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Krasnoselskii iteration process for approximating fixed points of enriched generalized nonexpansive mappings in Banach spaces

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We consider the class of enriched generalized nonexpansive mappings which includes enriched Kannan mappings, nonexpansive enriched Chatterjea mappings and enriched mappings. We prove some fixed point theorems for enriched generalized nonexpansive mappings using Krasnoselskii iteration process in Banach spaces. We also give stability result for such mappings under some appropriate conditions. The results presented in this paper improve and extend some works in literature.

Key words and phrases: enriched mapping, Krasnoselskii iteration, fixed point, stability.

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Introduction

There are many generalizations of the Banach Fixed Point Theory [1] in the literature. One of these generalizations is Hardy-Rogers' fixed point theorem [11] as follows.

Let $(X, \|\cdot\|)$ be a Banach space and $T : X \to X$ be a mapping such that

$$\|Tx - Ty\| \leq \theta \|x - y\| + \beta [\|x - Ty\| + \|y - Tx\|] + \gamma [\|x - Tx\| + \|y - Ty\|] \tag{1}$$

for all $x, y \in X$, where θ, β, γ are nonnegative constants such that $\theta + 2\beta + 2\gamma < 1$. Then T has a fixed point in X.

Note that Hardy-Rogers' fixed point theorem generalize Kannan's fixed point theorem [9], Chatterjea's fixed point theorem [7], Reich's fixed point theorem [12], Ciric's fixed point theorem [11], Reich-Rus-Ciric's fixed point theorem [13–16].

Inspired by the above studies, V. Berinde and M. Păcurar [2] introduced the concept of enriched contraction mapping as follows. And, they proved strong convergence theorem for the Krasnoselskii iteration used to approximate the fixed points of enriched Banach contractions. After, some authors [3–5] introduced the enriched Kannan mappings, enriched Chatterjea mappings and enriched nonexpansive mappings and they gave convergence results for Krasnoselskii iteration used to approximate fixed points of such mappings in Banach spaces.

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Definition 1 ([2]). *Let $(X, \|\cdot\|)$ be a linear normed space. A mapping $T : X \to X$ is said to be an enriched contraction mapping if there exist $\theta \in [0, k+1)$ and $k \in [0, \infty)$ such that*

$$\|k(x-y) + Tx - Ty\| \leq \theta \|x - y\|, \tag{2}$$

for all $x, y \in X$. We will also call T a (k, θ)-enriched contraction for the constants included in the inequality (2).

Definition 2 ([4]). *Let $(X, \|\cdot\|)$ be a linear normed space. A mapping $T : X \to X$ is said to be an enriched Kannan mapping if there exist $\gamma \in [0, 1/2)$ and $k \in [0, \infty)$ such that*

$$\|k(x-y) + Tx - Ty\| \leq \gamma [\|x - Tx\| + \|y - Ty\|], \tag{3}$$

for all $x, y \in X$. We will also call it (k, γ)-enriched Kannan mapping for the constants included in the inequality (3).

Definition 3 ([3]). *Let $(X, \|\cdot\|)$ be a linear normed space. A mapping $T : X \to X$ is said to be an enriched Chatterjea mapping if there exist $\beta \in [0, 1/2)$ and $k \in [0, \infty)$ such that*

$$\|k(x-y) + Tx - Ty\| \leq \beta\big(\|(k+1)(x-y) + y - Ty\| + \|(k+1)(y-x) + x - Tx\|\big) \tag{4}$$

for all $x, y \in X$. We will call T a (k, β)-enriched Chatterjea mapping for the constants included in the inequality (4).

Definition 4 ([5]). *Let $(X, \|\cdot\|)$ be a linear normed space. A mapping $T : X \to X$ is said to be an enriched nonexpansive mapping if there exists $k \in [0, \infty)$ such that*

$$\|k(x-y) + Tx - Ty\| \leq (k+1)\|x - y\|, \tag{5}$$

for all $x, y \in X$. We will also call T as a k-enriched nonexpansive mapping for the constant included in the inequality (5).

Definition 5 ([8]). *Let $(X, \|\cdot\|)$ be a linear normed space. A mapping $T : X \to X$ is said to be an enriched generalized nonexpansive mapping if there exist the nonegative real numbers θ, β, γ satisfying $\theta + 2\beta + 2\gamma < 1$ and $k \in [0, \infty)$ such that*

$$\begin{aligned}\|k(x-y) + Tx - Ty\| \leq{} & (k+1)\theta\|x-y\| \\ & + \beta\big(\|(k+1)(y-x) + x - Tx\| + \|(k+1)(x-y) + y - Ty\|\big) \\ & + \gamma\big(\|x - Tx\| + \|y - Ty\|\big),\end{aligned} \tag{6}$$

for all $x, y \in X$. It is also called T a $(k, \theta, \beta, \gamma)$-enriched generalized nonexpansive mapping for the constants included in the inequality (6).

Remark 1. *(i) If we take $k = 0$ in* (6)*, we obtain the Hardy-Rogers mapping* (1)*. That is, any enriched generalized nonexpansive mapping for constants $(0, \theta, \beta, \gamma)$ is a Hardy-Rogers mapping.*

(ii) If we take $\beta = \theta = 0$ in (6)*, we obtain the enriched Kannan mapping* (3)*. Also, taking $\gamma = \theta = 0$ in* (6)*, it reduces to enriched Chatterjea mapping* (4).

(iii) Taking $\theta = 1$ and $\beta = \gamma = 0$ in (6)*, it reduces to enriched nonexpansive mapping* (5).

1 Main Result

In this section, we will start by first giving an example for enriched generalized nonexpansive mappings.

Example 1. *Let $X = [0,4] \cup [5,6] \subseteq \mathbb{R}$ and $T : X \to X$ be defined by*

$$Tx = \begin{cases} -\frac{x}{3}, & x \in [2,4], \\ 0, & x \in [5,6]. \end{cases}$$

Then T is not enriched nonexpansive mapping but it is $\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{12}, \frac{1}{4}\right)$-enriched generalized nonexpansive mapping.

Proof. Case 1. Let $x, y \in [5,6]$. Then, we get

$$\begin{aligned} |k(x-y) + Tx - Ty| &= \left|\frac{1}{3}(x-y) + Tx - Ty\right| = \left|\frac{1}{3}(x-y)\right| \leq \left(\frac{1}{3}+1\right)\frac{1}{3}|x-y| \\ &= (k+1)\theta|x-y| \leq (k+1)\theta|x-y| \\ &\quad + \beta(|(k+1)(y-x) + x - Tx| + |(k+1)(x-y) + y - Ty|) \\ &\quad + \gamma(|x - Tx| + |y - Ty|). \end{aligned}$$

Case 2. For $x, y \in [2,4]$ it is obvious.

Case 3. Let $x \in [2,4]$ and $y \in [5,6]$. Then we have the following inequalities

$$|k(x-y) + Tx - Ty| = \left|\frac{1}{3}(x-y) - \frac{x}{3}\right| \leq 2, \qquad (k+1)\theta|x-y| = \frac{4}{9}|x-y| \geq \frac{4}{9},$$

and

$$\gamma(|x - Tx| + |y - Ty|) = \frac{1}{4}\left(\left|x + \frac{x}{3}\right| + |y|\right) \geq \frac{1}{4}\left(2 + \frac{2}{3} + 5\right).$$

From the above inequalities, we obtain

$$\begin{aligned} |k(x-y) + Tx - Ty| &\leq (k+1)\theta|x-y| \\ &\quad + \beta(|(k+1)(y-x) + x - Tx| + |(k+1)(x-y) + y - Ty|) \\ &\quad + \gamma(|x - Tx| + |y - Ty|). \end{aligned}$$

Therefore T is an enriched generalized nonexpansive mapping. Next, we show that T is not enriched nonexpansive. Let us take two elements 4 and 5 in X. Then

$$|k(x-y) + Tx - Ty| = \left|\frac{1}{3}(4-5) - \frac{4}{3}\right| > \left|\left(\frac{1}{3}+1\right)(4-5)\right| = (k+1)|x-y|.$$

That is, T is not enriched nonexpansive mapping. □

Now we give convergence theorem for enriched generalized nonexpansive mapping using the following Krasnoselskii type iterative scheme under appropriate assumptions

$$x_{n+1} = (1-\lambda)x_n + \lambda T x_n, \quad n \geq 0, \tag{7}$$

where $\lambda \subset (0,1)$ and x_0 is the starting point.

Let $Fix(T)$ denotes the set of fixed points of a mapping $T : C \to C$, i.e.

$$Fix(T) = \{x \in C : Tx = x\}.$$

Theorem 1. *Let C be a nonempty closed convex subset of a Banach space X and $T : C \to C$ be a $(k, \theta, \beta, \gamma)$-enriched generalized nonexpansive mapping. Then*

(i) $Fix(T) = \{p\}$;

(ii) the Krasnoselskii iteration $\{x_n\}_{n=0}^{\infty}$ given by $x_{n+1} = (1-\lambda)x_n + \lambda T x_n$ converges strongly to the fixed point of T;

(iii) the following estimate holds

$$\|x_{n+i-1} - p\| \leq \frac{\delta^i}{1-\delta} \|x_n - x_{n-1}\|, \quad n = 0, 1, 2, \dots, \; i = 1, 2, \dots, \tag{8}$$

where $\delta = \frac{\theta+\beta+\gamma}{1-\beta-\gamma}$.

Proof. *(i)* We consider the following mapping $S : C \to C$ given by

$$S(x) = (1-\lambda)x + \lambda T x \tag{9}$$

for $\lambda \subset (0,1)$. We know that for the mappings S and T the following property $Fix(S) = Fix(T)$ holds.

If $k > 0$ in (6), then let us put $\lambda = \frac{1}{k+1}$. Since $k \in [0, \infty)$, we have that $0 < \lambda < 1$ for all $n \in \mathbb{N}$. Then the contractive condition (1) reduces the following inequality

$$\begin{aligned} \left\| \left(\frac{1}{\lambda} - 1 \right)(x-y) + Tx - Ty \right\| \leq {} & \frac{1}{\lambda}\theta \|x-y\| \\ & + \beta \left(\left\| \frac{1}{\lambda}(y-x) + x - Tx \right\| + \left\| \frac{1}{\lambda}(x-y) + y - Ty \right\| \right) \\ & + \gamma \left(\|x - Tx\| + \|y - Ty\| \right), \end{aligned}$$

which implies that

$$\|Sx - Sy\| \leq \theta \|x-y\| + \beta \left(\|y - Sx\| + \|x - Sy\| \right) + \gamma \left(\|x - Sx\| + \|y - Sy\| \right). \tag{10}$$

From the above inequality, we say that S is a Hardy-Rogers mapping.

Consider the mapping (9). The iterative process $\{x_n\}_{n=0}^{\infty}$ defined by (7) is Picard iteration associated to S, that is $x_{n+1} = Sx_n$. If we take $x = x_n$ and $y = x_{n-1}$ in (10), we write

$$\begin{aligned} \|Sx_n - Sx_{n-1}\| \leq {} & \theta \|x_n - x_{n-1}\| + \beta \left(\|x_{n-1} - Sx_n\| + \|x_n - Sx_{n-1}\| \right) \\ & + \gamma \left(\|x_n - Sx_n\| + \|x_{n-1} - Sx_{n-1}\| \right), \end{aligned}$$

which implies that

$$\begin{aligned} \|x_{n+1} - x_n\| & \leq \theta \|x_n - x_{n-1}\| + \beta \left(\|x_{n-1} - x_{n+1}\| + \|x_n - x_n\| \right) \\ & \quad + \gamma \left(\|x_n - x_{n+1}\| + \|x_{n-1} - x_n\| \right) \\ & \leq \theta \|x_n - x_{n-1}\| + \beta \left(\|x_{n-1} - x_n\| + \|x_n - x_{n+1}\| \right) \\ & \quad + \gamma \left(\|x_n - x_{n+1}\| + \|x_{n-1} - x_n\| \right). \end{aligned}$$

Then

$$\|x_{n+1} - x_n\| \leq \delta \|x_{n-1} - x_n\|, \tag{11}$$

where $\delta = \frac{\theta+\beta+\gamma}{1-\beta-\gamma}$. Since $\theta + 2\beta + 2\gamma < 1$, we get that $0 < \delta < 1$.

Applying (11), it easy to see that the following estimates

$$\|x_{n+m} - x_n\| \leq \delta^n \frac{1-\delta^m}{1-\delta} \|x_1 - x_0\|, \tag{12}$$

and

$$\|x_{n+m} - x_n\| \leq \delta \frac{1-\delta^m}{1-\delta} \|x_n - x_{n-1}\| \tag{13}$$

hold. It follows from (12) that $\{x_n\}_{n=0}^{\infty}$ is a Cauchy sequence. Since C is a Banach space, there exists $p \in C$ such that $\lim_{n\to\infty} x_n = p$. Now, we will show that p is a fixed point of S. We have

$$\|p - Sp\| \leq \|p - x_{n+1}\| + \|x_{n+1} - Sp\| = \|p - x_{n+1}\| + \|Sx_n - Sp\|. \tag{14}$$

Using (10), we get

$$\|Sx_n - Sp\| \leq \theta \|x_n - p\| + \beta (\|x_n - Sx_n\| + \|p - Sp\|) + \gamma (\|x_n - Sp\| + \|p - Sx_n\|),$$

and therefore, from (14) we get

$$\|p - Sp\| \leq \frac{1+\beta+\gamma}{1-\beta-\gamma} \|p - x_{n+1}\| + \delta \|x_n - p\|.$$

Taking limit as $n \to \infty$ in the above inequality, we have that $\|p - Sp\| = 0$, that is $p = Sp$. So, $p \in Fix(S)$.

We will show the uniqueness of the fixed point of the mapping S. Suppose on the contrary that q is another fixed point of S. Then from (10)

$$\begin{aligned}\|p - q\| = \|Sp - Sq\| &\leq \theta \|p - q\| + \beta (\|q - Sp\| + \|p - Sq\|) + \gamma (\|p - Sp\| + \|q - Sq\|) \\ &= \theta \|p - q\| + \beta (\|q - p\| + \|p - q\|) + \gamma (\|p - p\| + \|q - q\|) \\ &\leq (\theta + 2\beta) \|p - q\|.\end{aligned}$$

This is a contradiction because of $\theta + 2\beta \neq 0$. This shows the uniqueness of the fixed point of the mapping S. Since $Fix(T) = Fix(S)$, claim *(i)* is proven.

(ii) Let $\{x_n\}_{n=0}^{\infty}$ be the Krasnoselskii iteration (7). Then

$$\|x_{n+1} - p\| = \|(1-\lambda)x_n + \lambda T x_n - p\| = \|Sx_n - p\|. \tag{15}$$

Taking $x := p$ and $y := x_n$ in (10), we have that

$$\begin{aligned}\|Sx_n - p\| &\leq \theta \|x_n - p\| + \beta (\|p - Sx_n\| + \|x_n - Sp\|) + \gamma (\|x_n - Sx_n\| + \|p - Sp\|) \\ &\leq \theta \|x_n - p\| + \beta (\|p - Sx_n\| + \|x_n - p\|) + \gamma (\|x_n - p\| + \|p - Sx_n\|)\end{aligned}$$

which implies that

$$(1 - \beta - \gamma) \|Sx_n - p\| \leq (\theta + \beta + \gamma) \|x_n - p\|.$$

From above inequality, we get

$$\|Sx_n - p\| \leq \frac{\theta+\beta+\gamma}{1-\beta-\gamma} \|x_n - p\| = \delta \|x_n - p\|. \tag{16}$$

Combining (15) and (16), we obtain

$$\|x_{n+1} - p\| = \|Sx_n - p\| \leq \delta \|x_n - p\|.$$

Inductively we obtain $\|x_{n+1}-p\| \le \delta^n \|x_0-p\|$. Since $\delta<1$, we have that

$$\lim_{n\to\infty} \|x_{n+1}-p\| = 0.$$

This shows that $\{x_n\}_{n=0}^{\infty}$ converges strongly to p.

(iii) If we take limit as $m\to\infty$ in (12) and (13), we obtain

$$\|x_n-p\| \le \frac{\delta^n}{1-\delta}\|x_1-x_0\|, \tag{17}$$

and

$$\|x_n-p\| \le \frac{\delta}{1-\delta}\|x_n-x_{n-1}\|. \tag{18}$$

Combining (17) and (18), we can obtain the error estimate (8). □

If we take $\gamma=\theta=0$ in Theorem 1, we obtain the following result.

Corollary 1 ([3]). *Let C be a nonempty closed convex subset of a Banach space X and $T: C\to C$ be a (k,β)-enriched Chatterjea mapping. Then*

(i) $Fix(T)=\{p\}$;

(ii) the Kransnoselskii iteration $\{x_n\}_{n=0}^{\infty}$ given by $x_{n+1}=(1-\lambda)x_n+\lambda Tx_n$ converges strongly to the fixed point of T;

(iii) the following estimate holds

$$\|x_{n+i-1}-p\| \le \frac{\delta^i}{1-\delta}\|x_n-x_{n-1}\|, \quad n=0,1,2,\ldots,\ i=1,2,\ldots,$$

where $\delta=\frac{\beta}{1-\beta}$.

If we take $\beta=\theta=0$ in Theorem 1, we obtain the following result.

Corollary 2 ([4]). *Let C be a nonempty closed convex subset of a Banach space X and $T: C\to C$ be a (k,γ)-enriched Kannan mapping. Then*

(i) $Fix(T)=\{p\}$;

(ii) the Kransnoselskii iteration $\{x_n\}_{n=0}^{\infty}$ given by $x_{n+1}=(1-\lambda)x_n+\lambda Tx_n$ converges strongly to the fixed point of T;

(iii) the following estimate holds

$$\|x_{n+i-1}-p\| \le \frac{\delta^i}{1-\delta}\|x_n-x_{n-1}\|, \quad n=0,1,2,\ldots,\ i=1,2,\ldots,$$

where $\delta=\frac{\gamma}{1-\gamma}$.

Next, we will consider the problem for the T-stability of Krasnoselskii iteration process for the enriched generalized nonexpansive mappings in Banach spaces. Let us recall the following definitions deal with stability.

Let $(X,\|\cdot\|)$ be a normed linear space, T a self map of X with $F_T=\{x\in X: Tx=x\}\neq 0$. Consider a fixed point iteration procedure, i.e. a sequence $\{x_n\}_{n=0}^{\infty}$ defined by $x_0\in X$ and

$$x_{n+1}=f(T,x_n), \quad n=0,1,2,\ldots, \tag{19}$$

where f is some function.

Examples of such iterations are: Picard iteration, obtained for $f(T,x_n)=Tx_n$ and Krasnoselskii iteration, obtained for $f(T,x_n)=(1-\lambda)x_n+\lambda Tx_n$, $\lambda\in[0,1]$.

Suppose $\{x_n\}_{n=0}^{\infty}$ converges strongly to some $p \in F_T$. In concrete applications, when computing $\{x_n\}_{n=0}^{\infty}$ we usually follow the next two steps.

1. Choose the initial guess (approximation) $x_0 \in X$.

2. Compute $x_1 = f(T, x_0)$. But due to various errors that occur during calculations (rounding errors, numerical approximations of functions, derivatives, integrals etc.), we do not obtain the exact value of x_1, but a different one, say y_1, which is however close enough to x_1, i.e. $y_1 \approx x_1$.

Consequently, when computing $x_2 = f(T, x_1)$ we will actually obtain x_2 as $x_2 = f(T, y_1)$. So, again, instead of the theoretical value of x_2 we expect, another value y_2 will be obtained, y_2 being close enough to x_2, i.e. $y_2 \approx x_2$, and so on. In this way, instead of the theoretical sequence $\{x_n\}_{n=0}^{\infty}$ defined by the iteration (19), we obtain practically an approximate sequence $\{y_n\}_{n=0}^{\infty}$. We shall consider the given fixed point iteration method to be numerically stable if and only if for y_n close enough to x_n at each stage, the approximate sequence $\{y_n\}_{n=0}^{\infty}$ still converges to the fixed point p of T. The next definition, due to Harder and Hicks [10], expresses basically the previous idea.

Definition 6 ([10]). *Let X be a linear normed space, T be a self map of X, and $\{x_n\}_{n=0}^{\infty} \subset X$ be a sequence defined by* (19)*, where $x_0 \in X$ is the initial approximation and f is some function. Suppose that the sequence $\{x_n\}_{n=0}^{\infty}$ converges to a fixed point p of T. Let $\{y_n\}_{n=0}^{\infty} \subset X$ be an arbitary sequence and set*

$$\epsilon_n = \|y_{n+1} - f(T, y_n)\|, \quad n = 0, 1, \ldots .$$

Then iteration procedure (19) *is said to be T-stable or stable with respect to T if and only if* $\lim_{n\to\infty} \epsilon_n = 0 \Rightarrow \lim_{n\to\infty} y_n = p$.

Lemma 1 ([6]). *If δ is a real number such that $0 \leq \delta < 1$, and $\{\epsilon_n\}_{n=0}^{\infty}$ is a sequence of positive numbers such that* $\lim_{n\to\infty} \epsilon_n = 0$*, then for any sequence of positive numbers $\{u_n\}_{n=0}^{\infty}$ satisfying*

$$u_{n+1} \leq \delta u_n + \epsilon_n, \quad n = 0, 1, \ldots,$$

we have

$$\lim_{n\to\infty} u_n = 0.$$

Theorem 2. *Let C be a nonempty closed convex subset of a Banach space X and $T : C \to C$ be a $(k, \theta, \beta, \gamma)$-enriched generalized nonexpansive mapping. Let $x_0 \in C$ and suppose that $x_{n+1} = f(T, x_n) = (1 - \lambda) x_n + \lambda T x_n$, $n \geq 0$, where λ is a real number in* $[0, 1]$*. If $\frac{1+\delta}{2} < \lambda$, then the Krasnoselskii iteration is T-stable.*

Proof. From Theorem 1, we know that T has a unique fixed point. Called $p \in X$, i.e. $Tp = p$. Let $\{y_n\}_{n=0}^{\infty} \subset C$ and define

$$\epsilon_n = \|y_{n+1} - (1 - \lambda) y_n - \lambda T y_n\|, \quad n \geq 0.$$

Assume that $\lim_{n\to\infty} \epsilon_n = 0$. Then

$$\begin{aligned} \|y_{n+1} - p\| &\leq \|y_{n+1} - (1 - \lambda) y_n - \lambda T y_n\| + \|(1 - \lambda) y_n + \lambda T y_n - p\| \\ &= \epsilon_n + \|(1 - \lambda) y_n + \lambda T y_n - ((1 - \lambda) + \alpha_n) p\| \\ &= \epsilon_n + \|(1 - \lambda)(y_n - p) + \lambda (T y_n - p)\| \\ &\leq \epsilon_n + (1 - \lambda) \|y_n - p\| + \lambda \|T y_n - p\| . \end{aligned} \tag{20}$$

We again consider the following mapping $S : C \to C$ given by

$$S(x) = (1-\lambda)x + \lambda Tx$$

for $\lambda \subset (0,1)$. From Theorem 1, we know that S is a Hardy-Rogers mapping. Then

$$\|Sy_n - p\| = \|(1-\lambda)y_n + \lambda Ty_n - p\| \geq \lambda \|Ty_n - p\| - (1-\lambda)\|y_n - p\|,$$

which implies that

$$\lambda \|Ty_n - p\| \leq \|Sy_n - p\| + (1-\lambda)\|y_n - p\|. \tag{21}$$

Combining (20) and (21), we obtain that

$$\begin{aligned} \|y_{n+1} - p\| &\leq \epsilon_n + (1-\lambda)\|y_n - p\| + \lambda \|Ty_n - p\| \\ &\leq \epsilon_n + (1-\lambda)\|y_n - p\| + \|Sy_n - p\| + (1-\lambda)\|y_n - p\| \\ &\leq \epsilon_n + 2(1-\lambda)\|y_n - p\| + \delta\|y_n - p\| \\ &= (2(1-\lambda) + \delta)\|y_n - p\| + \epsilon_n. \end{aligned}$$

Since $\frac{1+\delta}{2} < \lambda$, we have that $2(1-\lambda) + \delta < 1$. From Lemma 1, we obtain

$$\lim_{n\to\infty} \|y_n - p\| = 0,$$

which implies that

$$\lim_{n\to\infty} y_n = p.$$

□

References

[1] Banach S. *Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales*. Fund. Math. 1922, **3** (1), 133–181.

[2] Berinde V., Păcurar M. *Approximating fixed points of enriched contractions in Banach spaces*. J. Fixed Point Theory Appl. 2020, **22**, article number 38. doi:10.1007/s11784-020-0769-9

[3] Berinde V., Păcurar M. *Approximating fixed points of enriched Chatterjea contractions by Krasnoselskij iterative method in Banach spaces*. 2019, 1–13. doi:10.48550/arXiv.1909.03494

[4] Berinde V., Păcurar M. *Fixed point theorems for Kannan type mappings with applications to split feasibility and variational inequality problems*. 2019, 1–13. doi:10.48550/arXiv.1909.02379

[5] Berinde V. *Approximating fixed points of enriched nonexpansive mappings by Krasnoselskij iteration in Hilbert spaces*. Carpathian J. Math. 2019, **35** (3), 293–304. doi:10.37193/CJM.2019.03.04

[6] Berinde V. *On the stability of some fixed point procedures*. Bul. Ştiinţ. Univ. Baia Mare, Ser. B, Matematică-Informatică. 2002, **18** (1), 7–14.

[7] Chatterjea S.K. *Fixed-point theorems*. C. R. Acad. Bulgare Sci. 1972, **25**, 727–730.

[8] Deshmukh A. *Study of self-maps, their fixed points and iterative procedures for approximation*. 2020, 1–82. doi:10.13140/RG.2.2.35917.05600

[9] Kannan R. *Some results on fixed points*. Bull. Calcutta Math. Soc. 1968, **100**, 71–76.

[10] Harder A.M., Hicks T.L. *Stability results for fixed point iteration procedures*. Math. Japonica 1988, **33** (5), 693–706.

[11] Hardy G.E., Rogers T.D. *A generalisation of fixed point theorem of Reich*. Canad. Math. Bull. 1973, **16** (2), 201–208. doi:10.4153/CMB-1973-036-0

[12] Reich S. *Some remarks concerning contraction mappings*. Can. Math. Bull. 1971, **14** (1), 121–124. doi: 10.4153/CMB-1971-024-9

[13] Reich S. *Fixed point of contractive functions*. Boll. Unione Mat. Ital. 1972, **5**, 26–42.

[14] Reich S. *Kannanś fixed point theorem*. Boll. Unione Mat. Ital. 1971, **4** (4), 1–11.

[15] Rus I.A. Principles and Applications of the Fixed Point Theory. Editura Dacia: Clui-Napoca, Romania, 1979.

[16] Rus I.A. Generalized Contractions and Applications. Cluj University Press: Clui-Napoca, Romania, 2001.

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Ми розглядаємо клас збагачених узагальнених нерозширюючих відображень, яких включає збагачені відображення Каннана, нерозширюючі збагачені відображення Чаттерджі та збагачені відображення. Ми доводимо деякі теореми про нерухому точку для збагачених узагальнених нерозширюючих відображень використовуючи ітераційний процес Красносельського у банахових просторах. Також ми доводимо результат про стабільність для таких відображень за деяких відповідних умов. Результати статті покращують та розширюють деякі роботи в літературі.

Ключові слова і фрази: збагачене відображення, ітераційний процес Красносельського, нерухома точка, стабільність.