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Fekete-Szegö inequality for a subclass of analytic functions associated with Gegenbauer polynomials

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In this paper, we define a subclass of analytic functions by denote $T_\beta H\left(z, C_n^{(\lambda)}(t)\right)$ satisfying the following subordinate condition

$$(1-\beta)\left(\frac{zf'(z)}{f(z)}\right)+\beta\left(1+\frac{zf''(z)}{f'(z)}\right)\prec\frac{1}{(1-2tz+z^2)^\lambda},$$

where $\beta\geq 0$, $\lambda\geq 0$ and $t\in\left(\frac{1}{2},1\right]$. We give coefficient estimates and Fekete-Szegö inequality for functions belonging to this subclass.

Key words and phrases: analytic and univalent function, typically real function, subordination, Gegenbauer polynomial, coefficient estimate, Fekete-Szegö inequality.

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1 Introduction

Let D be the open unit disk $\{z\in C : |z|<1\}$ and A be the family of all analytic functions defined on D normalized by the conditions $f(0)=0$ and $f'(0)-1=0$. Then each function f in A has the following Taylor expansion

$$f(z)=z+\sum_{n=2}^{\infty}a_nz^n. \tag{1}$$

Furthermore, by S we shall denote the class of all functions A that are univalent in D.

In 1933, M. Fekete and G. Szegö [5] obtained a sharp bound of the functional $\left|a_3-\mu a_2^2\right|$ with real μ, $0\leq\mu\leq 1$, for a univalent function f. Since then, the problem of finding the sharp bounds for this functional of any compact family of functions $f\in A$ with any complex μ is known as the classical Fekete-Szegö problem or inequality.

Let f and g be analytic functions in D. We define that the function f is subordinate to g in D and denoted by

$$f(z)\prec g(z),\quad z\in D,$$

if there exists a Schwarz function ω, which is analytic in D with $\omega(0)=0$ and $|\omega(z)|<1$, $z\in D$, such that

$$f(z)=g\left(\omega(z)\right),\quad z\in D.$$

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If g is a univalent function in D, then

$$f(z) \prec g(z) \Longleftrightarrow f(0) = g(0) \quad \text{and} \quad f(D) \subset g(D).$$

A function $f \in A$ maps D onto a starlike domain with respect to $w_0 = 0$ if and only if

$$\frac{zf'(z)}{f(z)} \prec \frac{1-z}{1+z}, \quad z \in D. \tag{2}$$

A function $f \in A$ maps D onto a convex domain if and only if

$$1 + \frac{zf''(z)}{f'(z)} \prec \frac{1-z}{1+z}, \quad z \in D.$$

It is well known that if a function $f \in A$ satisfies (2), then f is univalent and starlike in D.

Let $\beta \in [0,1)$. A function $f \in A$ is said to be starlike of order β and convex of order β if

$$\frac{zf'(z)}{f(z)} \prec \frac{1-(1-2\beta)z}{1+z}, \quad z \in D,$$

and

$$1 + \frac{zf''(z)}{f'(z)} \prec \frac{1-(1-2\beta)z}{1+z}, \quad z \in D,$$

are satisfied, respectively.

In recent years, many studies have been made on the coefficient estimates of functions belonging to this class and the Fekete-Szegö inequality by defining sublasses of analytic functions associated with Chebyshev polynomials. Some of these studies are presented below.

In 2015, J. Dziok et al. [4] have studied the coefficient bounds and Fekete-Szegö inequality for the functions $f \in H(t)$, $t \in \left(\frac{1}{2}, 1\right]$, satisfying the following condition

$$1 + \frac{zf''(z)}{f'(z)} \prec H(z,t) = \frac{1}{1-2tz+z^2}.$$

In 2016, Ş. Altınkaya and S. Yalçın [1] have defined and studied the coefficient estimates, Fekete-Szegö inequality for the functions $f \in K(\lambda, t)$, $t \in \left(\frac{1}{2}, 1\right]$, $\lambda \geq 0$, satisfying the following condition

$$(1-\lambda)\frac{zf'(z)}{f(z)} + \lambda\left(1 + \frac{zf''(z)}{f'(z)}\right) \prec H(z,t).$$

In 2018, Ş. Altınkaya and S. Yalçın [2] have defined and studied the coefficient estimates, Fekete-Szegö inequality for the functions $f \in L(\alpha, t)$, $\alpha \geq 0$, $t \in \left(\frac{1}{2}, 1\right]$, satisfying the following condition

$$\left(\frac{zf'(z)}{f(z)}\right)^{\alpha}\left(1 + \frac{zf''(z)}{f'(z)}\right)^{1-\alpha} \prec H(z,t), \quad z \in D.$$

In 2019, E. Szatmari and Ş. Altınkaya [7] have defined and studied the coefficient estimates, Fekete-Szegö inequality for the functions $f \in F(H, \alpha, \delta, \mu)$, $0 \leq \alpha \leq 1$, $1 \leq \delta \leq 2$, $0 \leq \mu \leq 1$, satisfying the following condition

$$\left[\alpha\left(\frac{zf'(z)}{f(z)}\right)^{\delta} + (1-\alpha)\left(\frac{zf'(z)}{f(z)}\right)^{\mu}\left(1 + \frac{zf''(z)}{f'(z)}\right)^{1-\mu}\right] \prec H(z,t), \quad z \in D.$$

In 2020, M. Çağlar et al. [3] have defined and studied the coefficient estimates, Fekete-Szegö inequality for the functions $f \in N(\lambda, \beta, t)$, $0 \leq \beta \leq \lambda \leq 1$, $t \in \left(\frac{1}{2}, 1\right]$, satisfying the following condition

$$\frac{\lambda\beta z^3 f'''(z) + (2\lambda\beta + \lambda - \beta) z^2 f''(z) + z f'(z)}{\lambda\beta z^2 f''(z) + (\lambda - \beta) z f'(z) + (1 - \lambda + \beta) f(z)} \prec H(z,t), \quad z \in D.$$

In 1994, J. Szynal [8] introduced and investigated the class $T(\lambda)$, $\lambda \geq 0$, as the subclass of A consisting of functions of the form

$$f(z) = \int_{-1}^{1} k(z,t)\, d\mu(t), \tag{3}$$

where

$$k(z,t) = \frac{z}{(1 - 2tz + z^2)^{\lambda}}, \quad z \in D, \ -1 \leq t \leq 1, \tag{4}$$

and μ is a probability measure on the interval $[-1, 1]$. The collection of such measures on $[a, b]$ is denoted by $P_{[a,b]}$. The function $k(z,t)$ has the Taylor series expansion

$$k(z,t) = z + C_1^{(\lambda)}(t) z^2 + C_2^{(\lambda)}(t) z^3 + C_3^{(\lambda)}(t) z^4 + \ldots$$

where $C_n^{(\lambda)}(t)$ denotes the Gegenbauer polynomial of degree n.

First polynomials of this type are the following:

$$\begin{aligned} &C_0^{(\lambda)}(t) = 1, \\ &C_1^{(\lambda)}(t) = 2\lambda t, \\ &C_2^{(\lambda)}(t) = 2\lambda(\lambda+1)t^2 - \lambda, \\ &C_3^{(\lambda)}(t) = \tfrac{4}{3}\lambda(\lambda+1)(\lambda+2)t^3 - 2\lambda(\lambda+1)t. \end{aligned} \tag{5}$$

If $f \in T(\lambda)$ is given by (3), then the coefficients of this function can be written as follows

$$a_n = \int_{-1}^{1} C_{n-1}^{(\lambda)}(t)\, d\mu(t).$$

Note that $T(1) = T$ is the well-known class of typically real functions. For $\lambda = \frac{1}{2}$ we obtain the class of typically real functions related to Legendre polynomials $P_n(t) = C_n^{\left(\frac{1}{2}\right)}(t)$.

Now, we define a subclass of analytic functions in D, satisfying the subordination condition, associated with Gegenbauer polynomials.

Definition 1. *We define the class of analytic functions, and denote it by $T_\beta H\left(z, C_n^{(\lambda)}(t)\right)$, satisfying the following subordinate condition*

$$(1-\beta)\left(\frac{zf'(z)}{f(z)}\right) + \beta\left(1 + \frac{zf''(z)}{f'(z)}\right) \prec H\left(z, C_n^{(\lambda)}(t)\right) = \frac{k(z,t)}{z} = \frac{1}{(1 - 2tz + z^2)^{\lambda}}, \tag{6}$$

where $\beta \geq 0$, $\lambda \geq 0$ and $t \in \left(\frac{1}{2}, 1\right]$.

In this paper, we obtain initial coefficients $|a_2|$ and $|a_3|$ for subclass $T_\beta H\left(z, C_n^{(\lambda)}(t)\right)$ by means of Gegenbauer polynomials expansions of analytic functions in D. Also, we solve Fekete-Szegö problem for functions in this subclass.

2 Coefficient bounds for the functions, belonging to class $T_\beta H\left(z, C_n^{(\lambda)}(t)\right)$

We begin with the following result involving initial coefficient bounds $|a_2|$ and $|a_3|$ for a function from the class $T_\beta H\left(z, C_n^{(\lambda)}(t)\right)$.

Theorem 1. *Let the function $f(z)$ given by* (1) *be in the class $T_\beta H\left(z, C_n^{(\lambda)}(t)\right)$. Then*

$$|a_2| \leq \frac{2\lambda t}{1+\beta}$$

and

$$|a_3| \leq \frac{\lambda}{(2+4\beta)} \left\{ \left(\frac{(\lambda+1)\beta^2 + 2(1+4\lambda)\beta + (1+3\lambda)}{(1+\beta)^2} \right) 2t^2 - 1 \right\},$$

where $\beta \geq 0$, $\lambda \geq 0$ and $t \in \left(\frac{1}{\sqrt{2}}, 1\right]$.

Proof. From (4), we have

$$\frac{k(z,t)}{z} = 1 + C_1^{(\lambda)}(t)\,z + C_2^{(\lambda)}(t)\,z^2 + C_3^{(\lambda)}(t)\,z^3 + \ldots = \frac{1}{(1-2tz+z^2)^\lambda}.$$

From (1), we can write

$$\frac{zf'(z)}{f(z)} = \frac{z + 2a_2z^2 + 3a_3z^3 + \ldots}{z + a_2z^2 + a_3z^3 + \ldots} = 1 + a_2z + (2a_3 - a_2^2)z^2 + \ldots$$

and

$$1 + \frac{zf''(z)}{f'(z)} = 1 + \frac{2a_2z + 6a_3z^2 + \ldots}{1 + 2a_2z + 3a_3z^2 + \ldots} = 1 + 2a_2z + (6a_3 - 4a_2^2)z^2 + \ldots.$$

Thus, we obtain

$$\begin{aligned}(1-\beta)\left(\frac{zf'(z)}{f(z)}\right) &= \left\{1 + a_2z + (2a_3 - a_2^2)z^2 + \ldots\right\} \\ &= (1-\beta) + (1-\beta)\,a_2z + (1-\beta)\left\{2a_3 - a_2^2\right\}z^2 + \ldots\end{aligned}$$

and

$$\begin{aligned}\beta\left(1 + \frac{zf''(z)}{f'(z)}\right) &= \beta\left\{1 + 2a_2z + (6a_3 - 4a_2^2)z^2 + \ldots\right\} \\ &= \beta + 2\beta a_2z + \beta(6a_3 - 4a_2^2)z^2 + \ldots.\end{aligned}$$

With a simple calculation, we find

$$(1-\beta)\left(\frac{zf'(z)}{f(z)}\right)+\beta\left(1+\frac{zf''(z)}{f'(z)}\right)=1+(1+\beta)\,a_2z+\left\{(2+4\beta)\,a_3-(1+3\beta)\,a_2^2\right\}z^2+\dots.$$

From (6), we have

$$(1-\beta)\left(\frac{zf'(z)}{f(z)}\right)+\beta\left(1+\frac{zf''(z)}{f'(z)}\right)=1+C_1^{(\lambda)}(t)\,p(z)+C_2^{(\lambda)}(t)\,(p(z))^2+C_3^{(\lambda)}(t)\,(p(z))^3+\dots \tag{7}$$

for some analytic functions

$$p(z)=p_1z+p_2z^2+p_3z^3+\dots,\quad z\in D, \tag{8}$$

such that $p(0)=0$, $|p(z)|<1$, $z\in D$. For such functions, it is well known (see [6]), that

$$|p_j|\le 1,\quad j\in\mathbb{N}, \tag{9}$$

and for all $\nu\in\mathbb{R}$

$$\left|p_2-\nu p_1^2\right|\le\max\{1,|\nu|\}. \tag{10}$$

Therefore from (7) and (8) we have

$$\begin{aligned}(1-\beta)\left(\frac{zf'(z)}{f(z)}\right)+\beta\left(1+\frac{zf''(z)}{f'(z)}\right)&=1+C_1^{(\lambda)}(t)p(z)+C_2^{(\lambda)}(t)(p(z))^2+C_3^{(\lambda)}(t)(p(z))^3+\dots\\&=1+C_1^{(\lambda)}(t)\,p_1z+\left[C_1^{(\lambda)}(t)\,p_2+C_2^{(\lambda)}(t)\,p_1^2\right]z^2+\dots\end{aligned}$$

and thus

$$\begin{aligned}1+(1+\beta)\,a_2z&+\left\{(2+4\beta)a_3-(1+3\beta)a_2^2\right\}z^2+\dots\\&=1+C_1^{(\lambda)}(t)\,p_1z+\left[C_1^{(\lambda)}(t)\,p_2+C_2^{(\lambda)}(t)\,p_1^2\right]z^2+\dots.\end{aligned} \tag{11}$$

It follow from (11) that

$$(1+\beta)\,a_2=C_1^{(\lambda)}(t)\,p_1 \tag{12}$$

and

$$(2+4\beta)\,a_3-(1+3\beta)\,a_2^2=C_1^{(\lambda)}(t)\,p_2+C_2^{(\lambda)}(t)\,p_1^2. \tag{13}$$

From (5), (9) and (12), we can write

$$(1+\beta)\,a_2=C_1^{(\lambda)}(t)\,p_1\Longrightarrow a_2=\frac{2\lambda tp_1}{1+\beta}\Longrightarrow|a_2|\le\frac{2\lambda t}{1+\beta}.$$

From (5), (9) and (13), we can write

$$\begin{aligned}&(2+4\beta)a_3-(1+3\beta)\,a_2^2=C_1^{(\lambda)}(t)\,p_2+C_2^{(\lambda)}(t)\,p_1^2\Longrightarrow\\&(2+4\beta)\,a_3=C_1^{(\lambda)}(t)\,p_2+C_2^{(\lambda)}(t)\,p_1^2+(1+3\beta)\left[\frac{C_1^{(\lambda)}(t)\,p_1}{1+\beta}\right]^2\Longrightarrow\\&(2+4\beta)\,a_3=2\lambda tp_2+\left\{2\lambda(\lambda+1)\,t^2-\lambda\right\}p_1^2+(1+3\beta)\left[\frac{2\lambda tp_1}{1+\beta}\right]^2\Longrightarrow\\&(2+4\beta)\,a_3=2\lambda tp_2+\left[2\lambda(\lambda+1)\,t^2-\lambda+\left(\frac{1+3\beta}{(1+\beta)^2}\right)4\lambda^2t^2\right]p_1^2\Longrightarrow\\&(2+4\beta)\,a_3=2\lambda t\left\{p_2-\frac{1}{2t}\left[1-\left((\lambda+1)+\left(\frac{1+3\beta}{(1+\beta)^2}\right)2\lambda\right)2t^2\right]p_1^2\right\}.\end{aligned}$$

Thus, from (10), we have

$$|a_3| \le \frac{2\lambda t}{(2+4\beta)} \max\left\{1, \frac{1}{2t}\left|\left((\lambda+1)+\left(\frac{(1+3\beta)\lambda}{(1+\beta)^2}\right)\right)2t^2-1\right|\right\}.$$

With a simple calculation, we find that

$$(\lambda+1)+\left(\frac{(2+6\beta)\lambda}{(1+\beta)^2}\right) \ge 1$$

for $\beta \ge 0$, $\lambda \ge 0$ and $t \in \left(\frac{1}{\sqrt{2}}, 1\right]$. Consequently, we obtain

$$|a_3| \le \frac{\lambda}{(2+4\beta)}\left\{\left((\lambda+1)+\left(\frac{(2+3\beta)\lambda}{(1+\beta)^2}\right)\right)2t^2-1\right\} \Longrightarrow$$
$$|a_3| \le \frac{\lambda}{(2+4\beta)}\left\{\left(\frac{(\lambda+1)\beta^2+2(1+4\lambda)\beta+(1+3\lambda)}{(1+\beta)^2}\right)2t^2-1\right\}.$$

The proof of Theorem 1 is completed. □

Taking $\lambda = \frac{1}{2}$ in Theorem 1, we obtain the following corollary.

Corollary 1. *Let the function $f(z)$ given by* (1) *be in the class* $T_\beta H\left(z, C_n^{(\frac{1}{2})}(t)\right) = T_\beta H(z, P_n(t))$. *Then*

$$|a_2| \le \frac{t}{1+\beta} \quad \textit{and} \quad |a_3| \le \frac{1}{4(1+2\beta)}\left\{\frac{(3\beta^2+12\beta+5)}{(1+\beta)^2}t^2-1\right\},$$

where $\beta \ge 0$ *and* $t \in \left(\frac{1}{\sqrt{2}}, 1\right]$.

Taking $\lambda = \frac{1}{2}, \beta = 1$ in Theorem 1, we obtain the following corollary.

Corollary 2. *Let the function $f(z)$ given by* (1) *be in the class* $T_1 H(z, C_n^{(\frac{1}{2})}(t)) = T_1 H(z, P_n(t))$. *Then*

$$|a_2| \le \frac{t}{2} \quad \textit{and} \quad |a_3| \le \frac{1}{12}(5t^2-1),$$

where $t \in \left(\frac{1}{\sqrt{2}}, 1\right]$.

Taking $\lambda = 1$ in Theorem 1, we obtain the following corollary.

Corollary 3. *Let the function $f(z)$ given by* (1) *be in the class* $T_\beta H(z, C_n^{(\lambda)}(t)) = L(\beta, t)$. *Then*

$$|a_2| \le \frac{2t}{1+\beta} \quad \textit{and} \quad |a_3| \le \frac{(2\beta^2+10\beta+4)}{(1+2\beta)(1+\beta)^2}t^2 - \frac{1}{2(1+2\beta)},$$

where $\beta \ge 0$ *and* $t \in \left(\frac{1}{\sqrt{2}}, 1\right]$.

Remark 1. *The estimate of $|a_3|$, obtained in Corollary 3, is better than the corresponding estimate of Ş. Altınkaya and S. Yalçın [1].*

Taking $\lambda = 1, \beta = 1$ in Theorem 1, we obtain result of J. Dziok et al. [4] the following corollary.

Corollary 4. *Let the function $f(z)$ given by* (1) *be in the class $T_1 H(z, C_n^{(1)}(t)) = H(t)$. Then*

$$|a_2| \leq t \quad \text{and} \quad |a_3| \leq \frac{4}{3}t^2 - \frac{1}{6},$$

where $t \in \left(\frac{1}{\sqrt{2}}, 1\right]$.

3 Fekete-Szegö inequality for the function class $T_\beta H(z, C_n^{(\lambda)}(t))$

Now, we are ready to find the sharp bounds of Fekete-Szegö functional $a_3 - \xi a_2^2$ defined for $T_\beta H\left(z, C_n^{(\lambda)}(t)\right)$ given by (1).

Theorem 2. *Let a function $f(z)$ given by* (1) *be in the class $T_\beta H\left(z, C_n^{(\lambda)}(t)\right)$. Then for some $\xi \in \mathbb{R}$ we have*

$$|a_3 - \xi a_2^2| \leq \begin{cases} \frac{\lambda t}{(1+2\beta)}, & \text{for } \xi \in [\xi_1, \xi_2], \\ \frac{\lambda t}{(1+2\beta)} \left| \frac{2\lambda(\lambda+1)t^2 - \lambda}{2\lambda t} + \left(\frac{1+3\beta}{(1+\beta)^2}\right) 2\lambda t - \xi \left(\frac{1+2\beta}{(1+\beta)^2}\right) 4\lambda t \right|, & \text{for } \xi \notin [\xi_1, \xi_2], \end{cases}$$

where

$$\xi_1 = \frac{(1+\beta)^2}{4\lambda t(1+2\beta)} \left\{ -1 + \frac{2\lambda(\lambda+1)t^2 - \lambda}{2\lambda t} + \left(\frac{1+3\beta}{(1+\beta)^2}\right) 2\lambda t \right\}$$

and

$$\xi_2 = \frac{(1+\beta)^2}{4\lambda t(1+2\beta)} \left\{ 1 + \frac{2\lambda(\lambda+1)t^2 - \lambda}{2\lambda t} + \left(\frac{1+3\beta}{(1+\beta)^2}\right) 2\lambda t \right\}.$$

Proof. From (12) and (13) we can easily see that

$$(1+\beta)\, a_2 = C_1^{(\lambda)}(t)\, p_1 \implies a_2 = \frac{C_1^{(\lambda)}(t)\, p_1}{1+\beta}$$

and

$$\begin{aligned} &(2+4\beta)\, a_3 - (1+3\beta)\, a_2^2 = C_1^{(\lambda)}(t)\, p_2 + C_2^{(\lambda)}(t) p_1^2 \implies \\ &a_3 = \frac{1}{(2+4\beta)} \left\{ C_1^{(\lambda)}(t) p_2 + C_2^{(\lambda)}(t) p_1^2 + (1+3\beta)\, a_2^2 \right\}. \end{aligned}$$

Thus, we can write

$$a_3 - \xi a_2^2 = \frac{1}{2+4\beta} \left\{ C_1^{(\lambda)}(t) p_2 + C_2^{(\lambda)}(t) p_1^2 + (1+3\beta) \left(\frac{C_1^{(\lambda)}(t) p_1}{1+\beta} \right)^2 \right\} - \xi \left(\frac{C_1^{(\lambda)}(t) p_1}{1+\beta} \right)^2 \implies$$

$$a_3 - \xi a_2^2 = \frac{C_1^{(\lambda)}(t)}{2+4\beta} \left\{ p_2 + \frac{C_2^{(\lambda)}(t)}{C_1^{(\lambda)}(t)} p_1^2 + (1+3\beta) \frac{C_1^{(\lambda)}(t) p_1^2}{(1+\beta)^2} - \xi \frac{C_1^{(\lambda)}(t)(2+4\beta) p_1^2}{(1+\beta)^2} \right\} \implies$$

$$a_3 - \xi a_2^2 = \frac{C_1^{(\lambda)}(t)}{2+4\beta} \left\{ p_2 + \left[\frac{C_2^{(\lambda)}(t)}{C_1^{(\lambda)}(t)} + (1+3\beta) \frac{C_1^{(\lambda)}(t)}{(1+\beta)^2} - \xi \frac{C_1^{(\lambda)}(t)(2+4\beta)}{(1+\beta)^2} \right] p_1^2 \right\}.$$

Then, in view of (10), we conclude that

$$\left|a_3-\xi a_2^2\right| = \frac{C_1^{(\lambda)}(t)}{(2+4\beta)}\left|p_2+\left\{\frac{C_2^{(\lambda)}(t)}{C_1^{(\lambda)}(t)}+(1+3\beta)\frac{C_1^{(\lambda)}(t)}{(1+\beta)^2}-\xi\frac{C_1^{(\lambda)}(t)(2+4\beta)}{(1+\beta)^2}\right\}p_1^2\right|$$

or

$$\left|a_3-\xi a_2^2\right| = \frac{C_1^{(\lambda)}(t)}{(2+4\beta)}\left|p_2-\left\{-\frac{C_2^{(\lambda)}(t)}{C_1^{(\lambda)}(t)}-(1+3\beta)\frac{C_1^{(\lambda)}(t)}{(1+\beta)^2}+\xi\frac{C_1^{(\lambda)}(t)(2+4\beta)}{(1+\beta)^2}\right\}p_1^2\right|. \quad (14)$$

Finally, by using (5) in (14), we get

$$\left|a_3-\xi a_2^2\right| \le \frac{\lambda t}{(1+2\beta)}\max\left\{1,\left|\frac{2\lambda(\lambda+1)t^2-\lambda}{2\lambda t}+\left(\frac{1+3\beta}{(1+\beta)^2}\right)2\lambda t-\xi\left(\frac{1+2\beta}{(1+\beta)^2}\right)4\lambda t\right|\right\}.$$

Because $t>0$, we have

$$\left|\frac{2\lambda(\lambda+1)t^2-\lambda}{2\lambda t}+\left(\frac{1+3\beta}{(1+\beta)^2}\right)2\lambda t-\xi\left(\frac{1+2\beta}{(1+\beta)^2}\right)4\lambda t\right|\le 1 \iff$$

$$-1\le\frac{2\lambda(\lambda+1)t^2-\lambda}{2\lambda t}+\left(\frac{1+3\beta}{(1+\beta)^2}\right)2\lambda t-\xi\left(\frac{1+2\beta}{(1+\beta)^2}\right)4\lambda t\le 1 \iff$$

$$\frac{(1+\beta)^2}{4\lambda t(1+2\beta)}\left\{-1+\frac{2\lambda(\lambda+1)t^2-\lambda}{2\lambda t}+\left(\frac{1+3\beta}{(1+\beta)^2}\right)2\lambda t\right\}$$
$$\le\xi\le\frac{(1+\beta)^2}{4\lambda t(1+2\beta)}\left\{1+\frac{2\lambda(\lambda+1)t^2-\lambda}{2\lambda t}+\left(\frac{1+3\beta}{(1+\beta)^2}\right)2\lambda t\right\} \iff$$

$$\xi_1\le\xi\le\xi_2.$$

The proof of Theorem 2 is completed. □

Taking $\lambda=\frac{1}{2}$ in Theorem 2, we obtain the following corollary.

Corollary 5. *Let the function $f(z)$ given by* (1) *be in the class $T_\beta H(z,C_n^{(\frac{1}{2})}(t))=T_\beta H(z,P_n(t))$. Then for some $\xi\in\mathbb{R}$ we have*

$$\left|a_3-\xi a_2^2\right|\le\begin{cases}\frac{t}{2(1+2\beta)}, & \text{for } \xi\in[\xi_1,\xi_2],\\ \frac{t}{2(1+2\beta)}\left|\frac{3t^2-1}{2t}+\left(\frac{1+3\beta}{(1+\beta)^2}\right)t-\xi\left(\frac{1+2\beta}{(1+\beta)^2}\right)2t\right|, & \text{for } \xi\notin[\xi_1,\xi_2],\end{cases}$$

where

$$\xi_1=\frac{(1+\beta)^2}{2t(1+2\beta)}\left\{-1+\frac{3t^2-1}{2t}+\left(\frac{1+3\beta}{(1+\beta)^2}\right)t\right\}$$

and

$$\xi_2=\frac{(1+\beta)^2}{2t(1+2\beta)}\left\{1+\frac{3t^2-1}{2t}+\left(\frac{1+3\beta}{(1+\beta)^2}\right)t\right\}.$$

Taking $\lambda=\frac{1}{2},\beta=1$ in Theorem 2, we obtain the following corollary.

Corollary 6. *Let the function* $f(z)$ *given by* (1) *be in the class* $T_1H(z, C_n^{(\frac{1}{2})}(t)) = T_1H(z, P_n(t))$. *Then for some* $\xi \in \mathbb{R}$ *we have*

$$\left|a_3 - \xi a_2^2\right| \leq \begin{cases} \frac{t}{6}, & \text{for } \xi \in [\xi_1, \xi_2], \\ \frac{t}{6}\left|\frac{3t^2-1}{2t} + t - \frac{3}{2}\xi t\right|, & \text{for } \xi \notin [\xi_1, \xi_2], \end{cases}$$

where

$$\xi_1 = \frac{2}{3t}\left\{\frac{5t^2 - 2t - 1}{2t}\right\}$$

and

$$\xi_2 = \frac{2}{3t}\left\{\frac{5t^2 + 2t - 1}{2t}\right\}.$$

Taking $\lambda = 1$ in Theorem 2, we obtain the result of Ş. Altınkaya and S. Yalçın (see [1]).

Corollary 7. *Let the function* $f(z)$ *given by* (1) *be in the class* $T_\beta H(z, C_n^{(1)}(t)) = L(\beta, t)$. *Then for some* $\xi \in \mathbb{R}$ *we have*

$$\left|a_3 - \xi a_2^2\right| \leq \begin{cases} \frac{t}{(1+2\beta)}, & \text{for } \xi \in [\xi_1, \xi_2], \\ \frac{t}{(1+2\beta)}\left|\frac{4t^2-1}{2t} + \left(\frac{1+3\beta}{(1+\beta)^2}\right)2t - \xi\left(\frac{1+2\beta}{(1+\beta)^2}\right)4t\right|, & \text{for } \xi \notin [\xi_1, \xi_2], \end{cases}$$

where

$$\xi_1 = \frac{(\beta^2 + 5\beta + 2)4t^2 - (1+\beta)^2(1+2t)}{8t^2(1+2\beta)}$$

and

$$\xi_2 = \frac{(\beta^2 + 5\beta + 2)4t^2 - (1+\beta)^2(1-2t)}{8t^2(1+2\beta)}.$$

For $\lambda = 1$ and $\beta = 1$ in Theorem 2, we obtain result of J. Dziok et al. (see [4]).

Corollary 8. *Let the function* $f(z)$ *given by* (1) *be in the class* $T_1H(z, C_n^{(1)}(t)) = H(t)$. *Then for some* $\xi \in \mathbb{R}$ *we have*

$$\left|a_3 - \xi a_2^2\right| \leq \begin{cases} \frac{t}{3}, & \text{for } \xi \in [\xi_1, \xi_2], \\ \frac{\left|8t^2 - 3\xi t^2 - 1\right|}{6}, & \text{for } \xi \notin [\xi_1, \xi_2], \end{cases}$$

where

$$\xi_1 = \frac{8t^2 - (1+2t)}{6t^2}$$

and

$$\xi_2 = \frac{8t^2 - (1-2t)}{6t^2}.$$

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Камалі М. *Нерівність Фекете-Сеґу для підкласу аналітичних функцій, асоційованих з поліномами Ґеґенбауера* // Карпатські матем. публ. — 2022. — Т.14, №2. — С. 582–591.

У цій статті ми визначаємо підклас аналітичних функцій $T_\beta H\left(z, C_n^{(\lambda)}(t)\right)$, що задовольняє умову підпорядкування

$$(1-\beta)\left(\frac{zf'(z)}{f(z)}\right)+\beta\left(1+\frac{zf''(z)}{f'(z)}\right)\prec\frac{1}{(1-2tz+z^2)^\lambda},$$

де $\beta\geq 0$, $\lambda\geq 0$ і $t\in\left(\frac{1}{2},1\right]$. Ми надаємо оцінки коефіцієнтів та наводимо нерівність Фекете-Сеґу для функцій, що належать до цього підкласу.

Ключові слова і фрази: аналітична та однолистна функція, типово дійсна функція, підпорядкування, поліном Ґеґенбауера, оцінка коефіцієнта, нерівність Фекете-Сеґу.