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Some results on η-Yamabe solitons in 3-dimensional trans-Sasakian manifold

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The object of the present paper is to study some properties of 3-dimensional trans-Sasakian manifold whose metric is η-Yamabe soliton. We have studied here some certain curvature conditions of 3-dimensional trans-Sasakian manifold admitting η-Yamabe soliton. Lastly, we construct a 3-dimensional trans-Sasakian manifold satisfying η-Yamabe soliton.

Key words and phrases: Yamabe soliton, η-Yamabe soliton, η-Einstein manifold, trans-Sasakian manifold.

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Introduction

The concept of Yamabe flow was first introduced by R. Hamilton [7] to construct Yamabe metrics on compact Riemannian manifolds. On a Riemannian or pseudo-Riemannian manifold M, a time-dependent metric $g(\cdot, t)$ is said to evolve by the Yamabe flow if the metric g satisfies the given equation

$$\frac{\partial}{\partial t} g(t) = -rg(t), \quad g(0) = g_0,$$

where r is the scalar curvature of the manifold M.

In 2-dimension case, the Yamabe flow is equivalent to the Ricci flow, which is defined by $\frac{\partial}{\partial t} g(t) = -2S(g(t))$, where S denotes the Ricci tensor. But in dimension > 2 the Yamabe and Ricci flows do not agree, since the Yamabe flow preserves the conformal class of the metric but the Ricci flow does not in general.

A Yamabe soliton [1, 13] corresponds to self-similar solution of the Yamabe flow, is defined on a Riemannian or pseudo-Riemannian manifold (M, g) by a vector field ξ satisfying the equation

$$\frac{1}{2}\pounds_V g = (r - \lambda) g, \tag{1}$$

where $\pounds_\xi g$ denotes the Lie derivative of the metric g along the vector field ξ, r is the scalar curvature and λ is a constant. Moreover a Yamabe soliton is said to be expanding if $\lambda > 0$, steady if $\lambda = 0$ and shrinking if $\lambda < 0$.

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Yamabe solitons on a three-dimensional Sasakian manifold were studied by R. Sharma [14]. If the potential vector field V is of gradient type, $V = grad(f)$, for f a smooth function on M, then (V, λ) is called a gradient Yamabe soliton.

Definition. *As a generalization of Yamabe soliton, a Riemannian metric on (M, g) is said to be a η-Yamabe soliton [3] if*

$$\frac{1}{2}\pounds_{\xi} g = (r - \lambda)g - \mu\eta \otimes \eta, \tag{2}$$

where λ and μ are contants and η is a 1-form.

If λ and μ are two smooth functions then (2) is said to be an almost η-Yamabe soliton or a quasi-Yamabe soliton [3].

Moreover if $\mu = 0$, the above equation (2) reduces to (1) and so the η-Yamabe soliton becomes Yamabe soliton. Similarly an almost η-Yamabe soliton reduces to almost Yamabe soliton if in (2), λ is a smooth function and $\mu = 0$.

Denote

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z, \tag{3}$$

$$H(X, Y)Z = R(X, Y)Z - \frac{1}{(n-2)}[g(Y, Z)QX - g(X, Z)QY + S(Y, Z)X - S(X, Z)Y], \tag{4}$$

$$P(X, Y)Z = R(X, Y)Z - \frac{1}{(n-1)}[g(QY, Z)X - g(QX, Z)Y], \tag{5}$$

$$\tilde{C}(X, Y)Z = R(X, Y)Z - \frac{r}{n(n-1)}[g(Y, Z)X - g(X, Z)Y], \tag{6}$$

$$\begin{aligned} C^*(X, Y)Z = aR(X, Y)Z + b[S(Y, Z)X - S(X, Z)Y + g(Y, Z)QX - g(X, Z)QY] \\ - \frac{r}{n}\left[\frac{a}{n-1} + 2b\right][g(Y, Z)X - g(X, Z)Y], \end{aligned} \tag{7}$$

where a, b are constants,

$$W_2(X, Y)Z = R(X, Y)Z + \frac{1}{n-1}[g(X, Z)QY - g(Y, Z)QX] \tag{8}$$

the Riemannian-Christoffel curvature tensor R [10], the conharmonic curvature tensor H [8], the projective curvature tensor P [15], the concircular curvature tensor $\tilde{C}$ [11], the quasi-conformal curvature tensor C^* [16] and the W_2-curvature tensor [11] respectively in a Riemannian manifold (M^n, g), where Q is the Ricci operator, defined by $S(X, Y) = g(QX, Y)$, S is the Ricci tensor, $r = tr(S)$ is the scalar curvature, where $tr(S)$ is the trace of S and $X, Y, Z \in \chi(M)$, $\chi(M)$ being the Lie algebra of vector fields of M.

Now in (7), if $a = 1$ and $b = -\frac{1}{n-2}$, then we get

$$\begin{aligned} C^*(X, Y)Z = R(X, Y)Z - \frac{1}{n-2}[S(Y, Z)X - S(X, Z)Y + g(Y, Z)QX - g(X, Z)QY] \\ + \frac{r}{(n-1)(n-2)}[g(Y, Z)X - g(X, Z)Y] = C(X, Y)Z, \end{aligned}$$

where C is the conformal curvature tensor [5]. Thus the conformal curvature tensor C is a particular case of the tensor C^*.

In the present paper, we study η-Yamabe soliton on 3-dimensional trans-Sasakian manifolds. The paper is organized as follows. After introduction, Section 2 is devoted for preliminaries on 3-dimensional trans-Sasakian manifolds. In Section 3, we have studied η-Yamabe soliton on 3-dimensional trans-Sasakian manifolds. Here we examine if a 3-dimensional trans-Sasakian manifold admits η-Yamabe soliton, then the scalar curvature is constant and the manifold becomes η-Einstein. We also characterized the nature of the manifold if the manifold is Ricci symmetric and the Ricci tensor is η-recurrent. Section 4 deals with the curvature properties of 3-dimensional trans-Sasakian manifold. In this section we have shown the nature of the η-Yamabe soliton, when the manifold is ξ-projectively flat, ξ-concircularly flat, ξ-conharmonically flat, ξ-quasi-conformally flat. Here we have obtained some results on η-Yamabe soliton satisfying the conditions $R(\xi, X) \cdot S = 0$ and $W_2(\xi, X) \cdot S = 0$. In last section we gave an example of a 3-dimensional trans-Sasakian manifold satisfying η-Yamabe soliton.

1 Preliminaries

Let M be a connected almost contact metric manifold with an almost contact metric structure (ϕ, ξ, η, g), where ϕ is a $(1,1)$ tensor field, ξ is a vector field, η is a 1-form and g is the compatible Riemannian metric such that

$$\phi^2(X) = -X + \eta(X)\xi, \quad \eta(\xi) = 1, \quad \eta \circ \phi = 0, \quad \phi\xi = 0, \tag{9}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{10}$$

$$g(X, \phi Y) = -g(\phi X, Y), \tag{11}$$

$$g(X, \xi) = \eta(X), \tag{12}$$

for all vector fields $X, Y \in \chi(M)$.

An almost contact metric structure (ϕ, ξ, η, g) on M is called a trans-Sasakian structure [9], if $(M \times R, J, G)$ belongs to the class W_4 [6], where J is the almost complex structure on $M \times R$ defined by $J(X, f\frac{d}{dt}) = (\phi X - f\xi, \eta(X)\frac{d}{dt})$ for all vector fields X on M and smooth functions f on $M \times R$. It can be expressed by the condition [2]

$$(\nabla_X \phi)Y = \alpha(g(X, Y)\xi - \eta(Y)X) + \beta(g(\phi X, Y)\xi - \eta(Y)\phi X), \tag{13}$$

for some smooth functions α, β on M and we say that the trans-Sasakian structure is of type (α, β). From the above expression we can write

$$\nabla_X \xi = -\alpha\phi X + \beta(X - \eta(X)\xi), \tag{14}$$

$$(\nabla_X \eta)Y = -\alpha g(\phi X, Y) + \beta g(\phi X, \phi Y). \tag{15}$$

For a 3-dimensional trans-Sasakian manifold the following relations hold [4,12]:

$$2\alpha\beta + \xi\alpha = 0, \quad S(X, \xi) = (2(\alpha^2 - \beta^2) - \xi\beta)\eta(X) - X\beta - (\phi X)\alpha,$$

$$\begin{aligned} S(X, Y) = \Big[\frac{r}{2} + \xi\beta - (\alpha^2 - \beta^2)\Big]g(X, Y) - \Big[\frac{r}{2} + \xi\beta - 3(\alpha^2 - \beta^2)\Big]\eta(X)\eta(Y) \\ - (Y\beta + (\phi Y)\alpha)\eta(X) - (X\beta + (\phi X)\alpha)\eta(Y), \end{aligned}$$

where S denotes the Ricci tensor of type $(0,2)$, r is the scalar curvature of the manifold M and α, β are defined as earlier.

For $\alpha, \beta = $ const, the following relations hold [4,12]:

$$S(X,Y) = \left[\frac{r}{2} - (\alpha^2 - \beta^2)\right] g(X,Y) - \left[\frac{r}{2} - 3(\alpha^2 - \beta^2)\right] \eta(X)\eta(Y), \tag{16}$$

$$S(X,\xi) = 2(\alpha^2 - \beta^2)\eta(X),$$

$$R(X,Y)\xi = (\alpha^2 - \beta^2)[\eta(Y)X - \eta(X)Y], \tag{17}$$

$$R(\xi,X)Y = (\alpha^2 - \beta^2)[g(X,Y)\xi - \eta(Y)X], \tag{18}$$

$$R(\xi,X)\xi = (\alpha^2 - \beta^2)[\eta(X)\xi - X],$$

$$\eta(R(X,Y)Z) = (\alpha^2 - \beta^2)[g(Y,Z)\eta(X) - g(X,Z)\eta(Y)],$$

where R is the Riemannian curvature tensor, and

$$QX = \left[\frac{r}{2} - (\alpha^2 - \beta^2)\right] X - \left[\frac{r}{2} - 3(\alpha^2 - \beta^2)\right] \eta(X)\xi,$$

where Q is the Ricci operator defined earlier.

Again,

$$(\pounds_\xi g)(X,Y) = (\nabla_\xi g)(X,Y) - \alpha g(\phi X, Y) + 2\beta g(X,Y) - 2\beta\eta(X)\eta(Y) - \alpha g(X, \phi Y).$$

Then using (11), the above equation becomes

$$(\pounds_\xi g)(X,Y) = 2\beta g(X,Y) - 2\beta\eta(X)\eta(Y), \tag{19}$$

where ∇ is the Levi-Civita connection associated with g and $\pounds_\xi$ denotes the Lie derivative along the vector field ξ.

2 η-Yamabe soliton on 3-dimensional trans-Sasakian manifold

Let M be a 3-dimensional trans-Sasakian manifold. Consider the η-Yamabe soliton on M as

$$\frac{1}{2}(\pounds_\xi g)(X,Y) = (r - \lambda)g(X,Y) - \mu\eta(X)\eta(Y), \tag{20}$$

for all vector fields X, Y on M.

Theorem 1. *If a 3-dimensional trans-Sasakian manifold M admits an η-Yamabe soliton (g, ξ), ξ being the Reeb vector field of M, then the scalar curvature is constant.*

Proof. From (19) and (20), we get

$$(r - \lambda - \beta)g(X,Y) = (\mu - \beta)\eta(X)\eta(Y).$$

Taking $Y = \xi$ in the above equation and using (9), we have

$$(r - \lambda - \mu)\eta(X) = 0.$$

Since $\eta(X) \neq 0$, so we get

$$r = \lambda + \mu. \tag{21}$$

Now as both λ and μ are constants, r is also constant. $\square$

Corollary 1. *If a 3-dimensional trans-Sasakian manifold M admits a Yamabe soliton (g, ξ), ξ being the Reeb vector field of M, then ξ becomes a Killing vector field.*

Proof. In (21), if $\mu = 0$, we get $r = \lambda$ and so (20) becomes, $\pounds_\xi g = 0$. Thus ξ is a Killing vector field. $\square$

Corollary 2. *If a 3-dimensional trans-Sasakian manifold M admits an η-Yamabe soliton (g, ξ), ξ being the Reeb vector field of M, then the manifold becomes η-Einstein manifold.*

Proof. From (16) and (21), we have

$$S(X,Y) = \Big[\frac{\lambda+\mu}{2} - (\alpha^2-\beta^2)\Big]g(X,Y) - \Big[\frac{\lambda+\mu}{2} - 3(\alpha^2-\beta^2)\Big]\eta(X)\eta(Y) \tag{22}$$

for all vector fields X, Y on M. This concludes the proof. $\square$

Proposition 1. *Let a 3-dimensional trans-Sasakian manifold M admits an η-Yamabe soliton (g, ξ), ξ being the Reeb vector field of M. If the manifold is Ricci symmetric then $\lambda + \mu = 6(\alpha^2 - \beta^2)$, where $\lambda, \mu, \alpha, \beta$ are constants.*

Proof. We know $(\nabla_X S)(Y,Z) = XS(Y,Z) - S(\nabla_X Y, Z) - S(Y, \nabla_X Z)$ for all vector fields X, Y, Z on M and ∇ is the Levi-Civita connection associated with g.

Now replacing the expression of S from (22), we obtain

$$(\nabla_X S)(Y,Z) = -\Big[\frac{\lambda+\mu}{2} - 3(\alpha^2-\beta^2)\Big][\eta(Z)(\nabla_X\eta)Y + \eta(Y)(\nabla_X\eta)Z] \tag{23}$$

for all vector fields X, Y, Z on M.

Now, if the manifold is Ricci symmetric, i.e. $\nabla S = 0$, then from (23) we have

$$\Big[\frac{\lambda+\mu}{2} - 3(\alpha^2-\beta^2)\Big][\eta(Z)(\nabla_X\eta)Y + \eta(Y)(\nabla_X\eta)Z] = 0$$

for all vector fields X, Y, Z on M.

Taking $Z = \xi$ in the above equation and using (15), (9), we get

$$\Big[\frac{\lambda+\mu}{2} - 3(\alpha^2-\beta^2)\Big][\beta g(\phi X, \phi Y) - \alpha g(\phi X, Y)] = 0$$

for all vector fields X, Y on M. Hence we get $\lambda + \mu = 6(\alpha^2 - \beta^2)$. $\square$

Proposition 2. *Let a 3-dimensional trans-Sasakian manifold M admits an η-Yamabe soliton (g, ξ), ξ being the Reeb vector field of M. If the Ricci tensor S is η-recurrent, then $\alpha = \pm\beta$.*

Proof. If the Ricci tensor S is η-recurrent, then we have $\nabla S = \eta \otimes S$, which implies that

$$(\nabla_X S)(Y,Z) = \eta(X)S(Y,Z)$$

for all vector fields X, Y, Z on M. Then using (23), we get

$$-\Big[\frac{\lambda+\mu}{2} - 3(\alpha^2-\beta^2)\Big][\eta(Z)(\nabla_X\eta)Y + \eta(Y)(\nabla_X\eta)Z] = \eta(X)S(Y,Z)$$

for all vector fields X, Y, Z on M.

Using (15), the above equation becomes

$$-\Big[\frac{\lambda+\mu}{2}-3(\alpha^2-\beta^2)\Big][\eta(Z)(-\alpha g(\phi X,Y)+\beta g(\phi X,\phi Y)) \\ +\eta(Y)(-\alpha g(\phi X,Z)+\beta g(\phi X,\phi Z))]=\eta(X)S(Y,Z).$$

Now, taking $Y=\xi$, $Z=\xi$ and using formulas (9), (22), the above equation becomes $2(\alpha^2-\beta^2)\eta(X)=0$. Since $\eta(X)\neq 0$, for all X on M, we have

$$\alpha=\pm\beta. \tag{24}$$

This completes the proof. □

Proposition 3. *Let a 3-dimensional trans-Sasakian manifold M admits an η-Yamabe soliton (g,ξ), ξ being the Reeb vector field of M. If the manifold is Ricci symmetric and the Ricci tensor S is η-recurrent, then the manifold becomes flat.*

Proof. If the manifold is Ricci symmetric and the Ricci tensor S is η-recurrent, then using (24) in $\lambda+\mu=6(\alpha^2-\beta^2)$ and from (21) we obtain the result. □

Theorem 2. *Let M be a 3-dimensional trans-Sasakian manifold admitting an η-Yamabe soliton (g,V), V being a vector field on M. If V is pointwise co-linear with ξ, then V is a constant multiple of ξ, where ξ being the Reeb vector field of M.*

Proof. Let an η-Yamabe soliton be defined on a 3-dimensional trans-Sasakian manifold M as

$$\frac{1}{2}\pounds_V g=(r-\lambda)g-\mu\eta\otimes\eta, \tag{25}$$

where $\pounds_V g$ denotes the Lie derivative of the metric g along a vector field V, r is defined by (1) and λ,μ are defined by (2). Let V be pointwise co-linear with ξ, i.e. $V=b\xi$, where b is a function on M.

Then the equation (25) becomes

$$(\pounds_{b\xi}g)(X,Y)=2(r-\lambda)g(X,Y)-2\mu\eta(X)\eta(Y)$$

for any vector fields X,Y on M.

Applying the property of Lie derivative and Levi-Civita connection we have

$$bg(\nabla_X\xi,Y)+(Xb)\eta(Y)+bg(\nabla_Y\xi,X)+(Yb)\eta(X)=2(r-\lambda)g(X,Y)-2\mu\eta(X)\eta(Y).$$

Using (14) and (11), the above equation reduces to

$$2b\beta[g(X,Y)-\eta(X)\eta(Y)]+(Xb)\eta(Y)+(Yb)\eta(X)=2(r-\lambda)g(X,Y)-2\mu\eta(X)\eta(Y).$$

Now taking $Y=\xi$ in the above equation and using (9), (12), we obtain

$$Xb+(\xi b)\eta(X)=2(r-\lambda)\eta(X)-2\mu\eta(X). \tag{26}$$

Again taking $X=\xi$, we get

$$\xi b=r-\lambda-\mu. \tag{27}$$

Then using (27), the equation (26) becomes

$$Xb = (r - \lambda - \mu)\eta(X). \tag{28}$$

Applying exterior differentiation in (28), we have $(r - \lambda - \mu)d\eta = 0$. Since $d\eta \neq 0$ [4], the last equation gives

$$r = \lambda + \mu. \tag{29}$$

Using (29), the equation (28) becomes $Xb = 0$, which implies that b is constant. This concludes the proof. □

Corollary 3. *Let M be a 3-dimensional trans-Sasakian manifold admitting an η-Yamabe soliton (g, V), V being a vector field on M, which is pointwise co-linear with ξ, where ξ being the Reeb vector field of M. V is a Killing vector field iff the soliton reduces to a Yamabe soliton.*

Proof. Using (29), the equation (25) becomes

$$(\pounds_V g)(X, Y) = 2\mu[g(X, Y) - \eta(X)\eta(Y)],$$

for all vector fields X, Y, Z on M. Hence the proof. □

Theorem 3. *Let M be a 3-dimensional trans-Sasakian manifold admitting an η-Yamabe soliton (g, ξ), ξ being the Reeb vector field on M. Then Q and S are parallel along ξ, where Q is the Ricci operator, defined by $S(X, Y) = g(QX, Y)$ and S is the Ricci tensor of M.*

Proof. From the equation (22), we get

$$QX = \Big[\frac{\lambda + \mu}{2} - (\alpha^2 - \beta^2)\Big]X - \Big[\frac{\lambda + \mu}{2} - 3(\alpha^2 - \beta^2)\Big]\eta(X)\xi \tag{30}$$

for any vector field X on M and Q is defined as earlier. We know

$$(\nabla_\xi Q)X = \nabla_\xi QX - Q(\nabla_\xi X) \tag{31}$$

for any vector field X on M. Then using (30), the equation (31) becomes

$$(\nabla_\xi Q)X = -\Big[\frac{\lambda + \mu}{2} - 3(\alpha^2 - \beta^2)\Big]((\nabla_\xi \eta)X)\xi.$$

Using (15) in the above equation, we get $(\nabla_\xi Q)X = 0$, for any vector field X on M. Hence Q is parallel along ξ.

Again from (23), we obtain

$$(\nabla_\xi S)(X, Y) = -\Big[\frac{\lambda + \mu}{2} - 3(\alpha^2 - \beta^2)\Big][\eta(Y)(\nabla_\xi \eta)X + \eta(X)(\nabla_\xi \eta)Y]$$

for any vector fields X, Y on M. Using (15) in the above equation, we get $(\nabla_\xi S)(X, Y) = 0$, for any vector fields X, Y on M. Hence, S is parallel along ξ. □

3 Curvature properties on 3-dimensional trans-Sasakian manifold admitting η-Yamabe soliton

In this section, we have discussed and proved some of the curvature properties on 3-dimensional trans-Sasakian manifold admitting η-Yamabe soliton.

Theorem 4. *A 3-dimensional trans-Sasakian manifold M admitting η-Yamabe soliton (g, ξ), ξ being the Reeb vector field on M, is ξ-projectively flat.*

Proof. From the definition of projective curvature tensor (5), defined on a 3-dimensional trans-Sasakian manifold, using the property $g(QX, Y) = S(X, Y)$, we have

$$P(X,Y)Z = R(X,Y)Z - \frac{1}{2}[S(Y,Z)X - S(X,Z)Y]$$

for any vector fields X, Y, Z on M. Putting $Z = \xi$ in the above equation and using (17) and (22), we obtain

$$P(X,Y)\xi = (\alpha^2 - \beta^2)[\eta(Y)X - \eta(X)Y] - \frac{1}{2}[2(\alpha^2 - \beta^2)\eta(Y)X - 2(\alpha^2 - \beta^2)\eta(X)Y],$$

which implies that $P(X, Y)\xi = 0$. Hence the proof. □

Theorem 5. *A 3-dimensional trans-Sasakian manifold M admitting η-Yamabe soliton (g, ξ), ξ being the Reeb vector field on M, is ξ-concircularly flat iff $\lambda + \mu = 6(\alpha^2 - \beta^2)$.*

Proof. From the definition of concircular curvature tensor (6), defined on a 3-dimensional trans-Sasakian manifold, we have

$$\tilde{C}(X,Y)Z = R(X,Y)Z - \frac{r}{6}[g(Y,Z)X - g(X,Z)Y]$$

for any vector fields X, Y, Z on M. Putting $Z = \xi$ in the above equation and using (12) and (17), we obtain

$$\tilde{C}(X,Y)\xi = (\alpha^2 - \beta^2)[\eta(Y)X - \eta(X)Y] - \frac{r}{6}[\eta(Y)X - \eta(X)Y]. \tag{32}$$

Now using (21), we get

$$\tilde{C}(X,Y)\xi = \left[(\alpha^2 - \beta^2) - \frac{\lambda + \mu}{6}\right][\eta(Y)X - \eta(X)Y].$$

This implies that $\tilde{C}(X, Y)\xi = 0$ iff $\lambda + \mu = 6(\alpha^2 - \beta^2)$. □

Corollary 4. *Let M be a 3-dimensional trans-Sasakian manifold admitting an η-Yamabe soliton (g, ξ), ξ being the Reeb vector field on M. If the manifold is ξ-concircularly flat and the Ricci tensor is η-recurrent, then the manifold M becomes flat.*

Proof. If the Ricci tensor S is η-recurrent, then using (24) in (32), we have the result. □

Theorem 6. *A 3-dimensional trans-Sasakian manifold M admitting η-Yamabe soliton (g, ξ), ξ being the Reeb vector field on M, is ξ-conharmonically flat iff $\lambda + \mu = 0$.*

Proof. From the definition of conharmonic curvature tensor (4), defined on a 3-dimensional trans-Sasakian manifold, we have

$$H(X,Y)Z = R(X,Y)Z - [g(Y,Z)QX - g(X,Z)QY + S(Y,Z)X - S(X,Z)Y]$$

for any vector fields X, Y, Z on M. Putting $Z = \xi$ in the above equation and using (12), (17), (22) and (30), the above equation becomes

$$H(X,Y)\xi = (\alpha^2 - \beta^2)[\eta(Y)X - \eta(X)Y] - \Big[\frac{\lambda+\mu}{2} + (\alpha^2 - \beta^2)\Big][\eta(Y)X - \eta(X)Y].$$

Hence we get

$$H(X,Y)\xi = -\frac{\lambda+\mu}{2}[\eta(Y)X - \eta(X)Y].$$

This implies that $H(X,Y)\xi = 0$ iff $\lambda + \mu = 0$. □

Theorem 7. *A 3-dimensional trans-Sasakian manifold M admitting η-Yamabe soliton (g, ξ), ξ being the Reeb vector field on M, is ξ-quasi-conformally flat iff either $a + b = 0$ or $\lambda + \mu = 6(\alpha^2 - \beta^2)$.*

Proof. From the definition of quasi-conformal curvature tensor (7), defined on a 3-dimensional trans-Sasakian manifold, we have

$$\begin{aligned} C^*(X,Y)Z = {} & aR(X,Y)Z + b[S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY] \\ & - \frac{r}{3}\Big[\frac{a}{2} + 2b\Big][g(Y,Z)X - g(X,Z)Y] \end{aligned}$$

for any vector fields X, Y, Z on M and a, b are constants. Putting $Z = \xi$ in the above equation and using (12), (17), (21), (22) and (30), the above equation becomes

$$\begin{aligned} C^*(X,Y)\xi = {} & a(\alpha^2 - \beta^2)[\eta(Y)X - \eta(X)Y] + b\Big[\frac{\lambda+\mu}{2} + (\alpha^2 - \beta^2)\Big][\eta(Y)X - \eta(X)Y] \\ & - \frac{\lambda+\mu}{3}\Big[\frac{a}{2} + 2b\Big][\eta(Y)X - \eta(X)Y]. \end{aligned}$$

Hence we have

$$C^*(X,Y)\xi = \Big[a(\alpha^2 - \beta^2) + b\Big[\frac{\lambda+\mu}{2} + (\alpha^2 - \beta^2)\Big] - \frac{\lambda+\mu}{3}\Big[\frac{a}{2} + 2b\Big]\Big][\eta(Y)X - \eta(X)Y]. \quad (33)$$

This implies that $C^*(X,Y)\xi = 0$ iff $a(\alpha^2 - \beta^2) + b[\frac{\lambda+\mu}{2} + (\alpha^2 - \beta^2)] - \frac{\lambda+\mu}{3}[\frac{a}{2} + 2b] = 0$. Then by simplifying, we obtain $C^*(X,Y)\xi = 0$ iff $(a+b)[(\alpha^2 - \beta^2) - \frac{\lambda+\mu}{6}] = 0$, i.e. either $a + b = 0$ or $\lambda + \mu = 6(\alpha^2 - \beta^2)$. This concludes the proof. □

Corollary 5. *Let a 3-dimensional trans-Sasakian manifold M admits an η-Yamabe soliton (g, ξ), ξ being the Reeb vector field on M. If the manifold is ξ-quasi-conformally flat and the Ricci tensor is η-recurrent, then the manifold M becomes flat, provided $a + b \neq 0$.*

Proof. If the Ricci tensor S is η-recurrent, then using (24) in (33), we get

$$C^*(X,Y)\xi = -\frac{a+b}{6}(\lambda + \mu)[\eta(Y)X - \eta(X)Y]. \tag{34}$$

Hence using (21) in (34), we have the result. □

Theorem 8. *If a 3-dimensional trans-Sasakian manifold M admitting η-Yamabe soliton (g, ξ), ξ being the Reeb vector field on M, is ξ-semi symmetric, then either $(\alpha^2 - \beta^2) = 0$ or $\lambda + \mu = 6(\alpha^2 - \beta^2)$.*

Proof. We know

$$R(\xi, X) \cdot S = S(R(\xi, X)Y, Z) + S(Y, R(\xi, X)Z) \tag{35}$$

for any vector fields X, Y, Z on M.

Now let the manifold be ξ-semi symmetric, i.e. $R(\xi, X) \cdot S = 0$. Then from (35), we have $S(R(\xi, X)Y, Z) + S(Y, R(\xi, X)Z) = 0$ for any vector fields X, Y, Z on M. Using (18), the last equation becomes

$$S((\alpha^2 - \beta^2)(g(X,Y)\xi - \eta(Y)X), Z) + S(Y, (\alpha^2 - \beta^2)(g(X,Z)\xi - \eta(Z)X)) = 0.$$

Replacing the expression of S from (22) and simplifying we get

$$(\alpha^2 - \beta^2)\Big[\frac{\lambda + \mu}{2} - 3(\alpha^2 - \beta^2)\Big][g(X,Y)\eta(Z) + g(X,Z)\eta(Y) - 2\eta(X)\eta(Y)\eta(Z)] = 0.$$

Taking $Z = \xi$ in the above equation and using (9), (12), we obtain

$$(\alpha^2 - \beta^2)\Big[\frac{\lambda + \mu}{2} - 3(\alpha^2 - \beta^2)\Big][g(X,Y) - \eta(x)\eta(Y)] = 0$$

for any vector fields X, Y on M. Using (10), the above equation becomes

$$(\alpha^2 - \beta^2)\Big[\frac{\lambda + \mu}{2} - 3(\alpha^2 - \beta^2)\Big]g(\phi X, \phi Y) = 0$$

for any vector fields X, Y on M. Hence we get $(\alpha^2 - \beta^2)\Big[\frac{\lambda+\mu}{2} - 3(\alpha^2 - \beta^2)\Big] = 0$. Then either $(\alpha^2 - \beta^2) = 0$ or $\lambda + \mu = 6(\alpha^2 - \beta^2)$. □

Theorem 9. *If a 3-dimensional trans-Sasakian manifold M admits an η-Yamabe soliton (g, ξ), ξ being the Reeb vector field on M and satisfies $W_2(\xi, X) \cdot S = 0$, where W_2 is the W_2-curvature tensor and S is the Ricci tensor, then either $\lambda + \mu = 2(\alpha^2 - \beta^2)$ or $\lambda + \mu = 6(\alpha^2 - \beta^2)$.*

Proof. From the definition of W_2-curvature tensor (8), defined on a 3-dimensional trans-Sasakian manifold, we have

$$W_2(X,Y)Z = R(X,Y)Z + \frac{1}{2}[g(X,Z)QY - g(Y,Z)QX] \tag{36}$$

for any vector fields X, Y, Z on M.

Again we know, $W_2(\xi, X)\cdot S = S(W_2(\xi, X)Y, Z) + S(Y, W_2(\xi, X)Z)$ for any vector fields X, Y, Z on M. Replacing the expression of S from (22), on simplifying we get

$$W_2(\xi, X)\cdot S = \Big[\frac{\lambda+\mu}{2} - (\alpha^2-\beta^2)\Big][g(W_2(\xi, X)Y, Z) + g(Y, W_2(\xi, X)Z)] \\ - \Big[\frac{\lambda+\mu}{2} - 3(\alpha^2-\beta^2)\Big][\eta(W_2(\xi, X)Y)\eta(Z) + \eta(Y)\eta(W_2(\xi, X)Z)].$$

Now, from the definition of W_2-curvature tensor (36) and then by using (18), the property $g(QX, Y) = S(X, Y)$ and (22), the above equation becomes

$$W_2(\xi, X)\cdot S = \frac{1}{2}\Big[\frac{\lambda+\mu}{2} - (\alpha^2-\beta^2)\Big]\Big[\frac{\lambda+\mu}{2} - 3(\alpha^2-\beta^2)\Big] \\ \times [g(X, Y)\eta(Z) + g(X, Z)\eta(Y) - 2\eta(X)\eta(Y)\eta(Z)]$$

for any vector fields X, Y, Z on M. Let in this manifold M, $W_2(\xi, X)\cdot S = 0$. Then from the above equation, we get

$$\Big[\frac{\lambda+\mu}{2} - (\alpha^2-\beta^2)\Big]\Big[\frac{\lambda+\mu}{2} - 3(\alpha^2-\beta^2)\Big][g(X, Y)\eta(Z) + g(X, Z)\eta(Y) - 2\eta(X)\eta(Y)\eta(Z)] = 0,$$

for any vector fields X, Y, Z on M. Taking $Z = \xi$ in the above equation and using (9), (12), we obtain

$$\Big[\frac{\lambda+\mu}{2} - (\alpha^2-\beta^2)\Big]\Big[\frac{\lambda+\mu}{2} - 3(\alpha^2-\beta^2)\Big][g(X, Y) - \eta(X)\eta(Y)] = 0$$

for any vector fields X, Y on M. Using (10), the above equation becomes

$$\Big[\frac{\lambda+\mu}{2} - (\alpha^2-\beta^2)\Big]\Big[\frac{\lambda+\mu}{2} - 3(\alpha^2-\beta^2)\Big]g(\phi X, \phi Y) = 0$$

for any vector fields X, Y on M. Hence we get,

$$\Big[\frac{\lambda+\mu}{2} - (\alpha^2-\beta^2)\Big]\Big[\frac{\lambda+\mu}{2} - 3(\alpha^2-\beta^2)\Big] = 0. \tag{37}$$

Then either $\lambda+\mu = 2(\alpha^2-\beta^2)$ or $\lambda+\mu = 6(\alpha^2-\beta^2)$. □

Corollary 6. *If a 3-dimensional trans-Sasakian manifold M admits an η-Yamabe soliton (g, ξ), ξ being the Reeb vector field on M and satisfies $W_2(\xi, X)\cdot S = 0$, where W_2 is the W_2-curvature tensor and S is the Ricci tensor which is η- recurrent, then the manifold becomes flat.*

Proof. If the Ricci tensor S is η-recurrent then using (24) in (37) and from (21), we have the result. □

4 Example of a 3-dimensional trans-Sasakian manifold admitting η-Yamabe soliton

In this section, we give an example of a 3-dimensional trans-Sasakian manifold with α, β being constants. We consider the 3-dimensional manifold $M = \{(x, y, z) \in \mathbb{R}^3, z \neq 0\}$, where (x, y, z) are standard coordinates in $\mathbb{R}^3$. Let e_1, e_2, e_3 be a linearly independent system of vector fields on M given by

$$e_1 = z\frac{\partial}{\partial x}, \quad e_2 = z\frac{\partial}{\partial y}, \quad e_3 = z\frac{\partial}{\partial z}.$$

Let g be the Riemannian metric defined by

$$g(e_1,e_1)=g(e_2,e_2)=g(e_3,e_3)=1, \qquad g(e_1,e_2)=g(e_2,e_3)=g(e_3,e_1)=0.$$

Let η be the 1-form defined by $\eta(Z)=g(Z,e_3)$ for any $Z\in\chi(M)$, where $\chi(M)$ is the set of all differentiable vector fields on M and ϕ be the (1,1)-tensor field defined by $\phi e_1=-e_2$, $\phi e_2=e_1$, $\phi e_3=0$. Then, using the linearity of ϕ and g, we have

$$\eta(e_3)=1, \quad \phi^2(Z)=-Z+\eta(Z)e_3 \quad \text{and} \quad g(\phi Z,\phi W)=g(Z,W)-\eta(Z)\eta(W)$$

for any $Z,W\in\chi(M)$. Let ∇ be the Levi-Civita connection with respect to the Riemannian metric g. Then we have $[e_1,e_2]=0$, $[e_2,e_3]=-e_2$, $[e_1,e_3]=-e_1$. The connection ∇ of the metric g is given by

$$2g(\nabla_X Y,Z)=Xg(Y,Z)+Yg(Z,X)-Zg(X,Y)-g(X,[Y,Z])-g(Y,[X,Z])+g(Z,[X,Y]),$$

which is known as Koszul's formula. Using Koszul's formula, we can easily calculate,

$$\begin{gathered}\nabla_{e_1}e_3=-e_1, \quad \nabla_{e_2}e_3=-e_2, \quad \nabla_{e_3}e_3=0,\\ \nabla_{e_1}e_1=e_3, \quad \nabla_{e_2}e_1=0, \quad \nabla_{e_3}e_1=0,\\ \nabla_{e_1}e_2=0, \quad \nabla_{e_2}e_2=e_3, \quad \nabla_{e_3}e_2=0.\end{gathered}$$

We see that

$$\begin{aligned}(\nabla_{e_1}\phi)e_1&=\nabla_{e_1}\phi e_1-\phi\nabla_{e_1}e_1=-\nabla_{e_1}e_2-\phi e_3=0\\ &=0(g(e_1,e_1)e_3-\eta(e_1)e_1)-1(g(\phi e_1,e_1)e_3-\eta(e_1)\phi e_1).\end{aligned} \tag{38}$$

$$\begin{aligned}(\nabla_{e_1}\phi)e_2&=\nabla_{e_1}\phi e_2-\phi\nabla_{e_1}e_2=\nabla_{e_1}e_1-0=e_3\\ &=0(g(e_1,e_2)e_3-\eta(e_2)e_1)-1(g(\phi e_1,e_2)e_3-\eta(e_2)\phi e_1).\end{aligned} \tag{39}$$

$$\begin{aligned}(\nabla_{e_1}\phi)e_3&=\nabla_{e_1}\phi e_3-\phi\nabla_{e_1}e_3=0+\phi e_1=-e_2\\ &=0(g(e_1,e_3)e_3-\eta(e_3)e_1)-1(g(\phi e_1,e_3)e_3-\eta(e_3)\phi e_1).\end{aligned} \tag{40}$$

Hence from (38), (39) and (40) we can see that the manifold M satisfies (13) for $X=e_1$, $\alpha=0$, $\beta=-1$ and $e_3=\xi$. Similarly, it can be shown that for $X=e_2$ and $X=e_3$ the manifold also satisfies (13) for $\alpha=0$, $\beta=-1$ and $e_3=\xi$.

Hence the manifold M is a 3-dimensional trans-Sasakian manifold of type $(0,-1)$. Also, from the definition of the Riemannian curvature tensor R (3), we get

$$\begin{gathered}R(e_1,e_2)e_2=-e_1, \quad R(e_1,e_3)e_3=-e_1, \quad R(e_2,e_1)e_1=-e_2,\\ R(e_2,e_3)e_3=-e_2, \quad R(e_3,e_1)e_1=-e_3, \quad R(e_3,e_2)e_2=-e_3.\end{gathered}$$

Then the Ricci tensor S is given by

$$S(e_1,e_1)=-2, \quad S(e_2,e_2)=-2, \quad S(e_3,e_3)=-2. \tag{41}$$

Then the scalar curvature is $r=-6$. From (22), we have

$$S(e_1,e_1)=\frac{\lambda+\mu}{2}-(\alpha^2-\beta^2), \quad S(e_2,e_2)=\frac{\lambda+\mu}{2}-(\alpha^2-\beta^2), \quad S(e_3,e_3)=2(\alpha^2-\beta^2). \tag{42}$$

Then from (41) and (42), we get $\frac{\lambda+\mu}{2}-(\alpha^2-\beta^2)=-2$ and $\alpha^2-\beta^2=-1$. This implies the equality $\lambda+\mu=-6$. Then the value of $\lambda+\mu$ is same as the value of r and so it satisfies Theorem 1. Hence g defines an η-Yamabe soliton on a 3-dmensional trans-Sasakian manifold M.

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Рой С., Дей С., Бхаттачарія А. *Деякі результати про η-Ямабе солітони у трьохвимірному транс-Сасакяновому многовиді* // Карпатські матем. публ. — 2022. — Т.14, №1. — С. 158–170.

Метою цієї статті є вивчення деяких властивостей трьохвимірного транс-Сасакянового многовиду, чиєю метрикою є η-Ямабе солітон. Ми вивчили деякі умови кривизни трьохвимірного транс-Сасакянового многовиду, що допускає η-Ямабе солітон. Нарешті, ми будуємо трьохвимірний транс-Сасакяновий многовид, що задовольняє η-Ямабе солітон.

Ключові слова і фрази: солітон Ямабе, η-Ямабе солітон, η-айнштайнівський многовид, транс-Сасакяновий многовид.