edge crossings over all drawings of G in the plane. The main aim of this paper is to give the crossing numbers of the join products of eight graphs on six vertices with paths and cycles on n vertices. The proofs are done with the help of several well-known auxiliary statements, the idea of which is extended by a suitable classification of subgraphs that do not cross the edges of the examined graphs.

Key words and phrases: graph, join product, crossing number, path, cycle.

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1 Introduction

The problem of reducing the number of crossings on edges of graphs is interesting in many areas. One of the most popular areas is the implementation of the VLSI layout, which has revolutionized circuit design and had a strong impact on parallel computing. Crossing numbers were also studied to improve the readability of hierarchical structures and automated graphs. The visualized graph should be easy to read and understand. For the sake of clarity of the graphical drawings, the reduction of crossings is likely the most important. Therefore, the investigation on the crossing number of simple graphs is a classical, but very difficult problem. M.R. Garey and D.S. Johnson [7] proved that crossing number determining is an NP-complete problem. Nevertheless, many researchers are trying to solve this problem. Note that the exact values of the crossing numbers are known for some families of graphs, see K. Clancy et al. [4].

The *crossing number* $\mathrm{cr}(G)$ of a simple graph G with the vertex set $V(G)$ and the edge set $E(G)$ is the minimum possible number of edge crossings over all drawings of G in the plane (for the definition of a *drawing* see M. Klešč [9]). A drawing with a minimum number of crossings (an optimal drawing) is always a *good* drawing, meaning that no edge crosses itself, no two edges cross more than once, and no two edges incident with the same vertex cross.

Let $D = D(G)$ be a good drawing of the graph G. We denote the number of crossings in D by $\mathrm{cr}_D(G)$. Let G_i and G_j be edge-disjoint subgraphs of G. We denote the number of crossings between edges of G_i and edges of G_j by $\mathrm{cr}_D(G_i, G_j)$, and the number of crossings among edges of G_i in D by $\mathrm{cr}_D(G_i)$. For any three mutually edge-disjoint subgraphs G_i, G_j and G_k of G the following equations hold (see [9]):

$$\mathrm{cr}_D(G_i \cup G_j) = \mathrm{cr}_D(G_i) + \mathrm{cr}_D(G_j) + \mathrm{cr}_D(G_i, G_j),$$
$$\mathrm{cr}_D(G_i \cup G_j, G_k) = \mathrm{cr}_D(G_i, G_k) + \mathrm{cr}_D(G_j, G_k).$$

УДК 519.1
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Some parts of proofs will be based on D.J. Kleitman's result [8] on the crossing numbers for some complete bipartite graphs $K_{m,n}$. He showed that

$$\mathrm{cr}(K_{m,n}) = \left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{m-1}{2} \right\rfloor \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor, \quad \text{if} \quad m \leq 6.$$

The join product of two graphs G_i and G_j, denoted $G_i + G_j$, is obtained from vertex-disjoint copies of G_i and G_j by adding all edges between $V(G_i)$ and $V(G_j)$. For $|V(G_i)| = m$ and $|V(G_j)| = n$, the edge set of $G_i + G_j$ is the union of the disjoint edge sets of the graphs G_i, G_j and the complete bipartite graph $K_{m,n}$. Let P_n and C_n be the *path* and the *cycle* on n vertices, respectively, and let D_n denote the *discrete graph* (sometimes called *empty graph*) on n vertices. The crossing numbers of the join products of the paths and the cycles with all graphs of order at most four have been well-known for a long time by M. Klešč [10, 11], and M. Klešč and Š. Schrötter [14], and therefore it is understandable that our immediate goal is to establish the exact values for the crossing numbers of $G + P_n$ and $G + C_n$ also for all graphs G of order five and six. Of course, the crossing numbers of $G + P_n$ and $G + C_n$ are already known for a lot of graphs G of order five and six (see [3, 5, 6, 9, 12, 15, 17–20, 24]). In all these cases, the graph G is connected and contains usually at least one cycle. Note that the crossing numbers of the join products $G + P_n$ and $G + C_n$ are known only for some disconnected graphs G on five or six vertices [2, 16, 22, 23].

In this paper, we will use definitions and notation of the crossing numbers of graphs presented by M. Klešč [10]. Let G^* be the disconnected graph of order six consisting of one 5-cycle and one isolated vertex. The crossing numbers of $G^* + D_n$ and $G^* + P_n$ were determined by Š. Berežný and M. Staš [2] using the properties of cyclic permutations. The required result of Theorem 3 for $G^* + C_n$ is established mainly using the mentioned results. By adding new edges to the graph G^*, the crossing numbers of $G_i + C_n$ for two other graphs G_i of order six are given in Corollary 2. The third section is devoted to the connected graph H^* of order six consisting of one 4-cycle and two leaves adjacent with two opposite vertices of the 4-cycle, and also to four different graphs H_i containing H^* as a subgraph. The crossing number of $H^* + D_n$ was also determined by Š. Berežný and M. Staš [1] using the properties of cyclic permutations. Due to the special drawings of $H^* + P_n$ in Figures 3 and 4 for n even and odd, respectively, $\mathrm{cr}(H^* + D_n) = \mathrm{cr}(H^* + P_n)$ can be presented as the result of Theorem 5 and the crossing number of $H^* + C_n$ with two additional crossings in Theorem 12. The paper concludes by giving the crossing numbers of $H_i + C_n$ in Theorems 13, 14 and Corollary 3. Also in this paper, some proofs are done with the help of several well-known auxiliary statements as Lemmas 1, 2 and Corollary 1.

The result in Theorem 3 has already been claimed by Z. Zhou et al. [25]. Since that paper does not appear to be available in English, we were unable to verify that proof. K. Clancy et al. [4] also placed an asterisk on a number of the results in their survey to essentially indicate that the mentioned results appeared in journals do not have a sufficiently rigorous peer-review process. The results in Theorems 11 and 13 have also been claimed by Z. Zhou and L. Li [26], but again not in English.

Let us suppose a graph G with $V(G) = \{v_1, v_2, \ldots, v_6\}$ and the cycle C_n with the vertices $c_1, c_2, \ldots, c_n$. The join product $G + C_n$ consists of one copy of the graph G, one copy of the cycle C_n, and the edges joining each vertex of G with each vertex of C_n. Let C_n^* denote the subgraph of $G + C_n$ induced on the vertices $c_1, c_2, \ldots, c_n$. For the vertices $v_1, v_2, \ldots, v_6$ of the graph G, let

T^{v_i} denote the subgraph induced by n edges joining the vertex v_i with the vertices $c_1, c_2, \dots, c_n$ of C_n^*. The edges joining the vertices of G with the vertices of C_n^* form the complete bipartite graph $K_{6,n}$, and so

$$G + C_n = G \cup K_{6,n} \cup C_n^* = G \cup \left(\bigcup_{i=1}^{6} T^{v_i} \right) \cup C_n^*.$$

Similarly, let T^i, $1 \leq i \leq n$, denote the subgraph induced by the six edges incident with the vertex c_i. This means that the graph $T^1 \cup \dots \cup T^n$ is isomorphic to the complete bipartite graph $K_{6,n}$ and therefore, we can write

$$G + C_n = G \cup K_{6,n} \cup C_n^* = G \cup \left(\bigcup_{i=1}^{n} T^{i} \right) \cup C_n^*.$$

In the proofs of theorems, the following three statements related to some restricted subdrawings of the graphs $G + C_n$ will be helpful.

Lemma 1 ([10, Lemma 2.2])**.** *Let D be a good drawing of $D_m + C_n$, $m \geq 2$, $n \geq 3$, in which no edge of C_n^* is crossed, and C_n^* does not separate the other vertices of the graph. Then, for all $i, j \in \{1, 2, \dots, m\}$, two different subgraphs T^{v_i} and T^{v_j} cross each other in D at least $\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor$ times.*

Corollary 1 ([13, Lemma 2.2])**.** *Let D be a good drawing of the join product $D_m + C_n$, $m \geq 2$, $n \geq 3$, in which the edges of C_n^* do not cross each other and C_n^* does not separate r vertices $v_1, v_2, \dots, v_r$, $2 \leq r \leq m$. Let $T^{v_1}, T^{v_2}, \dots, T^{v_s}$, $s < r$, be the subgraphs induced on the edges incident with the vertices $v_1, v_2, \dots, v_s$ that do not cross C_n^*. If k edges of some subgraph T^{v_j} induced on the edges incident with the vertex v_j, $j \in \{s+1, s+2, \dots, r\}$, cross the cycle C_n^*, then the subgraph T^{v_j} crosses each of the subgraphs $T^{v_1}, T^{v_2}, \dots, T^{v_s}$ at least $\lfloor \frac{n-k}{2} \rfloor \lfloor \frac{(n-k)-1}{2} \rfloor$ times in D.*

Lemma 2 ([13, Lemma 2.2])**.** *Let G be a graph of order m, $m \geq 1$. In an optimal drawing of the join product $G + C_n$, $n \geq 3$, the edges of C_n^* do not cross each other.*

2 The Crossing Number of $G^* + C_n$

Let G^* be the disconnected graph of order six consisting of one 5-cycle and one isolated vertex. In the rest of the paper, let $v_1v_2v_3v_4v_5v_1$ and v_6 be the vertex notation of the 5-cycle and the isolated vertex of G^*, respectively. The crossing numbers of $G^* + D_n$ and $G^* + P_n$ are given by Š. Berežný and M. Staš [2].

Theorem 1 ([2, Theorem 3.1])**.** $\mathrm{cr}(G^* + D_n) = 6\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + \lfloor \frac{n}{2} \rfloor$ *for* $n \geq 1$.

Theorem 2 ([2, Theorem 5.2])**.** $\mathrm{cr}(G^* + P_n) = 6\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + \lfloor \frac{n}{2} \rfloor + 1$ *for* $n \geq 2$.

Theorem 3. $\mathrm{cr}(G^* + C_n) = 6\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + \lfloor \frac{n}{2} \rfloor + 2$ *for* $n \geq 3$.

Proof. In Figure 1, the edges of $K_{6,n}$ cross each other

$$6\binom{\lceil \frac{n}{2} \rceil}{2} + 6\binom{\lfloor \frac{n}{2} \rfloor}{2} = 6\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor$$

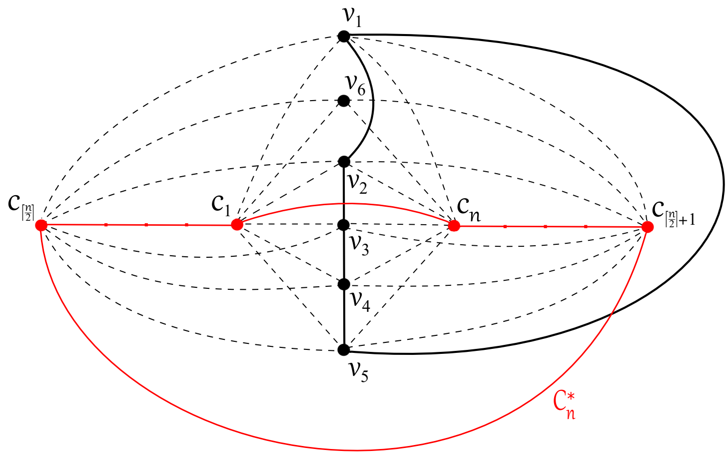

Figure 1. The good drawing of $G^* + C_n$ with $6\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + \lfloor \frac{n}{2} \rfloor + 2$ crossings

times, each subgraph T^i, $i \in \{1, \dots, \lceil \frac{n}{2} \rceil\}$ on the left side does not cross the edges of G^* and each subgraph T^i, $i \in \{\lceil \frac{n}{2} \rceil + 1, \dots, n\}$ on the right side crosses the edges of G^* exactly once. The cycle C_n^* crosses G^* twice, and so $6\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + \lfloor \frac{n}{2} \rfloor + 2$ crossings appear among the edges of the graph $G^* + C_n$ in this drawing. Thus, $\mathrm{cr}(G^* + C_n) \leq 6\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + \lfloor \frac{n}{2} \rfloor + 2$.

To prove the reverse inequality assume that there is a drawing of the graph $G^* + C_n$ with at most $6\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + \lfloor \frac{n}{2} \rfloor + 1$ crossings and let D be such a good drawing. By Theorem 2, none of the edges of C_n^* is crossed in D, because otherwise removing the crossed edge from C_n^* results in a good drawing of the graph $G^* + P_n$ with less than $6\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + \lfloor \frac{n}{2} \rfloor + 1$ crossings. Since there is no crossing on the edges of C_n^*, the edges of C_n^* do not cross each other. The subdrawing of C_n^* induced by D divides the plane into two regions and at least five vertices v_1, v_2, v_3, v_4 and v_5 of G^* must be placed in one of them. For all $i, j \in \{1, \dots, 5\}$ by Lemma 1, any two different subgraphs T^{v_i} and T^{v_j} cross each other at least $\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor$ times, and therefore, there are at least $\binom{5}{2} \lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor \geq 6\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + \lfloor \frac{n}{2} \rfloor + 2$ crossings in $D(T^{v_1} \cup T^{v_2} \cup T^{v_3} \cup T^{v_4} \cup T^{v_5})$. This contradiction completes the proof. $\square$

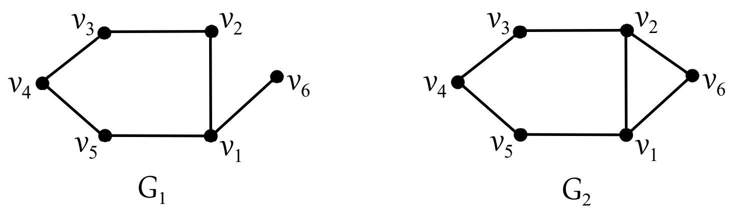

Figure 2. Two graphs G_1 and G_2 by adding new edges to the graph G^*

In Figure 2, let G_1 be the graph obtained from G^* by adding the edge v_1v_6 and G_2 be the graph obtained from G^* by adding the edges v_1v_6 and v_2v_6. Since we can add both edges v_1v_6 and v_2v_6 to the graph G^* without additional crossings in Figure 1, the drawings of the graphs $G_1 + C_n$ and $G_2 + C_n$ with exactly $6\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + \lfloor \frac{n}{2} \rfloor + 2$ crossings are obtained. On the other hand, $G^* + C_n$ is a subgraph of each $G_i + C_n$, and therefore, $\mathrm{cr}(G_i + C_n) \geq \mathrm{cr}(G^* + C_n)$ for each $i = 1, 2$. Thus, the following result is obvious.

Corollary 2. $\mathrm{cr}(G_i + C_n) = 6\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + \lfloor \frac{n}{2} \rfloor + 2$ *for* $n \geq 3$, *where* $i = 1, 2$.

Note that the crossing number of the graph $G_2 + C_n$ was obtained by M. Klešč et al. [12].

3 The Crossing Number of $H^* + C_n$

Let H^* be the connected graph consisting of one 4-cycle and two leaves adjacent with two opposite vertices of the 4-cycle. In the rest of the paper, let $v_1v_2v_3v_4v_1$ and v_5, v_6 be the vertex notation of the 4-cycle and two leaves of H^*, respectively. The crossing number of $H^* + D_n$ was established by Š. Berežný and M. Staš [1].

Theorem 4 ([1, Theorem 3.1])**.** $\mathrm{cr}(H^* + D_n) = 6\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + 2\lfloor \frac{n}{2} \rfloor$ *for* $n \geq 1$.

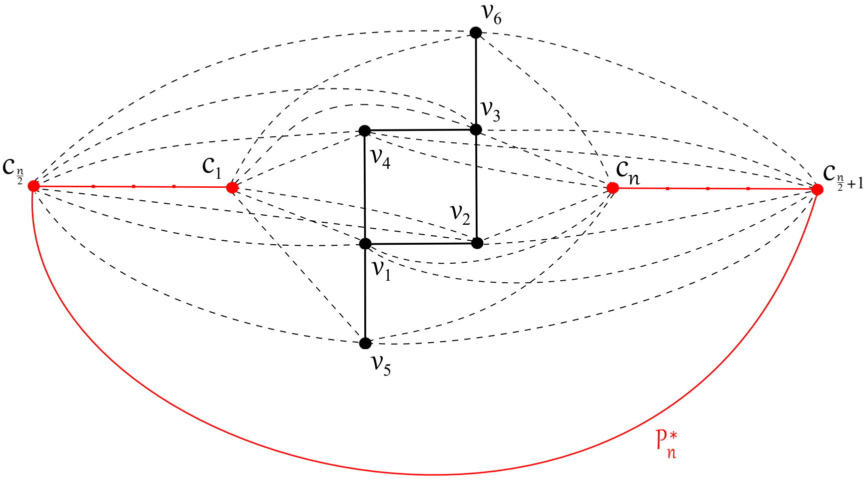

Figure 3. The good drawing of $H^* + P_n$ with $6\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + 2\lfloor \frac{n}{2} \rfloor$ crossings for n even

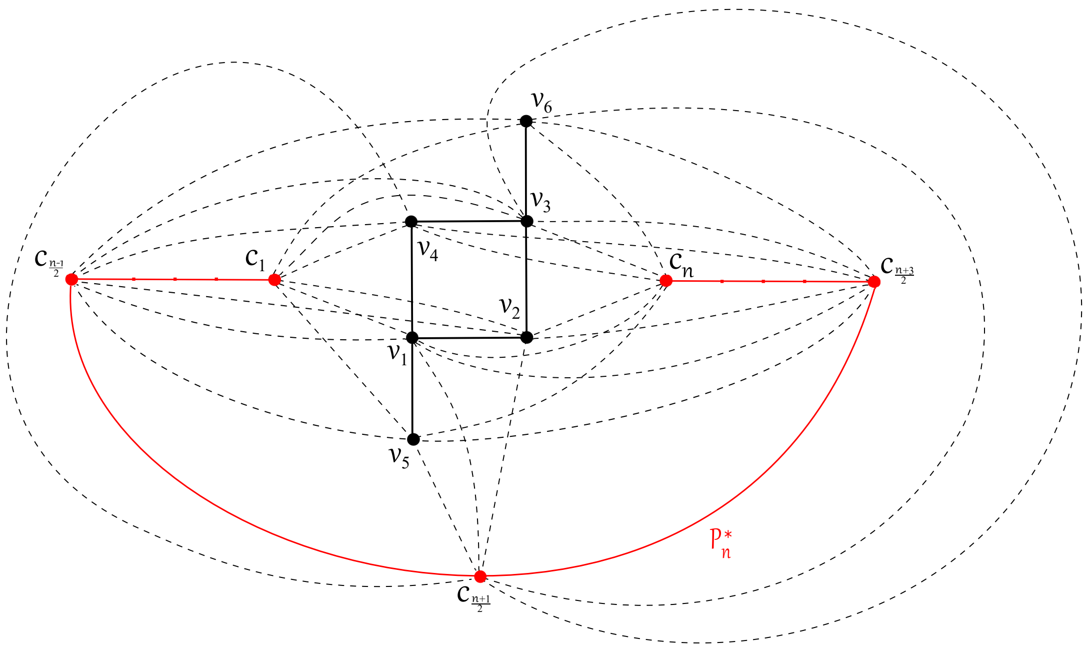

Figure 4. The good drawing of $H^* + P_n$ with $6\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + 2\lfloor \frac{n}{2} \rfloor$ crossings for n odd

For n even, Figure 3 offers the drawing of $H^* + P_n$ with $6\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + 2\lfloor \frac{n}{2} \rfloor$ crossings provided by the edges of $K_{6,n}$ cross each other $6\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor$ times and each subgraph T^i crosses the edges of H^* exactly once. For n odd at least 3, Figure 4 shows the drawing also with

$6\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor+2\lfloor\frac{n}{2}\rfloor$ crossings by adding one subgraph $T^{\frac{n+1}{2}}$ by which the edges of each of the $n-1$ graphs T^i, $i \neq \frac{n+1}{2}$, are crossed exactly three times, that is,

$$6\,\frac{n-1}{2}\,\frac{n-3}{2}+2\,\frac{n-1}{2}+3(n-1)=6\,\frac{n-1}{2}\,\frac{n-1}{2}+2\,\frac{n-1}{2}.$$

As H^*+D_n is a subgraph of H^*+P_n, the lower bound is the same based on Theorem 4 and so, the next result is obvious.

Theorem 5. $\mathrm{cr}(H^*+P_n)=6\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor+2\lfloor\frac{n}{2}\rfloor$ *for* $n\geq 2$.

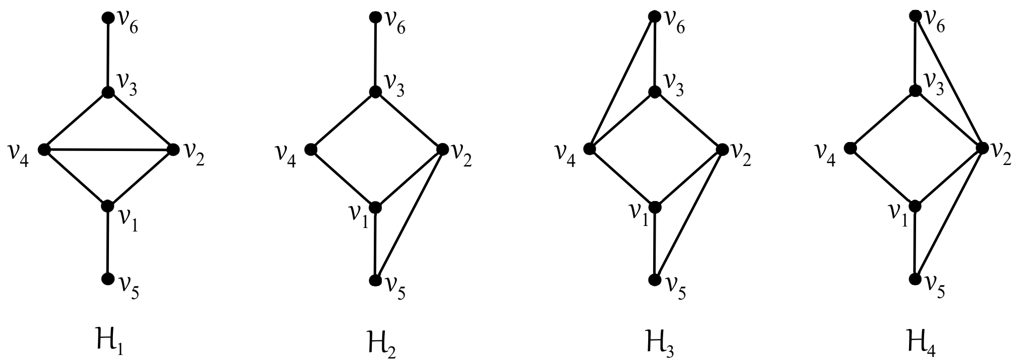

Figure 5. Four graphs H_1, H_2, H_3 and H_4 by adding new edges to the graph H^*

In Figure 5, let H_1 be the graph obtained from H^* by adding the edge v_2v_4, i.e. $H_1=H^*\cup\{v_2v_4\}$. Similarly, let $H_2=H^*\cup\{v_2v_5\}$, $H_3=H^*\cup\{v_2v_5,v_4v_6\}$ and $H_4=H^*\cup\{v_2v_5,v_2v_6\}$. Since we can add the edge v_2v_4 to the graph H^* without additional crossings in Figures 3 and 4, the drawings of the graph H_1+P_n with exactly $6\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor+2\lfloor\frac{n}{2}\rfloor$ crossings are obtained for all n at least two.

Theorem 6. $\mathrm{cr}(H_1+D_n)=6\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor+2\lfloor\frac{n}{2}\rfloor$ *for* $n\geq 1$.

Theorem 7. $\mathrm{cr}(H_1+P_n)=6\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor+2\lfloor\frac{n}{2}\rfloor$ *for* $n\geq 2$.

The crossing numbers of the join products of the graphs H_2 and H_3 with the paths P_n have already been investigated by E. Draženská [6] and M. Klešč [9], respectively.

Theorem 8 ([6, Theorem 1])**.** $\mathrm{cr}(H_2+P_n)=6\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor+2\lfloor\frac{n}{2}\rfloor+1$ *for* $n\geq 2$.

Theorem 9 ([9, Theorem 3.1])**.** $\mathrm{cr}(H_3+P_n)=6\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor+2\lfloor\frac{n}{2}\rfloor+1$ *for* $n\geq 2$.

Theorem 10 ([21, Corollary 4.1])**.** $\mathrm{cr}(H_4+D_n)=6\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor+2\lfloor\frac{n}{2}\rfloor$ *for* $n\geq 1$.

In Figure 6, there is the good drawing of H_4+P_n with $6\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor+2\lfloor\frac{n}{2}\rfloor+1$ crossings. Clearly, H_2 is a subgraph of H_4, and therefore, $\mathrm{cr}(H_4+P_n)\geq\mathrm{cr}(H_2+P_n)$.

Theorem 11. $\mathrm{cr}(H_4+P_n)=6\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor+2\lfloor\frac{n}{2}\rfloor+1$ *for* $n\geq 2$.

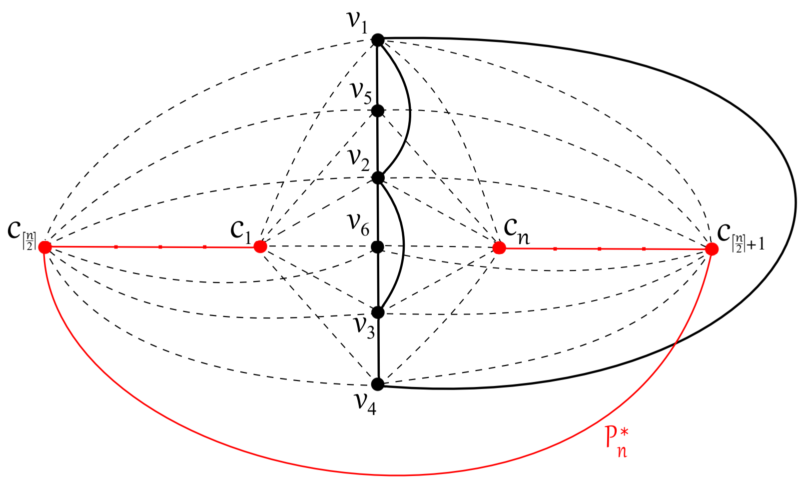

Figure 6. The good drawing of $H_4 + P_n$ with $6\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + 2\lfloor \frac{n}{2} \rfloor + 1$ crossings

Theorem 12. $\mathrm{cr}(H^* + C_n) = 6\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + 2\lfloor \frac{n}{2} \rfloor + 2$ *for* $n \geq 3$.

Proof. The proof proceeds in a similar way as for the graph $G^* + C_n$ in Theorem 3. Into both drawings in Figures 3 and 4, it is possible to add the edge c_1c_n which forms the cycle C_n^* on the vertices of the path P_n^* with exactly two another crossings. Thus, the crossing number of the graph $H^* + C_n$ is at most $6\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + 2\lfloor \frac{n}{2} \rfloor + 2$. To prove the reverse inequality assume that there is a drawing of $H^* + C_n$ with at most $6\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + 2\lfloor \frac{n}{2} \rfloor + 1$ crossings and let D be such a good drawing. By Theorems 4 and 5, at most one edge of the cycle C_n^* can be crossed in D, which yields that the edges of C_n^* do not cross each other. Again, the subdrawing of C_n^* induced by D divides the plane into two regions and the four vertices v_1, v_2, v_3, v_4 with at least one vertex v_5 or v_6 must be placed in one of them. By Lemma 1, there are at least $\binom{5}{2}\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + 1 \geq 6\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + 2\lfloor \frac{n}{2} \rfloor + 2$ crossings in D, because the graph H^* is connected. This completes the proof. □

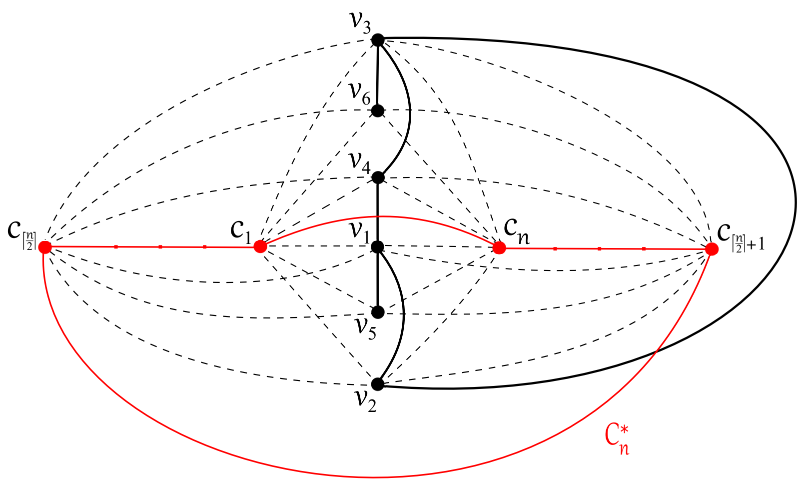

Figure 7. The good drawing of $H^* + C_n$ with $6\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + 2\lfloor \frac{n}{2} \rfloor + 2$ crossings

Due to Theorem 12, the good drawing of $H^* + C_n$ in Figure 7 is optimal. Clearly, we can add both edges v_2v_5 and v_4v_6 to the graph H^* without additional crossings, and therefore, the crossing numbers of the join products $H_2 + C_n$ and $H_3 + C_n$ are at most $6\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + 2\lfloor \frac{n}{2} \rfloor + 2$.

As H^* is a subgraph of the graph H_2, which is also a subgraph of H_3, we have

$$\mathrm{cr}(H_3 + C_n) \geq \mathrm{cr}(H_2 + C_n) \geq \mathrm{cr}(H^* + C_n) = 6\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor + 2\left\lfloor\frac{n}{2}\right\rfloor + 2.$$

Corollary 3. $\mathrm{cr}(H_i + C_n) = 6\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor + 2\lfloor\frac{n}{2}\rfloor + 2$ *for* $n \geq 3$*, where* $i = 2, 3$.

We also remark that the crossing number of the graph $H_3 + C_n$ was already obtained by M. Klešč [9].

Theorem 13. $\mathrm{cr}(H_4 + C_n) = 6\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor + 2\lfloor\frac{n}{2}\rfloor + 3$ *for* $n \geq 3$.

Proof. Into the drawing in Figure 6, it is possible to add the edge c_1c_n which forms the cycle C_n^* on the vertices of the path P_n^* with just two another crossings, and so the crossing number of the graph $H_4 + C_n$ is at most $6\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor + 2\lfloor\frac{n}{2}\rfloor + 3$. Let D be a good drawing of $H_4 + C_n$ with at most $6\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor + 2\lfloor\frac{n}{2}\rfloor + 2$ crossings. By Theorem 10, at most two edges of the cycle C_n^* can be crossed in D, but we can suppose that the edges of C_n^* do not cross each other using Lemma 2. The subdrawing of C_n^* induced by D divides the plane into two regions with at least five vertices of H_4 in one of them, because all three vertices of degree 2 are adjacent only with the vertices of degree at least 3. The case $\mathrm{cr}_D(H_4, C_n^*) = 2$ using Lemma 1 implies at least $\binom{5}{2}\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor + 2$ crossings in D. Now, let us assume that $\mathrm{cr}_D(H_4, C_n^*) = 0$ in the following three possible subcases. If there is no subgraph T^{v_i} by which is crossed any edge of C_n^*, then there are at least $\binom{6}{2}\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor$ crossings in D. Similarly, if there is only one T^{v_i} by which is crossed some edge of C_n^*, then we obtain at least $\binom{5}{2}\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor + 1$ crossings in D. Now, let us turn to the possibility of an existence of two different subgraphs T^{v_i} and T^{v_j} with $\mathrm{cr}_D(T^{v_i}, C_n^*) = 1$ and $\mathrm{cr}_D(T^{v_j}, C_n^*) = 1$. This, by Corollary 1 for $r = 6$, $s = 4$ and $k = 1$, enforces at least

$$\binom{4}{2}\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor + 4\left\lfloor\frac{n-1}{2}\right\rfloor\left\lfloor\frac{n-2}{2}\right\rfloor + 4\left\lfloor\frac{n-1}{2}\right\rfloor\left\lfloor\frac{n-2}{2}\right\rfloor + 2 \tag{1}$$

crossings in D. The last number of crossings thus determined confirms a contradiction with the assumption in D for all n at least 4. For $n = 3$, if all three subgraphs T^i cross the edges of H_4 at least once, then we can add three additional crossings on edges of H_4 in (1). Finally, for $n = 3$, if at least one of T^1, T^2 and T^3, say T^1, does not cross H_4, it is not difficult to verify that $\mathrm{cr}_D(H_4 \cup T^1, T^2) \geq 6$ and $\mathrm{cr}_D(H_4 \cup T^1, T^3) \geq 6$ hold for all possible placements of two vertices c_2 and c_3 of C_3^* in the subdrawing $D(H_4 \cup T^1)$, which yields at least 12 crossings in D. This also contradicts the assumption of D, and the proof of Theorem 13 is done. □

In a good drawing D of the graph $H_1 + C_n$, we separate the subgraphs T^i, $i \in \{1, 2, \ldots, n\}$, of $H_1 + C_n$ into two subsets. Let us denote by R_0 the set of subgraphs T^i for which $\mathrm{cr}_D(H_1, T^i) = 0$. Every other subgraph T^i crosses H_1 at least once in D.

Lemma 3. *Let D be a good drawing of $H_1 + C_n$, $n \geq 3$, with* $\mathrm{cr}_D(H_1, C_n^*) = 2$*. Let $T^i \in R_0$ be any subgraph of $H_1 + C_n$ and let* $|R_0| \geq \left\lceil\frac{n+1-(-1)^n+\mathrm{cr}_D(H_1)}{2}\right\rceil$*. If both conditions*

$$\mathrm{cr}_D(H_1 \cup T^i, T^j) \geq 5 \qquad \text{for any } T^j \in R_0,\ j \neq i, \tag{2}$$

and

$$\mathrm{cr}_D(H_1 \cup T^i, T^k) \geq 3 \qquad \text{for any } T^k \notin R_0 \tag{3}$$

hold, or $\mathrm{cr}_D(H_1) \geq 1$ *and both conditions*

$$\mathrm{cr}_D(H_1 \cup T^i, T^j) \geq 6 \qquad \textit{for any } T^j \in R_0,\ j \neq i, \tag{4}$$

and

$$\mathrm{cr}_D(H_1 \cup T^i, T^k) \geq 2 \qquad \textit{for any } T^k \notin R_0 \tag{5}$$

hold, then there are at least $6\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + 2\lfloor \frac{n}{2} \rfloor + 3$ *crossings in* D.

Proof. For easier reading, let $r = |R_0|$. By the assumption, $r \geq \lceil \frac{n+1-(-1)^n+\mathrm{cr}_D(H_1)}{2} \rceil$. The number of T^k that cross the graph H_1 at least once is equal to $n - r$. By fixing of the graph $H_1 \cup T^i$ with the assumptions of the conditions (2) and (3), we have

$$\begin{aligned} \mathrm{cr}_D(H_1 + C_n) &\geq 6\left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{n-2}{2} \right\rfloor + 5(r-1) + 3(n-r) + 2 = 6\left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{n-2}{2} \right\rfloor + 3n + 2r - 3 \\ &\geq 6\left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{n-2}{2} \right\rfloor + 3n + 2\left\lceil \frac{n+1-(-1)^n+0}{2} \right\rceil - 3 \geq 6\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + 2\left\lfloor \frac{n}{2} \right\rfloor + 3. \end{aligned}$$

Similarly, if the conditions (4) and (5) are fulfilled, then

$$\begin{aligned} \mathrm{cr}_D(H_1 + C_n) &\geq 6\left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{n-2}{2} \right\rfloor + 6(r-1) + 2(n-r) + 2 = 6\left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{n-2}{2} \right\rfloor + 2n + 4r - 4 \\ &\geq 6\left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{n-2}{2} \right\rfloor + 2n + 4\left\lceil \frac{n+1-(-1)^n+1}{2} \right\rceil - 4 \geq 6\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor + 2\left\lfloor \frac{n}{2} \right\rfloor + 3. \end{aligned}$$

This completes the proof. □

Theorem 14. $\mathrm{cr}(H_1 + C_n) = 6\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + 2\lfloor \frac{n}{2} \rfloor + 3$ *for* $n \geq 3$.

Proof. The proof proceeds in a similar way as for the graph $H_4 + C_n$ in Theorem 13, but many more cases will need to be discussed. Into both drawings in Figures 3 and 4 by adding the edge v_2v_4, it is possible to add the edge c_1c_n which forms the cycle C_n^* on the vertices of the path P_n^* with just three another crossings, i.e. C_n^* is crossed by three edges v_1v_4, v_2v_3 and v_2v_4 of the graph H_1. Thus, $\mathrm{cr}(H_1 + C_n) \leq 6\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + 2\lfloor \frac{n}{2} \rfloor + 3$. Let D be a good drawing of $H_1 + C_n$ with at most $6\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + 2\lfloor \frac{n}{2} \rfloor + 2$ crossings. By Theorem 6, at most two edges of the cycle C_n^* can be crossed in D, and we can also suppose that the edges of C_n^* do not cross each other using Lemma 2. The subdrawing of C_n^* induced by D divides the plane into two regions with at least four vertices of H_1 in one of them, and so the following three possible cases may occur.

Case 1: $\mathrm{cr}_D(H_1, C_n^*) = 0$. In this case, we can follow the same discussion as in the proof of Theorem 13 for all n at least 4. For $n = 3$, $\mathrm{cr}_D(H_1, T^1 \cup T^2 \cup T^3) \geq 2$ enforces at least two additional crossings on edges of H_1 in (1), which yields a contradiction with the assumption in D. Therefore, let at least two of T^1, T^2 and T^3, say T^1 and T^2, do not cross the edges of H_1, and let T^3 cross the edges of H_1 at most once. For a $T^i \in R_0$, we have four ways of obtaining the subdrawing of $H_1 \cup T^i$ depending on which region of $D(H_1 \cup T^i \setminus \{v_1, v_3\})$ the edges c_iv_1 and c_iv_3 are placed in. Using cyclic permutations, it is not difficult to verify that $\mathrm{cr}_D(T^i, T^j) \geq 4$ holds for any two different $T^i, T^j \in R_0$. Now, let the subgraph T^3 cross the edges of H_1 just once, otherwise, we obtain at least 12 crossings in $D(T^1 \cup T^2 \cup T^3)$. In possible regions of $D(H_1 \cup T^i)$, one can easily determine that $\mathrm{cr}_D(T^i, T^3) \geq 2$ is fulfilling for any $i = 1, 2$. All subcases confirm a contradiction with the assumption in D.

Case 2: $\mathrm{cr}_D(H_1, C_n^*) = 1$. In the rest of the proof, based on the symmetry of the graph H_1, let the edge v_3v_6 cross some edge of the cycle C_n^*. If there is no subgraph T^{v_i}, $i \in \{1, \dots, 5\}$, by which is crossed any edge of C_n^*, then there are at least $\binom{5}{2}\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor + 1$ crossings in D using Lemma 1. Obviously, the case $\mathrm{cr}_D(T^{v_6}, C_n^*) = 1$ also contradicts the assumption of D, because all edges of five subgraphs T^{v_i}, $i \in \{1, \dots, 5\}$, must be placed in one region of C_n^*. Now, assume $\mathrm{cr}_D(T^{v_i}, C_n^*) = 1$ for only one $i \in \{1, \dots, 5\}$. Since some edges of T^{v_i} and T^{v_6} are crossed in the second region of C_n^* and there are two crossings on the edges of C_n^*, we obtain at least

$$\binom{4}{2}\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor + 4\left\lfloor\frac{n-1}{2}\right\rfloor\left\lfloor\frac{n-2}{2}\right\rfloor + 3 \tag{6}$$

crossings in D by Corollary 1 for $r = 5$, $s = 4$, and $k = 1$. The last number of crossings thus determined confirms a contradiction with the assumption in D for all n at least 4. For $n = 3$, $\mathrm{cr}_D(H_1, T^1 \cup T^2 \cup T^3) \geq 2$ enforces at least two additional crossings on edges of H_1 in (6), and therefore, we can apply the same discussion for the subgraphs T^1, T^2 and T^3 as in the previous Case 1.

Case 3: $\mathrm{cr}_D(H_1, C_n^*) = 2$. If one of the seven edges of the graph H_1 crosses the edges of the cycle C_n^* up to twice, then there are at least $\binom{6}{2}\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor + 2$ crossings in D. Now, suppose that the edges of C_n^* are crossed by two different edges of H_1, that is, either by both bridges v_1v_5, v_3v_6 or two edges v_jv_2, v_jv_4 for only one $j \in \{1, 3\}$. In all mentioned subcases, we obtain at least

$$\binom{4}{2}\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor + \left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor + 2 + n - |R_0| + \mathrm{cr}_D(H_1) \tag{7}$$

crossings in D. Clearly, $|R_0| = 0$ contradicts the assumption of D for all n at least 3. For $|R_0| \geq 1$, let us suppose drawings of H_1 only with the possibility of obtaining a drawing of $H_1 \cup T^i$ for a subgraph $T^i \in R_0$ (not necessarily planar drawings of H_1) with respect to the restriction that the edges of all subgraphs T^k cannot cross the edges of C_n^*. For this purpose, we will further deal with only two possible cases of drawings of the graph H_1 with respect to the cycle C_n^* presented in Figure 8.

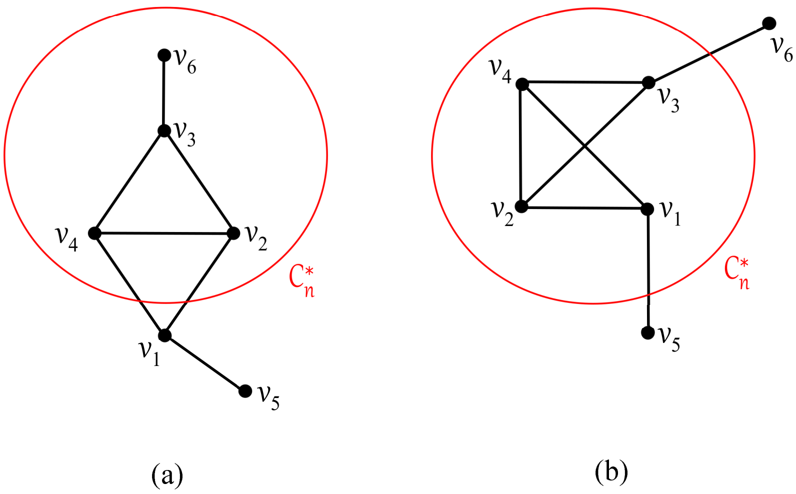

Figure 8. Two possible drawings of the graph H_1 with respect to the cycle C_n^* with the possibility to obtain a subgraph $T^i \in R_0$ such that $\mathrm{cr}_D(T^i, C_n^*) = 0$. (**a**) the planar drawing of H_1 with $\mathrm{cr}_D(v_jv_2 \cup v_jv_4, C_n^*) = 2$ for $j = 1$; (**b**) the nonplanar drawing of H_1 for which both bridges of H_1 are crossed by C_n^*

If $|R_0| \geq \lceil \frac{n+1-(-1)^n+\mathrm{cr}_D(H_1)}{2} \rceil$, it is not difficult to verify that the conditions (2), (3) and (4), (5) are fulfilled for the drawing of H_1 in Figure 8 (a) and (b), respectively. Consequently, Lemma 3 forces at least $6\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + 2\lfloor \frac{n}{2} \rfloor + 3$ crossings in D. Finally, if

$$1 \leq |R_0| < \left\lceil \frac{n+1-(-1)^n+\mathrm{cr}_D(H_1)}{2} \right\rceil,$$

the number of crossings in (7) confirms a contradiction with the assumption in D for all n at least 4. For $n = 3$, we obtain also the contradiction with the number of crossings in D except for the case of the drawing of H_1 in Figure 8 (a) with $T^1, T^2 \in R_0$ and T^3 by which the edges of H_1 are crossed exactly once, but the same discussion as in Case 1 forces at least 11 crossings in D again.

Thus, it was shown in all mentioned cases that there is no good drawing D of the graph $H_1 + C_n$ with fewer than $6\lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor + 2\lfloor \frac{n}{2} \rfloor + 3$ crossings. This completes the proof. □

Conclusions

We suppose that similar forms of discussions can be used to estimate the unknown values of the crossing numbers of the remaining graphs on six vertices with a much larger number of edges in the join products with the paths, and also with the cycles. We expect the same for other symmetric graphs of order five.

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Сташ М. *Число схрещень об'єднаних добутків восьми графів шостого порядку зі шляхами та циклами* // Карпатські матем. публ. — 2023. — Т.15, №1. — С. 66–77.

Число схрещень $\mathrm{cr}(G)$ графа G — це найменше число перетинів ребер плоского зображення графа G. Головним завданням цієї статті є знайти число схрещень об'єднаних добутків восьми графів на шести вершинах з шляхами і циклами на n вершинах. Доведення ґрунтуються на кількох відомих допоміжних твердженнях, ідея яких поглиблена відповідною класифікацією підграфів, що не перетинають ребра досліджуваних графів.

Ключові слова і фрази: граф, об'єднаний добуток, число схрещень, шлях, цикл.