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Approximation properties of modified Jain-Gamma operators

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In the present paper, we study some approximation properties of a modified Jain-Gamma operator. Using Korovkin type theorem, we first give approximation properties of such operator. Secondly, we compute the rate of convergence of this operator by means of the modulus of continuity and we present approximation properties of weighted spaces. Finally, we obtain the Voronovskaya type theorem of this operator.

Key words and phrases: Jain operator, Gamma operator, weighted space, modulus of continuity, Peetre K-function, Voronovskaya theorem.

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Introduction

Up to now, various linear positive operators via special functions of mathematics, distributions in statistics etc. and their generalizations and modifications have been studied intensively by many authors and they are still being worked on (see [3, 4, 7–10, 14, 16, 20, 22–25, 27]).

It is well known that one of the distributions in statistics is the Poisson distribution. This distribution is defined as

$$w_\beta(k\,;\alpha)=\frac{\alpha}{k!}(\alpha+k\beta)^{k-1}e^{-(\alpha+k\beta)},\quad k\in\mathbb{N}_0,\quad 0<\alpha<\infty,\quad |\beta|<1,$$

on $\mathbb{R}_+=[0,\infty)$. In 1972, based on this distribution, G.C. Jain [14] defined a class of positive linear operators as follows

$$P_n^\beta(f,x)=\sum_{v=0}^{\infty}w_\beta(v,nx)f\left(\frac{v}{n}\right),\quad 0\le\beta<1,\quad n\in\mathbb{N}_0,\quad x\in\mathbb{R}_+, \tag{1}$$

and studied various convergence properties, where

$$w_\beta(v,nx)=nx(nx+v\beta)^{v-1}\frac{e^{-(nx+v\beta)}}{v!},\quad v\in\mathbb{N}_0, \tag{2}$$

and $w_\beta\,(v,nx)=1$. In 1973, P.C. Consul and G.C. Jain [7] examined the features of the more general version of this operator. In 2012, S. Tarabie [24] defined sequences of the Jain-Beta operator using Jain operator and examined various convergence and statistical convergence properties. In 2013, A. Farcaş [12] proved a Voronovkaya type theorem for Jain operators. Again, in 2013, V.N. Mishra and P. Patel [19] generalized the operator further and examined

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the convergence properties. In 2014, O. Agratini [2] studied uniform convergence properties of linear positive operator sequences of integral type including Jain operator with the help of continuity modulus. In 2015, A. Olgun, F. Taşdelen and A. Erençin [20] gave a different generalization of the Jain operator with the help of a ρ function, and studied the convergence properties of this operator and the Voronovskaya type theorem. In 2015, V.N. Mishra, P. Sharma and M.M. Birou [18] gave the convergence properties of a different form of Jain-Baskakov operators. In 2017, A. Kumar and L.N. Mishra [16] discussed the convergence properties of the Stancu type of Jain-Baskakov operators.

If $\beta = 0$ in (1), well-known Szász-Mirakyan operators are obtained as follows

$$S_n(f,x) = \sum_{v=0}^{\infty} e^{-nx}\frac{(nx)^v}{v!} f\Big(\frac{v}{n}\Big).$$

This operator is a well-known operator and has many generalizations. For this, we refer a reader to [5,7,11,15]. Many researchers have still working on generalization of Jain operators.

One of the well-known important operators is the Gamma operator. In [17], A. Lupaş and M. Müller defined Gamma operator as

$$G_n(f,x) = \int_0^{\infty} \frac{x^{n+1}}{n!} e^{-xy} y^n f\Big(\frac{n}{y}\Big) dy, \quad x \in (0,\infty), \quad n \in \mathbb{N}, \tag{3}$$

and examined various convergence properties. Later, some studies have been given on the various convergence properties of the Gamma operator [23,26].

In 2007, L. Rempulska and M. Skorupka [22] extended a modified version of Gamma operator as follows

$$G_{n,p}(f,x) = \int_0^{\infty} \frac{x^{n+1}}{n!} e^{-xy} y^n F_p\Big(x, \frac{n}{y}\Big) dy$$

and investigated the approximation properties for differentiable functions in the polynomial weighted spaces. Recently, new operators have been defined by using the Gamma operator and the convergence properties of these operators are examined [1,6,10,21].

In this paper, we define the Jain-Gamma operator by using the expressions (1)–(3) and examine the convergence properties of this operator.

1 Constructions of operators

Let $x \in [0,\infty)$, $\{\beta_n\}$ be a sequence, such that

$$\beta_n \in [0,1) \ \ \forall n \in \mathbb{N}, \qquad \lim_{n\to\infty} \beta_n = 0, \tag{4}$$

and f be defined on the space $C_B[0,\infty)$ of all continuous bounded functions. We define Jain-Gamma operators as follows

$$A_{n,\alpha}^{\beta_n}(f;x) = \sum_{k=1}^{\infty} \frac{nx(nx+k\beta_n)^{k-1}}{k!} e^{-(nx+k\beta_n)} \frac{x^{n+1}}{n!} \int_0^{\infty} e^{-xy} y^n f\Big(\frac{k}{xy}\Big) dy + e^{-nx} f(0). \tag{5}$$

The following recurrence relations are proved in [12]. Let

$$S(r,\alpha,\beta_n) = \sum_{k=0}^{\infty} \frac{(\alpha+k\beta_n)^{k+r-1}}{k!} e^{-(\alpha+k\beta_n)}, \quad r = 0,1,2,\ldots, \tag{6}$$

and $\alpha S(0,\alpha,\beta_n) = 1$ for $0 < \alpha < \infty$, $|\beta_n| < 1$, $x \in [0,\infty)$. Then, it follows

$$S(r,\alpha,\beta_n) = \sum_{k=0}^{\infty} \beta_n^k (\alpha + k\beta_n) S(r-1,\alpha + k\beta_n,\beta_n),$$

$$S(r,\alpha,\beta_n) = \alpha S(r-1,\alpha,\beta_n) + \beta_n S(r,\alpha+\beta_n,\beta_n),$$

$$S(1,\alpha,\beta_n) = \sum_{k=0}^{\infty} \beta_n^k = \frac{1}{1-\beta_n}, \tag{7}$$

$$S(2,\alpha,\beta_n) = \sum_{k=0}^{\infty} \frac{\beta_n^k(\alpha+k\beta_n)}{1-\beta_n} = \frac{\alpha}{(1-\beta_n)^2} + \frac{\beta_n^2}{(1-\beta_n)^3}, \tag{8}$$

$$S(3,\alpha,\beta_n) = \frac{\alpha^3}{(1-\beta_n)^3} + \frac{3\alpha\beta_n^2}{(1-\beta_n)^4} + \frac{\beta_n^3 + 2\beta_n^4}{(1-\beta_n)^5}, \tag{9}$$

$$S(4,\alpha,\beta_n) = \frac{\alpha^3}{(1-\beta_n)^4} + \frac{6\alpha^2\beta_n^k}{(1-\beta_n)^5} + \frac{\alpha\beta_n^3(11\beta_n+4)}{(1-\beta_n)^6} + \frac{+6\beta_n^6 + 8\beta_n^5 + \beta_n^4}{(1-\beta_n)^7}. \tag{10}$$

Now, by taking $\alpha = nx$ in (6), we can give some auxiliary lemmas for the operator (5).

2 Auxiliary results

Lemma 1. *For the operators* (5)*, we have:*

i)

$$A_{n,\alpha}^{\beta_n}(1;x) = 1;$$

ii)

$$A_{n,\alpha}^{\beta_n}(t;x) = x/(1-\beta_n);$$

iii)

$$A_{n,\alpha}^{\beta_n}(t^2;x) = \frac{nx^2}{(1-\beta_n)^2(n-1)} + \frac{x}{(1-\beta_n)^3(n-1)};$$

iv)

$$A_{n,\alpha}^{\beta_n}(t^3;x) = \frac{n^2x^3}{(n-1)(n-2)(1-\beta_n)^3} + \frac{3nx^2}{(n-1)(n-2)(1-\beta_n)^4} + \frac{x(2\beta_n+1)}{(n-1)(n-2)(1-\beta_n)^5};$$

v)

$$A_{n,\alpha}^{\beta_n}(t^4;x) = \frac{n^3x^4}{(n-1)(n-2)(n-3)(1-\beta_n)^4} + \frac{6n^2x^3}{(n-1)(n-2)(n-3)(1-\beta_n)^5} - \frac{nx^2(8\beta_n+7)}{(n-1)(n-2)(n-3)(1-\beta_n)^6} + \frac{x(6\beta_n^2+8\beta_n+1)}{(n-1)(n-2)(n-3)(1-\beta_n)^7}.$$

Proof. *i)* Using the operators (5), we obtain

$$A_{n,\alpha}^{\beta_n}(1;x) = \sum_{k=1}^{\infty} \frac{nx(nx+k\beta_n)^{k-1}}{k!} e^{-(nx+k\beta_n)} \frac{x^{n+1}}{n!} \int_0^{\infty} e^{-xy} y^n dy.$$

If $xy = t$, then it follows

$$A_{n,\alpha}^{\beta_n}(1;x) = \sum_{k=1}^{\infty} \frac{nx(nx+k\beta_n)^{k-1}}{k!} e^{-(nx+k\beta_n)} \frac{x^{n+1}}{n!} \int_0^{\infty} e^{-t} \frac{t^n}{x^n} \frac{dt}{x}$$
$$= \sum_{k=1}^{\infty} \frac{nx(nx+k\beta_n)^{k-1}}{k!} e^{-(nx+k\beta)} = 1,$$

which proves the first result.

ii) If (7) is used for $f(t) = t$, we have

$$A_{n,\alpha}^{\beta_n}(t;x) = \sum_{k=1}^{\infty} \frac{nx(nx+k\beta_n)^{k-1}}{k!} e^{-(nx+k\beta_n)} \frac{x^{n+1}}{n!} \int_0^{\infty} e^{-xy} y^n \frac{k}{xy} dy$$
$$= \sum_{k=1}^{\infty} \frac{nx(nx+k\beta_n)^{k-1}}{k!} e^{-(nx+k\beta_n)} \frac{x^{n+1}}{n!} \int_0^{\infty} e^{-t} \frac{t^n}{x^n} \frac{k}{t} \frac{dt}{x}$$
$$= \sum_{k=1}^{\infty} \frac{nx(nx+k\beta_n)^{k-1}}{k!} e^{-(nx+k\beta_n)} \frac{k}{n} = \sum_{k=0}^{\infty} \frac{x(nx+\beta_n+k\beta_n)^{k}}{k!} e^{-(nx+\beta_n+k\beta_n)}$$
$$= xS(1, nx+\beta_n, \beta_n) = \frac{x}{1-\beta_n}.$$

iii) If (8) is used for $f(t) = t^2$,we write

$$A_{n,\alpha}^{\beta_n}(t^2;x) = \sum_{k=1}^{\infty} \frac{nx(nx+k\beta_n)^{k-1}}{k!} e^{-(nx+k\beta_n)} \frac{x^{n+1}}{n!} \int_0^{\infty} e^{-xy} y^n \frac{k^2}{x^2y^2} dy$$
$$= \sum_{k=1}^{\infty} \frac{nx(nx+k\beta_n)^{k-1}}{k!} e^{-(nx+k\beta_n)} \frac{k^2}{n(n-1)}.$$

Taking into account $k^2 = k(k-1) + k$, we can write last equation in the following form

$$A_{n,\alpha}^{\beta_n}(t^2;x) = \frac{x}{n-1} \sum_{k=2}^{\infty} \frac{(nx+k\beta)^{k-1}}{(k-2)!} e^{-(nx+k\beta)} + \frac{x}{n-1} \sum_{k=1}^{\infty} \frac{(nx+k\beta)^{k-1}}{(k-1)!} e^{-(nx+k\beta)}$$
$$= \frac{x}{n-1} S(2, nx+2\beta, \beta) + \frac{x}{n-1} S(1, nx+\beta, \beta)$$
$$= \frac{nx^2}{(1-\beta)^2(n-1)} + \frac{x}{(1-\beta)^3(n-1)}.$$

iv) If (9) is applied for $f(t) = t^3$, we have

$$A_{n,\alpha}^{\beta_n}(t^3;x) = \sum_{k=1}^{\infty} \frac{nx(nx+k\beta_n)^{k-1}}{k!} e^{-(nx+k\beta_n)} \frac{x^{n+1}}{n!} \int_0^{\infty} e^{-xy} y^n \frac{k^3}{x^3y^3} dy$$
$$= \sum_{k=1}^{\infty} \frac{nx(nx+k\beta_n)^{k-1}}{k!} e^{-(nx+k\beta_n)} \frac{k^3}{n(n-1)(n-2)}.$$

Since $k^3 = k(k-1)(k-2) + 3k^2 - 2k$, the last equation is written as follows

$$A_{n,\alpha}^{\beta_n}(t^3;x) = \frac{x}{(n-1)(n-2)} \left[\sum_{k=3}^{\infty} \frac{(nx+k\beta_n)^{k-1}}{(k-3)!} e^{-(nx+k\beta_n)} \right.$$
$$\left. + 3 \sum_{k=2}^{\infty} \frac{(nx+k\beta_n)^{k-1}}{(k-2)!} e^{-(nx+k\beta_n)} + \sum_{k=1}^{\infty} \frac{(nx+k\beta_n)^{k-1}}{(k-1)!} e^{-(nx+k\beta_n)} \right]$$
$$= \frac{x}{(n-1)(n-2)} \left[S(3, nx+3\beta_n, \beta_n) + 3S(2, nx+2\beta_n, \beta_n) + S(1, nx+\beta_n, \beta_n) \right]$$
$$= \frac{n^2x^3}{(n-1)(n-2)(1-\beta_n)^3} + \frac{3nx^2}{(n-1)(n-2)(1-\beta_n)^4} + \frac{x(2\beta_n+1)}{(n-1)(n-2)(1-\beta_n)^5}.$$

v) If (10) is used for $f(t) = t^4$, we have

$$
\begin{aligned}
A_{n,\alpha}^{\beta_n}(t^4;x) &= \sum_{k=1}^{\infty} \frac{nx(nx+k\beta_n)^{k-1}}{k!} e^{-(nx+k\beta_n)} \frac{x^{n+1}}{n!} \int_0^{\infty} e^{-xy} y^n \frac{k^4}{x^4 y^4}\, dy \\
&= \sum_{k=1}^{\infty} \frac{nx(nx+k\beta_n)^{k-1}}{k!} e^{-(nx+k\beta_n)} \frac{k^4}{n(n-1)(n-2)(n-3)}.
\end{aligned}
$$

Taking into account $k^4 = k(k-1)(k-2)(k-3) + 6k^3 - 11k^2 + 6k$, we can write last equation as

$$
\begin{aligned}
A_{n,\alpha}^{\beta_n}(t^4;x) &= \frac{x}{(n-1)(n-2)(n-3)} \Bigg[\sum_{k=4}^{\infty} \frac{(nx+k\beta_n)^{k-1}}{(k-4)!} e^{-(nx+k\beta\beta_n)} \\
&\quad + 6 \sum_{k=3}^{\infty} \frac{(nx+k\beta_n)^{k-1}}{(k-3)!} e^{-(nx+k\beta_n)} + 7 \sum_{k=2}^{\infty} \frac{(nx+k\beta_n)^{k-1}}{(k-2)!} e^{-(nx+k\beta_n)} \\
&\quad + \sum_{k=1}^{\infty} \frac{(nx+k\beta_n)^{k-1}}{(k-1)!} e^{-(nx+k\beta_n)} \Bigg] \\
&= \frac{x}{(n-1)(n-2)(n-3)} [S(4, nx+4\beta_n, \beta_n) + 6S(3, nx+3\beta_n, \beta_n) \\
&\quad + 7S(2, nx+2\beta_n, \beta_n) + S(1, nx+\beta_n, \beta_n)] \\
&= \frac{n^3x^4}{(n-1)(n-2)(n-3)(1-\beta_n)^4} + \frac{6n^2x^3}{(n-1)(n-2)(n-3)(1-\beta_n)^5} \\
&\quad - \frac{nx^2(8\beta_n+7)}{(n-1)(n-2)(n-3)(1-\beta_n)^6} + \frac{x(6\beta_n^2+8\beta_n+1)}{(n-1)(n-2)(n-3)(1-\beta_n)^7},
\end{aligned}
$$

which completes the proof. □

Theorem 1. *Let $f \in C_B[0,\infty)$, $x \in [0,\infty)$ and $n \in \mathbb{N}$. Then we have*

$$\lim_{n\to\infty} (A_{n,\alpha}^{\beta_n}(f;x) - f(x)) = 0.$$

Proof. It is clear from Lemma 1. □

Lemma 2. *For the operators* (5)*, the inequality*

$$A_{n,\alpha}^{\beta}((t-x)^2;x) \le M^*(n,\beta_n) \frac{x^2+x}{n-1}$$

holds, where $M_i = (n,\beta_n)$, $i = 1,2$; $M^* = \max(M_i)$.

Proof. Note $x^s/(1+x)^l \le x^s$ for all $x \ge 0$, $l \le s$, $l,s = 1,2$. From Lemma 1 and from linearity of the operators (5) it follows

$$
\begin{aligned}
A_{n,\alpha}^{\beta_n}((t-x)^2;x) &= \Big(\frac{n}{(n-1)(1-\beta_n)^2} - \frac{2}{1-\beta_n} + 1 \Big) x^2 + \frac{1}{(n-1)(1-\beta_n)^3} x \\
&= \frac{1}{n-1} \frac{\beta_n^2(n-1)+1}{(\beta_n-1)^2} x^2 + \frac{1}{n-1} \frac{1}{(1-\beta_n)^3} x \\
&= \frac{x^2}{n-1} M_1(n,\beta_n) + \frac{x}{n-1} M_2(n,\beta_n) \le M^*(n,\beta_n) \frac{x^2+x}{n-1}.
\end{aligned}
$$

□

3 Rates of convergence

Now, let us give the convergence properties of the operators defined by (5) with the help of the modulus of continuity and Peetre's K-functional. We also present the convergence of the operator for $f \in \mathrm{Lip}_M(\gamma)$.

Theorem 2. *Let $x \in [0,\infty)$, $n \in \mathbb{N}$ and $f \in C_B[0,\infty)$. Then we have*

$$|A_{n,\alpha}^{\beta_n}(f;x) - f(x)| \leq M^{**} w\left(f; \sqrt{\frac{x^2+x}{n-1}}\right),$$

where $M^{**} = 1 + \sqrt{M^*}$.

Proof. Using the definition of the continuity modulus, well-known properties and the linearity of the operators (5), we may write

$$\begin{aligned}
|A_{n,\alpha}^{\beta_n}(f;x) - f(x)| &= \left| \sum_{k=1}^{\infty} \frac{nx(nx+k\beta_n)^{k-1}}{k!} e^{-(nx+k\beta_n)} \frac{x^{n+1}}{n!} \int_0^\infty e^{-xy} y^n f\left(\frac{k}{xy}\right) - f(x)\, dy \right| \\
&\leq \sum_{k=1}^{\infty} \frac{nx(nx+k\beta_n)^{k-1}}{k!} e^{-(nx+k\beta_n)} \frac{x^{n+1}}{n!} \int_0^\infty e^{-xy} y^n \left| f\left(\frac{k}{xy}\right) - f(x) \right| dy \\
&\leq \sum_{k=1}^{\infty} \frac{nx(nx+k\beta_n)^{k-1}}{k!} e^{-(nx+k\beta_n)} \frac{x^{n+1}}{n!} \\
&\quad \times \int_0^\infty e^{-xy} y^n \left(1 + \frac{|k/(xy) - x|}{\delta}\right) w(f;\delta)\, dy \\
&= w(f;\delta) + \sum_{k=1}^{\infty} \frac{nx(nx+k\beta_n)^{k-1}}{k!} e^{-(nx+k\beta_n)} \frac{x^{n+1}}{n!} \frac{1}{\delta} w(f;\delta) \\
&\quad \times \int_0^\infty e^{-xy} y^n \left| \frac{k}{xy} - x \right| dy.
\end{aligned}$$

By applying the Cauchy–Schwarz inequality to the second expression on the right side of this inequality first for the integral and then for the sum, we get

$$\begin{aligned}
|A_{n,\alpha}^{\beta_n}(f;x) - f(x)| &\leq w(f;\delta) + \frac{1}{\delta} w(f;\delta) \left\{ \sum_{k=1}^{\infty} \frac{nx(nx+k\beta_n)^{k-1}}{k!} e^{-(nx+k\beta_n)} \right. \\
&\quad \left. \times \left(\int_0^\infty \frac{x^{n+1}}{n!} e^{-xy} y^n \left(\frac{k}{xy} - x\right)^2 dy \right)^{1/2} \left(\int_0^\infty \frac{x^{n+1}}{n!} e^{-xy} y^n 1^2 dy \right)^{1/2} \right\} \\
&= w(f;\delta) + \frac{1}{\delta} w(f;\delta) \sqrt{A_{n,\alpha}^{\beta_n}((t-x)^2;x)} \\
&\leq w(f;\delta) + \frac{1}{\delta} w(f;\delta) \sqrt{M^*} \sqrt{\frac{x^2+x}{n-1}}.
\end{aligned}$$

If we take $\delta = \sqrt{(x^2+x)/(n-1)}$, then it follows

$$|A_{n,\alpha}^{\beta_n}(f;x) - f(x)| \leq w\left(f; \sqrt{\frac{x^2+x}{n-1}}\right)(1+\sqrt{M^*}) \leq M^{**} w\left(f; \sqrt{\frac{x^2+x}{n-1}}\right),$$

which ends the proof. □

Let $C_B[0,\infty)$ denote the space of real valued continuous and bounded functions on the interval $[0,\infty)$, with the norm $\|f\| = \sup_{0\le x<\infty} |f(x)|$. For every $\delta > 0$, Peetre's K-functional is defined by

$$K_2(f;\delta) = \inf_{g\in C_B^2(0,\infty)} \{\|f-g\| + \delta\|g''\|\},$$

where $C_B^2(0,\infty) = \{g \in C_B(0,\infty) :\ g', g'' \in C_B(0,\infty)\}$. There exists an absolute constant $C > 0$ such that

$$K_2(f;\delta) \le Cw_2(f;\sqrt{\delta}) \tag{11}$$

holds, where w_2 is the second order modulus of smoothness of f, defined by

$$w(f;\delta) = \sup_{0<h\le\delta} \sup_{0<x<\infty} |f(x+2h) - 2f(x+h) + f(x)|.$$

Now, we consider $\hat{A}_{n,\alpha}^{\beta_n}(f;x)$ by means of the operator $A_{n,\alpha}^{\beta_n}$, namely

$$\hat{A}_{n,\alpha}^{\beta_n}(f;x) = A_{n,\alpha}^{\beta_n}(f;x) - f\Big(\frac{x}{1-\beta_n}\Big) + f(x). \tag{12}$$

Lemma 3. *Let $g \in C_B^2(0,\infty)$. Then we have*

$$|\hat{A}_{n,\alpha}^{\beta_n}(g;x) - g(x)| \le \delta_n(x)\|g''\|, \quad \textit{where} \quad \delta_n(x) = A_{n,\alpha}^{\beta_n}((t-x)^2;x) + \Big(\frac{\beta_n x}{1-\beta_n}\Big)^2.$$

Proof. For the operators $\hat{A}_{n,\alpha}^{\beta_n}(f;x)$, we get

$$\begin{aligned}\hat{A}_{n,\alpha}^{\beta_n}(t-x;x) &= A_{n,\alpha}^{\beta_n}(t-x;x) - \frac{\beta_n x}{1-\beta_n} + (x-x) = A_{n,\alpha}^{\beta_n}(t;x) - xA_{n,\alpha}^{\beta_n}(1;x) - \frac{\beta_n x}{1-\beta_n} + (x-x)\\ &= A_{n,\alpha}^{\beta_n}(t;x) - xA_{n,\alpha}^{\beta_n}(1;x) - A_{n,\alpha}^{\beta_n}(t;x) + xA_{n,\alpha}^{\beta_n}(1;x) = 0.\end{aligned}$$

Let $g \in C_B^2(0,\infty)$ and $x \in (0,\infty)$. By Taylor's formula of g, we may write

$$g(t) - g(x) = (t-x)g'(x) + \int_x^t (t-u)g''(u)\,du, \quad t \in [0,\infty).$$

If we apply the operator $\hat{A}_{n,\alpha}^{\beta_n}$ to this equality, we obtain

$$\begin{aligned}\hat{A}_{n,\alpha}^{\beta_n}(g(t)-g(x);x) &= \hat{A}_{n,\alpha}^{\beta_n}((t-x)g'(x);x) + \hat{A}_{n,\alpha}^{\beta_n}\Big(\int_x^t (t-u)g''(u)\,du;x\Big)\\ &= \hat{A}_{n,\alpha}^{\beta_n}\Big(\int_x^t (t-u)g''(u)\,du;x\Big) = A_{n,\alpha}^{\beta_n}\Big(\int_x^t (t-u)g''(u)\,du;x\Big)\\ &\quad - \Big(\int_x^{x/(1-\beta_n)} \Big(\frac{x}{1-\beta_n} - u\Big)g''(u)\,du;x\Big) + \int_x^x (x-u)\,du.\end{aligned}$$

By using the following inequality

$$\Big|\int_x^t (t-u)g''(u)\,du\Big| \le \int_x^t |t-u|\|g''(u)\|\,du \le \frac{(t-x)^2}{2}\|g''(u)\| \le (t-x)^2\|g''(u)\|,$$

we can write

$$\int_x^{x/(1-\beta_n)} \Big(\frac{x}{1-\beta_n} - u\Big)g''(u)\,du \le \Big(\frac{\beta_n x}{1-\beta_n} + \frac{ax}{(n+\beta_n)(1+x)}\Big)^2\|g''(u)\|.$$

In view of this inequality, we can conclude that

$$|\hat{A}_{n,\alpha}^{\beta_n}(g;x) - g(x)| \le \Big\{A_{n,\alpha}^{\beta_n}((t-x)^2;x) + \Big(\frac{\beta_n x}{1-\beta_n}\Big)^2\Big\}\|g''\| = \delta_n(x)\|g''\|.$$

$\square$

Theorem 3. *Let $f \in C_B[0,\infty)$. For all $x \in [0,\infty)$, there exists a constant $B > 0$ such that*

$$|A_{n,\alpha}^{\beta_n}(f;x) - f(x)| \leq Bw_2(f;\sqrt{\delta_n(x)}) + w\Big(f;\frac{\beta_n x}{1-\beta_n}\Big)$$

where

$$\delta_n(x) = A_{n,\alpha}^{\beta_n}((t-x)^2;x) + \Big(\frac{\beta_n x}{1-\beta_n}\Big)^2.$$

Proof. For the operators $\hat{A}_{n,\alpha}^{\beta_n}$, we write

$$\hat{A}_{n,\alpha}^{\beta_n}(f;x) - f(x) = \hat{A}_{n,\alpha}^{\beta_n}(f-g;x) + (f-g)(x) + \hat{A}_{n,\alpha}^{\beta_n}(g-g(x);x). \tag{13}$$

From the equality (12), it follows

$$A_{n,\alpha}^{\beta_n}(f;x) - f\Big(\frac{x}{1-\beta_n}\Big) + f(x) - f(x) = \hat{A}_{n,\alpha}^{\beta_n}(f-g;x) + (f-g)(x) + \hat{A}_{n,\alpha}^{\beta_n}(g;x) - g(x) \tag{14}$$

and

$$|A_{n,\alpha}^{\beta_n}(f;x) - f(x)| \leq |\hat{A}_{n,\alpha}^{\beta_n}(f-g;x)| + |(f-g)(x)| + |\hat{A}_{n,\alpha}^{\beta_n}(g;x) - g(x)| + \Big|f\Big(\frac{x}{1-\beta_n}\Big) - f(x)\Big|.$$

By taking the supremum of $\hat{A}_{n,\alpha}^{\beta_n}$ operators, we get

$$\begin{aligned}|\hat{A}_{n,\alpha}^{\beta_n}(f;x)| = \Big|A_{n,\alpha}^{\beta_n}(f;x) - f\Big(\frac{x}{1-\beta_n}\Big) + f(x)\Big| &\leq |A_{n,\alpha}^{\beta_n}(f;x)| + 2\|f\| \\ &\leq \|f\| A_{n,\alpha}^{\beta_n}(1;x) + 2\|f\| = 3\|f\|.\end{aligned}$$

Now if equality (13) is replaced by inequality (14), we have

$$|A_{n,\alpha}^{\beta_n}(f;x) - f(x)| \leq 3\|f-g\| + \|f-g\| + |\hat{A}_{n,\alpha}^{\beta_n}(g;x) - g(x)| + \Big|f\Big(\frac{x}{1-\beta_n}\Big) - f(x)\Big|.$$

From Lemma 3, we obtain

$$\begin{aligned}|A_{n,\alpha}^{\beta_n}(f;x) - f(x)| &\leq 4\|f-g\| + \delta_n(x)\|g''\| + w\Big(f;\frac{\beta_n x}{1-\beta_n}\Big) \\ &\leq 4\{\|f-g\| + \delta_n(x)\|g''\|\} + w\Big(f;\frac{\beta_n x}{1-\beta_n}\Big).\end{aligned}$$

By taking the infimum for all $g \in C_B^2(0,\infty)$ on the right-hand side of the last inequality and considering (11), we get

$$\begin{aligned}|A_{n,\alpha}^{\beta_n}(f;x) - f(x)| &\leq 4L_2(f;\delta_n(x)) + w\Big(f;\frac{\beta_n x}{1-\beta_n}\Big) \\ &\leq 4Cw_2(f;\sqrt{\delta_n(x)}) + w\Big(f;\frac{\beta_n x}{1-\beta_n}\Big) \\ &= Bw_2(f;\sqrt{\delta_n(x)}) + w\Big(f;\frac{\beta_n x}{1-\beta_n}\Big),\end{aligned}$$

which completes the proof. □

Theorem 4. *Let $f \in C_B[0,\infty)$. For all $x \in [0,\infty)$, there exists a constant $L > 0$ such that*

$$|A_{n,\alpha}^{\beta}(f;x) - f(x)| \leq Lw_2\big(f;\sqrt{\delta_n(x)}\big) + w\Big(f;\frac{\beta_n x}{1-\beta_n}\Big).$$

Proof. For $x \in [0,\infty)$, we consider the operators $\hat{A}_{n,\alpha}^{\beta_n}$, defined by

$$\hat{A}_{n,\alpha}^{\beta_n}(f;x) = A_{n,\alpha}^{\beta}(f;x) - f\Big(\frac{x}{1-\beta_n}\Big) + f(x). \tag{15}$$

By Taylor's expansion, we have $g(t) = g(x) + (t-x)g'(x) + \int_x^t (t-u)g''(u)\,du$. Applying $\hat{A}_{n,\alpha}^{\beta_n}$ on both sides of the above equation, we have

$$\hat{A}_{n,\alpha}^{\beta_n}(g;x) - g(x) = g'(x)\hat{A}_{n,\alpha}^{\beta_n}((t-x);x) + \hat{A}_{n,\alpha}^{\beta_n}\Big(\int_x^t (t-u)g''(u)\,du;x\Big).$$

Thus, by (15) we get

$$\begin{aligned}|\hat{A}_{n,\alpha}^{\beta_n}(g;x) - g(x)| &\leq \hat{A}_{n,\alpha}^{\beta_n}\Big(\Big|\int_x^t (t-u)g''(u)\,du\Big|;x\Big) \leq A_{n,\alpha}^{\beta_n}((t-x)^2;x)\|g''\| \\ &+ \Big(\int_x^{x/(1-\beta_n)} \Big(\frac{x}{1-\beta_n} - u\Big)g''(u)\,du;x\Big) = \delta_n(x)\|g''\|.\end{aligned} \tag{16}$$

Since

$$\begin{aligned}|A_{n,\alpha}^{\beta_n}(f;x)| \leq \sum_{k=1}^{\infty} &\frac{nx(nx+k\beta_n)^{k-1}}{k!}\, e^{-(nx+k\beta_n)}\frac{x^{n+1}}{n!} \\ &\int_0^{\infty} e^{-xy}y^n\Big|f\Big(\frac{k}{xy}\Big)\Big|dy + e^{-nx}|f(0)| \leq \|f\|,\end{aligned} \tag{17}$$

using (16) and (17) in (15), we obtain

$$\begin{aligned}|A_{n,\alpha}^{\beta_n}(f;x) - f(x)| &\leq |\hat{A}_{n,\alpha}^{\beta_n}(f-g;x) - (f-g)(x)| + |\hat{A}_{n,\alpha}^{\beta_n}(g;x) - g(x)| + \Big|\frac{x}{1-\beta_n} - f(x)\Big| \\ &\leq 2\|f-g\| + \delta_n(x)\|g''\| + w\Big(f;\frac{\beta_n x}{1-\beta_n}\Big).\end{aligned}$$

Taking infimum over all $g \in C_B^2(0,\infty)$, we get

$$|A_{n,\alpha}^{\beta_n}(f;x) - f(x)| \leq Lw_2(f;\delta_n(x)) + w\Big(f;\frac{\beta_n x}{1-\beta_n}\Big).$$

In view of (11), we get

$$|A_{n,\alpha}^{\beta_n}(f;x) - f(x)| \leq Lw_2\big(f;\sqrt{\delta_n(x)}\big) + w\Big(f;\frac{\beta_n x}{1-\beta_n}\Big),$$

which proves the theorem. □

Theorem 5. *Let $0 < \gamma \leq 1$ and $f \in C_B[0,\infty)$. Then if $f \in \mathrm{Lip}_M(\gamma)$, that is, the inequality*

$$|f(t) - f(x)| \leq M|t-x|^{\gamma}, \quad x,t \in (0,\infty),$$

holds, then for each $x \in [0,\infty)$ we have

$$|A_{n,\alpha}^{\beta_n}(f;x) - f(x)| \leq M\delta_n^{\frac{\gamma}{2}}(x),$$

where $\delta_n = A_{n,\alpha}^{\beta_n}((t-x)^2;x)$ and $M > 0$ is a constant.

Proof. Let $f \in C_B[0,\infty) \bigcap \mathrm{Lip}_M(\gamma)$. By the linearity and monotonicity of the operators $A^{\beta_n}_{n,\alpha}$, we get

$$\begin{aligned}|A^{\beta_n}_{n,\alpha}(f;x) - f(x)| &\leq A^{\beta_n}_{n,\alpha}(|f(t)-f(x)|;x) \leq MA^{\beta_n}_{n,\alpha}(|t-x|^{\gamma};x)\\ &= M\sum_{k=1}^{\infty}\frac{nx(nx+k\beta_n)^{k-1}}{k!}e^{-(nx+k\beta_n)}\frac{x^{n+1}}{n!}\int_0^{\infty}e^{-xy}y^n\left|\frac{k}{xy}-x\right|^{\gamma}dy.\end{aligned}$$

By applying the Hölder inequality twice succesively to the right side with $p = 2/\gamma$, $q = 2/(2-\gamma)$, we obtain

$$\begin{aligned}|A^{\beta_n}_{n,\alpha}(f;x) - f(x)| &\leq M\sum_{k=1}^{\infty}\frac{nx(nx+k\beta_n)^{k-1}}{k!}e^{-(nx+k\beta_n)}\frac{x^{n+1}}{n!}\int_0^{\infty}e^{-xy}y^n\left|\frac{k}{xy}-x\right|^{\gamma}1dy\\ &\leq M\left(\sum_{k=1}^{\infty}\frac{nx(nx+k\beta_n)^{k-1}}{k!}e^{-(nx+k\beta_n)}\frac{x^{n+1}}{n!}\int_0^{\infty}e^{-xy}y^n\left|\frac{k}{xy}-x\right|^{2}dy\right)^{\gamma/2}\\ &\quad\times\left(\sum_{k=1}^{\infty}\frac{nx(nx+k\beta_n)^{k-1}}{k!}e^{-(nx+k\beta_n)}\frac{x^{n+1}}{n!}\int_0^{\infty}e^{-xy}y^n dy\right)^{(2-\gamma)/2}\\ &\leq M\left(\sum_{k=1}^{\infty}\frac{nx(nx+k\beta_n)^{k-1}}{k!}e^{-(nx+k\beta_n)}\frac{x^{n+1}}{n!}\int_0^{\infty}e^{-xy}y^n\left|\frac{k}{xy}-x\right|^{2}dy\right)^{\gamma/2}\end{aligned}$$

and

$$\begin{aligned}|A^{\beta_n}_{n,\alpha}(f;x) - f(x)| &\leq M\sum_{k=1}^{\infty}\frac{nx(nx+k\beta_n)^{k-1}}{k!}e^{-(nx+k\beta_n)}\left(\int_0^{\infty}\frac{x^{n+1}}{n!}e^{-xy}y^n\left(\frac{k}{xy}-x\right)^{4/\gamma}dy\right)^{\gamma/2}\\ &\quad\times\left(\int_0^{\infty}\frac{x^{n+1}}{n!}e^{-xy}y^n\,dy\right)^{(2-\gamma)/2}\\ &\leq M\left(\sum_{k=1}^{\infty}\frac{nx(nx+k\beta_n)^{k-1}}{k!}e^{-(nx+k\beta_n)}\frac{x^{n+1}}{n!}\int_0^{\infty}e^{-xy}y^n\left(\frac{k}{xy}-x\right)^{2}dy\right)^{\gamma/2}\\ &= MA^{\beta_n}_{n,\alpha}((t-x)^2;x)^{\gamma/2} = M\delta_n^{\gamma/2}(x),\end{aligned}$$

which is the desired result. □

4 Weighted approximation properties

Now, we give some definitions and theorems that we will use in this section.

Let $\rho(x) = 1 + x^2$ and $B_\rho[0,\infty)$ denote the space of all functions having the property

$$|f(x)| \leq M_f\rho(x),$$

where $x \in [0,\infty)$ and M_f is a positive constant on f functions. The norm on $B_\rho[0,\infty)$ is defined as follows

$$\|f\|_\rho = \sup_{0\leq x<\infty}\frac{|f(x)|}{1+x^2}.$$

$C_\rho[0,\infty)$ denotes the space of all continuous functions belonging to $B_\rho[0,\infty)$ and $C^0_\rho[0,\infty)$ denotes the subspace of all functions $f \in C_\rho[0,\infty)$ for which

$$\lim_{x\to\infty}\frac{|f(x)|}{\rho(x)} = 0.$$

The basic theorem for approximation of weighted spaces is given by A.D. Gadjiev in [13].

Theorem 6. *Let $\{A_n\}$ be a sequence of positive linear operators defined from $C_\rho^0[0,\infty)$ to $B_\rho[0,\infty)$, and satisfying the conditions $\lim_{n\to\infty}\|A_n(t^v;x)-x^v\|_\rho = 0$, $v=0,1,2$. Then for any $f\in C_\rho^0[0,\infty)$,*

$$\lim_{n\to\infty}\|A_n(f;x)-f(x)\|_\rho = 0.$$

It is shown in [13] that a sequence of linear positive operators A_n is defined from $C_\rho^0[0,\infty)$ to $B_\rho[0,\infty)$ if and only if $\|A_n(\rho;x)\|_\rho \le M_\rho$, where M_ρ is a positive constant.

Theorem 7. *Let $\{A_{n,\alpha}^{\beta_n}\}$ be a sequence of positive linear operators. For each $f\in C_\rho^0[0,\infty)$, we have*

$$\lim_{n\to\infty}\|A_{n,\alpha}^{\beta_n}(f;x)-f(x)\|_\rho = 0.$$

Proof. Using Lemma 1, we get

$$\begin{aligned}
\sup_{0\le x<\infty}\frac{|A_{n,\alpha}^{\beta_n}(\rho;x)|}{1+x^2} &= \sup_{0\le x<\infty}\frac{|A_{n,\alpha}^{\beta_n}(1+t^2;x)|}{1+x^2} = \sup_{0\le x<\infty}\frac{|A_{n,\alpha}^{\beta_n}(1;x)+A_{n,\alpha}^{\beta_n}(t^2;x)|}{1+x^2}\\
&= \sup_{0\le x<\infty}\frac{\left|1+\dfrac{nx^2}{(n-1)(1-\beta_n)^2}+\dfrac{x}{(n-1)(1-\beta_n)^3}\right|}{1+x^2}\\
&\le 1+\frac{n}{(n-1)(1-\beta_n)^2}+\frac{1}{(n-1)(1-\beta_n)^3}.
\end{aligned}$$

There exists a positive constant D such that for each n and $0\le\beta_n<1$

$$\frac{n}{(n-1)(1-\beta_n)^2}+\frac{1}{(n-1)(1-\beta_n)^3} < D.$$

Hence we may write $\sup_{0\le x<\infty}\frac{|A_{n,\alpha}^{\beta_n}(\rho;x)|}{1+x^2} = \|A_{n,\alpha}^{\beta_n}(\rho;x)\|_\rho \le 1+D$, which shows that $\{A_{n,\alpha}^{\beta_n}\}$ is a sequence of positive linear operators defined from $C_\rho^0[0,\infty)$ to $B_\rho[0,\infty)$. Considering the results (7) and (8) obtained above, for $v=0,1,2$ it is clear that $\lim_{n\to\infty}\|A_{n,\alpha}^{\beta_n}(t^v;x)-x^v\|_\rho = 0$, $v=0,1,2$. Thus, the proof is completed. □

Theorem 8. *Let $x\in[0,\infty)$, $n\in\mathbb{N}$ and $f\in C_B[0,\infty)$. For the operators*

$$A_{n,\alpha}^{\beta_n}(f;x) = \sum_{k=1}^{\infty}\frac{nx(nx+k\beta_n)^{k-1}}{k!}e^{-(nx+k\beta_n)}\frac{x^{n+1}}{n!}\int_0^\infty e^{-xy}y^n f\left(\frac{k}{xy}\right)dy + e^{-nx}f(0)$$

and

$$B_{n,\alpha}^{\beta_n}(f;x) = \sum_{k=1}^{\infty}\frac{nx(nx+k\beta_n)^{k-1}}{k!}e^{-(nx+k\beta_n)}f\left(\frac{k}{n}\right),$$

the inequality

$$|A_{n,\alpha}^{\beta_n}(f;x)-B_{n,\alpha}^{\beta_n}(f;x)| \le w(f;\delta)\varphi(x)$$

holds true, where

$$\varphi(x) = 1+\frac{1}{\delta}\sqrt{\frac{x^2}{(n-1)(1-\beta_n)^2}+\frac{x}{n(n-1)(1-\beta_n)^3}}$$

and

$$\delta = \sqrt{\frac{x^2}{(n-1)(1-\beta_n)^2}+\frac{x}{n(n-1)(1-\beta_n)^3}}.$$

Proof. From the definition and properties of modulus of continuity, we have

$$A_{n,\alpha}^{\beta_n}(f;x) - B_{n,\alpha}^{\beta_n}(f;x) = \sum_{k=1}^{\infty} \frac{nx(nx+k\beta_n)^{k-1}}{k!} e^{-(nx+k\beta_n)} \frac{x^{n+1}}{n!} \int_0^\infty e^{-xy} y^n \left(f\left(\frac{k}{xy}\right) - f\left(\frac{k}{n}\right)\right) dy,$$

from which it follows

$$\begin{aligned}
|A_{n,\alpha}^{\beta_n}(f;x) - B_{n,\alpha}^{\beta_n}(f;x)| &\leq \sum_{k=1}^{\infty} \frac{nx(nx+k\beta_n)^{k-1}}{k!} e^{-(nx+k\beta_n)} \frac{x^{n+1}}{n!} \int_0^\infty e^{-xy} y^n \left| f\left(\frac{k}{xy}\right) - f\left(\frac{k}{n}\right)\right| dy \\
&\leq \sum_{k=1}^{\infty} \frac{nx(nx+k\beta_n)^{k-1}}{k!} e^{-(nx+k\beta_n)} \frac{x^{n+1}}{n!} \int_0^\infty e^{-xy} y^n w\left(f; \frac{\left|\frac{k}{xy} - \frac{k}{n}\right|}{\delta}\delta\right) dy \\
&\leq w(f,\delta) + \frac{1}{\delta} w(f,\delta) \sum_{k=1}^{\infty} \frac{nx(nx+k\beta_n)^{k-1}}{k!} e^{-(nx+k\beta_n)} \frac{x^{n+1}}{n!} \\
&\quad \times \int_0^\infty y^n e^{-xy} \left| \frac{k}{xy} - \frac{k}{n} \right| dy.
\end{aligned}$$

By applying the Cauchy-Schwarz inequality to the second expression on the right side of this inequality first for the integral and then for the sum, we get

$$|A_{n,\alpha}^{\beta_n}(f;x) - B_{n,\alpha}^{\beta_n}(f;x)| \leq w(f,\delta) + \frac{1}{\delta} w(f,\delta) \sqrt{A_{n,\alpha}^{\beta_n}\left(\left(\frac{k}{xy} - \frac{k}{n}\right)^2; x\right)}.$$

If we calculate $A_{n,\alpha}^{\beta_n}((k/(xy) - k/n)^2; x)$, we show that

$$A_{n,\alpha}^{\beta_n}\left(\left(\frac{k}{xy} - \frac{k}{n}\right)^2; x\right) = k^2 \sum_{k=1}^{\infty} \frac{nx(nx+k\beta_n)^{k-1}}{k!} e^{-(nx+k\beta_n)} \frac{x^{n+1}}{n!} \int_0^\infty e^{-xy} y^n \left(\frac{1}{x^2y^2} - \frac{2}{xyn} + \frac{1}{n^2}\right) dy.$$

If we say $xy = t$, then it follows

$$\begin{aligned}
A_{n,\alpha}^{\beta_n}\left(\left(\frac{k}{xy} - \frac{k}{n}\right)^2; x\right) &= \sum_{k=1}^{\infty} \frac{nx(nx+k\beta_n)^{k-1}}{k!} e^{-(nx+k\beta_n)} \frac{k^2}{n(n-1)} \\
&\quad - \frac{2}{n} \sum_{k=1}^{\infty} \frac{nx(nx+k\beta_n)^{k-1}}{k!} e^{-(nx+k\beta_n)} \frac{k^2}{n} \\
&\quad + \frac{1}{n^2} \sum_{k=1}^{\infty} \frac{nx(nx+k\beta_n)^{k-1}}{k!} e^{-(nx+k\beta_n)} k^2 \\
&= \left[1 - \frac{2(n-1)}{n} + \frac{n-1}{n}\right] \left[\frac{nx^2}{(n-1)(1-\beta_n)^2} + \frac{x}{(n-1)(1-\beta_n)^3}\right] \\
&= \frac{x^2}{(n-1)(1-\beta_n)^2} + \frac{x}{n(n-1)(1-\beta_n)^3},
\end{aligned}$$

from which, it follows

$$\lim_{n\to\infty} A_{n,\alpha}^{\beta_n}\left(\left(\frac{k}{xy} - \frac{k}{n}\right)^2; x\right) = 0.$$

Thus, we have

$$\begin{aligned}
|A_{n,\alpha}^{\beta_n}(f;x) - B_{n,\alpha}^{\beta_n}(f;x)| &\leq w(f,\delta) + \frac{1}{\delta} w(f,\delta) \sqrt{\frac{x^2}{(n-1)(1-\beta_n)^2} + \frac{x}{n(n-1)(1-\beta_n)^3}} \\
&\leq w(f,\delta)\varphi(x).
\end{aligned}$$

□

5 Voronovskaya type theorem

Lemma 4. *Since operator $A_{n,\alpha}^{\beta_n}(f;x)$ defined in (5) is linear and positive, we get:*

i)

$$A_{n,\alpha}^{\beta_n}(t-x;x) = \beta_n x/(1-\beta_n);$$

ii)

$$A_{n,\alpha}^{\beta_n}((t-x)^2;x) = \frac{\beta_n^2(n-1)+1}{(\beta_n-1)^2(n-1)}x^2 + \frac{1}{(1-\beta_n)^3(n-1)}x;$$

iii)

$$\begin{aligned}A_{n,\alpha}^{\beta_n}((t-x)^3;x) &= \frac{\beta_n^3(n^2-3n+2)+3\beta_n(n-2)+4}{(\beta_n-1)^3(n-1)(n-2)}x^3\\ &\quad + \frac{3\beta_n(2-n)-2}{(1-\beta_n)^4(n-1)(n-2)}x^2 + \frac{2\beta_n+1}{(1-\beta_n)^5(n-1)(n-2)}x;\end{aligned}$$

iv)

$$\begin{aligned}&A_{n,\alpha}^{\beta_n}((t-x)^4;x)\\ &= \frac{\beta_n^4(6n^2-n^3-11n+6)+6\beta_n^2(-n^2+5n-6)+16\beta_n(3-n)-3(n+6)}{(n-1)(n-2)(n-3)(1-\beta\beta_n)^4}x^4\\ &\quad + \frac{6(4\beta_n(n-3)+\beta_n^2(n^2-5n+6)+n+6)}{(n-1)(n-2)(n-3)(1-\beta_n)^5}x^3\\ &\quad + \frac{12\beta_n(n-1)+8\beta_n^2(3-n)+11n-12}{(n-1)(n-2)(n-3)(1-\beta_n)^6}x^2 + \frac{6\beta_n^2+8\beta_n+1}{(n-1)(n-2)(n-3)(1-\beta_n)^7}x.\end{aligned}$$

The proof can be easily done by using the linearity of operator $A_{n,\alpha}^{\beta_n}$ and Lemma 1.

Theorem 9. *Let $x \geq 0$, $0 \leq \alpha \leq \beta_n$ and $n \in N$. For $f \in C^2[0,\infty)$ and bounded, we have*

$$\lim_{n\to\infty} n[A_{n,\alpha}^{\beta_n}(f;x) - f(x)] = \frac{(x+x^2)}{2}f''(x).$$

Proof. Let $x,t \in [0,\infty)$, $f \in C^2(0,\infty)$. By Taylor's formula for f, we have

$$f(t) = f(x) + (t-x)f'(x) + \frac{(t-x)^2}{2!}f''(x) + (t-x)^2\phi(t;x), \tag{18}$$

where $\phi(t;x) \in C[0,\infty)$ and $\lim\limits_{t\to x}\phi(t;x) = 0$. By applying the operator $A_{n,\alpha}^{\beta_n}$ to the both sides of (18), we have

$$\begin{aligned}A_{n,\alpha}^{\beta_n}f(t) &= f(x)A_{n,\alpha}^{\beta_n}(1;x) + f'(x)A_{n,\alpha}^{\beta_n}(t-x;x)\\ &\quad + \frac{f''(x)}{2!}A_{n,\alpha}^{\beta_n}((t-x)^2;x) + A_{n,\alpha}^{\beta_n}((t-x)^2\phi(t;x);x),\end{aligned}$$

where

$$A_{n,\alpha}^{\beta_n}((t-x)^2\phi(t;x);x) = \sum_{k=1}^{\infty}\frac{nx(nx+k\beta_n)^{k-1}}{k!}e^{-(nx+k\beta_n)}\frac{x^{n+1}}{n!}\int_0^\infty e^{-xy}y^n\left(\frac{k}{xy}-x\right)^2\phi(t;x)dy.$$

By applying the Cauchy-Schwarz inequality twice succesively to the right side, we get

$$nA_{n,\alpha}^{\beta_n}((t-x)^2\phi(t;x);x) \le \sqrt{n^2 A_{n,\alpha}^{\beta_n}((t-x)^4;x)}\sqrt{A_{n,\alpha}^{\beta_n}(\phi(t;x);x)}.$$

From Lemma 4, we obtain $A_{n,\alpha}^{\beta_n}((t-x)^4;x) = O(n^{-2})$. Since $\phi(t;x) \in C[0,\infty)$ and $\lim\limits_{n\to\infty} \phi(t;x) = 0$, we get

$$\lim_{n\to\infty} nA_{n,\alpha}^{\beta_n}((t-x)^2\phi(t;x);x) = 0.$$

If (4) and Lemma 4 are used, the desired result is obtained. □

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У цій роботі досліджуються деякі апроксимаційні властивості модифікованого оператора Джейна-Гамма. Використовуючи теорему типу Коровкіна, спочатку наведено апроксимаційні властивості такого оператора. Потім обчислено швидкість збіжності цього оператора за допомогою модуля неперервності та представлено апроксимаційні властивості вагових просторів. Насамкінець отримано теорему типу Вороновської цього оператора.

Ключові слова і фрази: оператор Джейна, оператор Гамма, ваговий простір, модуль неперервності, K-функція Пеетре, теорема Вороновської.