ISSN 2075-9827 e-ISSN 2313-0210
Carpathian Math. Publ. 2021, **13** (1), 259–271
doi:10.15330/cmp.13.1.259-271

https://journals.pnu.edu.ua/index.php/cmp
Карпатські матем. публ. 2021, Т.13, №1, С.259–271

On k-Fibonacci balancing and k-Fibonacci Lucas-balancing numbers

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The balancing number n and the balancer r are solution of the Diophantine equation

$$1+2+\cdots+(n-1)=(n+1)+(n+2)+\cdots+(n+r).$$

It is well known that if n is balancing number, then $8n^2+1$ is a perfect square and its positive square root is called a Lucas-balancing number. For an integer $k\geq 2$, let $(F_n^{(k)})_n$ be the k-generalized Fibonacci sequence, which starts with $0,\ldots,0,1,1$ (k terms) and each term afterwards is the sum of the k preceding terms. The purpose of this paper is to show that 1, 6930 are the only balancing numbers and 1, 3 are the only Lucas-balancing numbers, which are a term of k-generalized Fibonacci sequence. This generalizes the result from [Fibonacci Quart. 2004, **42** (4), 330–340].

Key words and phrases: k-generalized Fibonacci numbers, balancing numbers, Lucas-balancing numbers, linear form in logarithms, reduction method.

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1 Introduction

The first definition of balancing numbers is essentially due to R.P. Finkelstein [8], although he called them numerical centers. A positive integer n is called balancing number if

$$1+2+\cdots+(n-1)=(n+1)+(n+2)+\cdots+(n+r)$$

holds for some positive integer r. Then r is called balancer corresponding to the balancing number n. The n-th term of the sequence of balancing numbers is denoted by B_n. A. Behera and G.K. Panda [2] proved that the balancing numbers fulfill the recurrence relation

$$B_0=1,\quad B_1=6,\quad B_n=6B_{n-1}-B_{n-2}\quad\text{for all}\quad n\geq 2.$$

It is well known that if n is a balancing number, then $8n^2+1$ is a perfect square, and the positive square root of $8n^2+1$ is called a Lucas-balancing number which is denoted by C_n (see [13]). The Lucas-balancing numbers C_n satisfy the recurrence relation

$$C_0=1,\quad C_1=3,\quad C_n=6C_{n-1}-C_{n-2}\quad\text{for all}\quad n\geq 2.$$

The balancing and Lucas-balancing numbers are indexed in *The On-Line Encyclopedia of Integer Sequences* (OEIS) as A001109 and A001541, respectively.

УДК 511.176
2020 *Mathematics Subject Classification:* 11B39, 11J86.

The Fibonacci sequence $(F_n)_{n\geq 0}$ is given by

$$F_0 = 0, \quad F_1 = 1, \quad F_n = F_{n-1} + F_{n-2} \quad \text{for all} \quad n \geq 2.$$

It is the sequence A000045 in OEIS.

A balancing number is called *Fibonacci balancing number* if it is a Fibonacci number (see [9]). In [9], K. Liptai has shown that 1 is the only Fibonacci balancing number.

Let $k \geq 2$ be an integer. We consider a generalization of Fibonacci sequence called the k-generalized Fibonacci sequence $F_n^{(k)}$ defined as

$$F_n^{(k)} = F_{n-1}^{(k)} + F_{n-2}^{(k)} + \cdots + F_{n-k}^{(k)} \quad \text{for all} \quad n \geq 2,$$

with the initial conditions $F_{-(k-2)}^{(k)} = F_{-(k-3)}^{(k)} = \cdots = F_0^{(k)} = 0$ and $F_1^{(k)} = 1$. If $k = 2$, we obtain the classical Fibonacci sequence. Below we present the values of these numbers for the first few values of k and $n \geq 1$. Note that the underlying terms are balancing or Lucas-balancing numbers.

k	Name	First non-zero terms
2	Fibonacci	$\underline{1}$, $\underline{1}$, 2, $\underline{3}$, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, ...
3	Tribonacci	$\underline{1}$, $\underline{1}$, 2, 4, 7, 13, 24, 44, 81, 149, 274, 504, 927, 1705, 3136, ...
4	Tetranacci	$\underline{1}$, $\underline{1}$, 2, 4, 8, 15, 29, 56, 108, 208, 401, 773, 1490, 2872, 5536, ...
5	Pentanacci	$\underline{1}$, $\underline{1}$, 2, 4, 8, 16, 31, 61, 120, 236, 464, 912, 1793, 3525, $\underline{6930}$, ...
6	Hexanacci	$\underline{1}$, $\underline{1}$, 2, 4, 8, 16, 32, 63, 125, 248, 492, 976, 1936, 3840, 7617, ...
7	Heptanacci	$\underline{1}$, $\underline{1}$, 2, 4, 8, 16, 32, 64, 127, 253, 504, 1004, 2000, 3984, 7936, ...
8	Octanacci	$\underline{1}$, $\underline{1}$, 2, 4, 8, 16, 32, 64, 128, 255, 509, 1016, 2028, 4048, 8080, ...
9	Nonanacci	$\underline{1}$, $\underline{1}$, 2, 4, 8, 16, 32, 64, 128, 256, 511, 1021, 2040, 4076, 8144, ...
10	Decanacci	$\underline{1}$, $\underline{1}$, 2, 4, 8,16, 32, 64, 128, 256, 512, 1023, 2045, 4088, 8172, ...

We say that a balancing number (Lucas-balancing number) is *k-Fibonacci balancing number* (*k-Fibonacci Lucas-balancing number*) if it is k-Fibonacci number too. The aim of the present work is to determine all the k-Fibonacci balancing and k-Fibonacci Lucas-balancing numbers. We prove the following results.

Theorem 1. *1 and 6930 are the only k-Fibonacci balancing number. Moreover, all the solutions of the Diophantine equation*

$$F_n^{(k)} = B_m \tag{1}$$

are given by $(n, k, m) = (1, k, 0), (2, k, 0), (15, 5, 6)$.

Theorem 2. *1 and 3 are the only k-Fibonacci Lucas-balancing number. Moreover, all the solutions of the Diophantine equation*

$$F_n^{(k)} = C_m \tag{2}$$

are given by $(n, k, m) = (1, k, 0), (2, k, 0), (4, 2, 1)$.

Our proofs of Theorems 1 and 2 are mainly based on linear forms in logarithms of algebraic numbers and a reduction algorithm originaly introduced by A. Baker and H. Davenport in [1]. Here, we use a version due to A. Dujella and A. Pethő in [7, Lemma 5 (a)].

2 Premilmeries and known results

This section is devoted to collect a few definitions, notations and theorems, which will be used in the rest of this work.

2.1 Linear forms in logarithms

For any non-zero algebraic number η of degree d over $\mathbb{Q}$, whose minimal polynomial over $\mathbb{Z}$ is $a\prod_{j=1}^{d}\left(X-\eta^{(j)}\right)$, we denote by

$$h(\eta)=\frac{1}{d}\left(\log|a|+\sum_{j=1}^{d}\log\max\left\{1,|\eta^{(j)}|\right\}\right)$$

the usual absolute logarithmic height of η. In particular, if $\eta=p/q$ is a rational number with $\gcd(p,q)=1$ and $q>0$, then $h(\eta)=\log\max\{|p|,q\}$. The following properties of the logarithmic height function $h(\cdot)$, which will be used in the next sections without special reference, are also known:

$$\begin{aligned} h(\eta\pm\gamma) &\leq h(\eta)+h(\gamma)+\log 2, \\ h(\eta\gamma^{\pm1}) &\leq h(\eta)+h(\gamma), \qquad (3)\\ h(\eta^{s}) &= |s|\,h(\eta), \quad s\in\mathbb{Z}. \qquad (4)\end{aligned}$$

The main approach to show Theorems 1 and 2 is the Baker's theory about lower bounds for linear forms in logarithms. In [10], E.M. Matveev proved the following theorem.

Theorem 3 ([10]). *Let $\eta_1,\ldots,\eta_s$ be a real algebraic numbers and let $b_1,\ldots,b_s$ be nonzero rational integer numbers. Let $d_{\mathbb{K}}$ be the degree of the number field $\mathbb{Q}(\eta_1,\ldots,\eta_s)$ over $\mathbb{Q}$. Define*

$$\Gamma:=\eta_1^{b_1}\cdots\eta_s^{b_s}-1.$$

If $\Gamma\neq 0$, then

$$|\Gamma|\geq\exp(-1.4\cdot 30^{s+3}s^{4.5}d_{\mathbb{K}}^{2}(1+\log d_{\mathbb{K}})(1+\log B)A_1\cdots A_s),$$

where $A_j=\max\{d_{\mathbb{K}}h(\eta),|\log\eta|,0.16\}$ for $j=1,\ldots,s$, and $B\geq\max\{|b_1|,\ldots,|b_s|\}$.

2.2 The de Weger reduction algorithm

Here, we present a variant of the reduction method of Baker and Davenport due to de Weger [14].

Let $\vartheta_1,\vartheta_2,\beta\in\mathbb{R}$ be given and let $x_1,x_2\in\mathbb{Z}$ be unknowns. Let

$$\Lambda=\beta+x_1\vartheta_1+x_2\vartheta_2. \qquad (5)$$

Set $X=\max\{|x_1|,|x_2|\}$. Let X_0,Y be positive. Assume that

$$|\Lambda|<c\exp(-\rho Y) \qquad (6)$$

and

$$Y\leq X\leq X_0, \qquad (7)$$

where c,ρ be positive constants.

When $\beta=0$ in (5), we get $\Lambda=x_1\vartheta_1+x_2\vartheta_2$. Put $\vartheta=-\vartheta_1/\vartheta_2$. We assume that x_1 and x_2 are coprime. Let the continued fraction expansion of ϑ be given by $[a_0,a_1,a_2,\ldots]$, and let the kth convergent of ϑ be p_k/q_k for $k=0,1,2,\ldots$. We may assume without loss of generality that $|\vartheta_1|<|\vartheta_2|$ and that $x_1>0$. We have the following results.

Lemma 1 ([14, Lemma 3.1]). *If* (6) *and* (7) *hold for* x_1, x_2 *with* $X \geq 1$ *and* $\beta = 0$*, then* $(-x_2, x_1) = (p_k, q_k)$ *for an index* k *that satisfies*

$$k \leq -1 + \frac{\log(1 + X_0\sqrt{5})}{\log\left(\frac{1+\sqrt{5}}{2}\right)} := Y_0.$$

Lemma 2 ([14, Lemma 3.2]). *Let* $A = \max_{0 \leq k \leq Y_0} a_{k+1}$*. If* (6) *and* (7) *hold for* x_1*,* x_2 *with* $X \geq 1$ *and* $\beta = 0$*, then*

$$Y < \frac{1}{\rho}\log\left(\frac{c(A+2)}{|\vartheta_2|}\right) + \frac{1}{\rho}\log X < \frac{1}{\rho}\log\left(\frac{c(A+2)X_0}{|\vartheta_2|}\right).$$

When $\beta \neq 0$ in (5), put $\vartheta = -\vartheta_1/\vartheta_2$ and $\psi = \beta/\vartheta_2$. Then we have $\frac{\Lambda}{\vartheta_2} = \psi - x_1\vartheta + x_2$. Let p/q be a convergent of ϑ with $q > X_0$. For a real number x we let

$$\|x\| = \min\{|x - n| : n \in \mathbb{Z}\}$$

be the distance from x to the nearest integer. We have the following result.

Lemma 3 ([14, Lemma 3.3]). *Suppose that* $\| \ q\psi \ \| > \frac{2X_0}{q}$*. Then, the solutions of* (6) *and* (7) *satisfy*

$$Y < \frac{1}{\rho}\log\left(\frac{q^2 c}{|\vartheta_2| X_0}\right).$$

2.3 The balancing and Lucas-balancing sequence

Let $\delta := (3 + 2\sqrt{2})$ and $\overline{\delta} := (3 - 2\sqrt{2})$ be the roots of the characteristic equation $x^2 - 6x + 1$ of both the balancing and Lucas-balancing sequences, the Binet formulas

$$B_n = \frac{\delta^n - \overline{\delta}^n}{4\sqrt{2}} \tag{8}$$

and

$$C_n = \frac{\delta^n + \overline{\delta}^n}{2} \tag{9}$$

hold for all nonnegative integer n's. Furthermore, the inequalities

$$\delta^{n-1} < B_n < \delta^n \tag{10}$$

and

$$\delta^{n-1} < C_n < \delta^n \tag{11}$$

hold for all $n \geq 1$.

2.4 Properties of k-generalized Fibonacci sequence

In this subsection, we recall some facts and properties of the k-generalized Fibonacci sequence which will be used later. The characteristic polynomial of the k-generalized Fibonacci numbers $(F_n^{(k)})_n$ is

$$\Psi_k(x) = x^k - x^{k-1} - \cdots - x - 1.$$

$\Psi_k(x)$ is irreducible over $\mathbb{Q}[x]$ and has just one root $\alpha(k)$ outside the unit circle (see, for example, [11, 12, 15]). It is real and positive, so it satisfies $\alpha(k) > 1$. The other root are strictly inside the unit circle. Furthermore, in [15] D.A. Wolfram showed that

$$2(1 - 2^{-k}) < \alpha(k) < 2 \quad \text{for all} \quad k \geq 2. \tag{12}$$

To simplify the notation, in general, we omit the dependence on k of ϕ. For $s \geq 2$, let

$$f_s(x) := \frac{x-1}{2 + (s+1)(x-2)}.$$

In [6], G.P.B. Dresden, Z. Du gave the Binet-type formula

$$F_n^{(k)} = \sum_{i=1}^{k} f_k(\alpha_i)\alpha_i^{n-1},$$

where α_i are the zeros of $\Psi_k(x)$, and proved that

$$\left| F_n^{(k)} - f_k(\alpha)\alpha^{n-1} \right| < \frac{1}{2} \tag{13}$$

hold for all $n \geq k - 2$. Furthermore, it was showed in [3] that

$$\alpha^{n-2} \leq F_n^{(k)} \leq \alpha^{n-1} \tag{14}$$

hold for all $n \geq 1$.

In [4], J.J. Bravo, C.A. Gómez and F. Luca proved that $1/2 < f_k(\alpha) < 3/4$ and $|f_k(\alpha_i)| < 1$, $2 \leq i \leq k$, hold. So, the number $f_k(\alpha)$ is not an algebraic integer. In addition, they proved that the logarithmic height of f satisfies

$$h(f_k(\alpha)) < \log(k+1) + \log 4 \quad \text{for all} \quad k \geq 2. \tag{15}$$

Finally, in [5, pp. 542, 543] the authors proved that for all $n \geq k + 2$ we have

$$F_n^{(k)} = 2^{n-2}(1 + \zeta), \quad \text{where } |\zeta| < \frac{1}{2^{k/2}}. \tag{16}$$

3 k-Fibonacci balancing numbers

This section is devoted to show Theorem 1.

3.1 An inequality for n and m versus k

If $2 \leq n \leq k + 1$, we have $F_n^{(k)} = 2^{n-2}$ and since 1 is the only perfect power in the balancing sequence, we deduce that equation (1) has only the solution $(n, k, m) = (2, k, 0)$ in this range. The fact that $F_1^{(k)} = F_2^{(k)}$ imply that $(1, k, 0)$ is also a solution of the Diophantine equation (1). From now, we assume that $n \geq k + 2$. Further we may suppose that $k \geq 3$ because that case $k = 2$ is already studied.

Using inequalities (14) and (10), we get from equation (1) that

$$\alpha^{n-2} \leq \delta^{m-1} \quad \text{and} \quad \delta^{m-2} \leq \alpha^{n-1}.$$

The above inequalities give

$$(n-2)\left(\frac{\log \alpha}{\log \delta}\right) + 1 \leq m \leq (n-1)\left(\frac{\log \alpha}{\log \delta}\right) + 2.$$

Using the fact that $7/4 < \alpha < 2$ for all $k \geq 3$ (see (12)), we deduce that

$$0.3n - 0.6 < m < 0.4n + 1.7. \tag{17}$$

Lemma 4. *If* (n, k, m) *is a solution in integers of equation* (1) *with* $k \geq 3$ *and* $n \geq k+2$*, then the inequalities* $2.4m < n < 6.8 \cdot 10^{15} k^4 \log^3 k$ *hold.*

Proof. From equation (1), estimate (13) and identity (8), we have

$$\left| f_k(\alpha)\alpha^{n-1} - \frac{\delta^m}{4\sqrt{2}} \right| < \frac{1}{2} + \frac{1}{4\sqrt{2}}.$$

If we multiply through by $4\sqrt{2}\delta^{-m}$ we arrive at

$$|\Gamma_1| < 3.9\delta^{-m}, \tag{18}$$

where $\Gamma_1 = (4\sqrt{2}f_k(\alpha))\alpha^{n-1}\delta^{-m} - 1$.

With the aim of applying Theorem 3 we choose

$$(\eta_1, b_1) := (4\sqrt{2}f_k(\alpha), 1), \quad (\eta_2, b_2) := (\alpha, n-1), \quad (\eta_3, b_3) := (\delta, -m).$$

For this choice, the field $\mathbb{K} := \mathbb{Q}(\alpha, \sqrt{2})$ contains η_1, η_2, η_3 and has $d_{\mathbb{K}} \leq 2k$. Since $h(\eta_2) = (\log\alpha)/k < (\log 2)/k$ and $h(\eta_3) = (\log\delta)/2$, we deduce that

$$\max\{2kh(\eta_2), |\log\eta_2|, 0.16\} = 2\log 2 := A_2$$

and

$$\max\{2kh(\eta_3), |\log\eta_3|, 0.16\} = k\log\delta := A_3.$$

On the other hand, by using the estimate (15) and the proprieties (3) together with (4), it follows that for all $k \geq 3$

$$h(\eta_1) \leq h(f_k(\alpha)) + h(4\sqrt{2}) < \log(k+1) + \log 4 + \log(4\sqrt{2}) < 4.2\log k.$$

Thus, we obtain

$$\max\{2kh(\eta_1), |\log\eta_1|, 0.16\} < 8.4k\log k := A_1.$$

The fact that $0.4n + 1.7 < n$ hold for all $n \geq 5$ and the inequality (17), imply that we can take $B := n$.

Before applying Theorem 3, we need to check that $\Gamma_1 \neq 0$. Indeed, if we assume that $\Gamma_1 = 0$, we get that

$$f_k(\alpha) = \frac{\delta^m}{4\sqrt{2}}\alpha^{-n+1},$$

and so $f_k(\alpha)$ would be an algebraic integer, contradicting some thing previously mentioned. Thus, $\Gamma_1 \neq 0$. Therefore, by Theorem 3, it result

$$|\Gamma_1| > \exp\Big(-1.432 \cdot 10^{11}(2k)^2(1+\log(2k))(1+\log n)(8.4k\log k)(2\log 2)(k\log\delta)\Big). \tag{19}$$

When we compare the lower bound (19) and the upper bound (18) of $|\Gamma_1|$ we obtain

$$m\log\delta - \log 3.9 < 1.18 \cdot 10^{13}k^4\log k(1+\log 2k)(1+\log n),$$

taking into account the facts $1 + \log 2k < 2.6\log k$ and $1 + \log n < 1.7\log n$ which hold for $k \geq 3$ and $n \geq 5$, we conclude that $m < 3 \cdot 10^{13}k^4\log^2 k\log n$. By the inequality (17), the last inequality becomes

$$\frac{n}{\log n} < 10^{14}k^4\log^2 k. \tag{20}$$

Since the function $x \mapsto x/\log x$ is increasing for all $x > e$, it is easy to check that

$$\frac{x}{\log x} < T \quad \Longrightarrow \quad x < 2T\log T \quad \text{whenever} \quad T \geq 3. \tag{21}$$

Thus, fixing $T := 10^{14}k^4\log^2 k$, inequality (20) together with $32.3 + 4\log k + 2\log\log k < 34\log k$, which holds for all $k \geq 2$, gives

$$\begin{aligned} n &< (2\cdot 10^{14}k^4\log^2 k)\log(10^{14}k^4\log^2 k) \\ &< (2\cdot 10^{14}k^4\log^2 k)(32.3 + 4\log k + 2\log\log k) < 6.8\cdot 10^{15}k^4\log^3 k. \end{aligned}$$

Whence the result. □

3.2 The case $3 \leq k \leq 220$

In this subsection, we treat the case $k \in [3, 220]$. We show the following result.

Lemma 5. *The Diophantine equation* (1) *has no solution, when* $k \in [3, 220]$ *and* $n \geq k+2$.

Proof. Let us set

$$\Lambda_1 = \log(\Gamma_1 + 1) = (n-1)\log\alpha - m\log\delta + \log(4\sqrt{2}f_k(\alpha)).$$

Then, (18) can be rewritten as

$$\left|e^{\Lambda_1} - 1\right| < 3.9\delta^{-m}. \tag{22}$$

Note that $\Lambda_1 \neq 0$, since $\Gamma_1 \neq 0$, so we distinguish the following cases. If $\Lambda_1 > 0$, then $e^{\Lambda_1} - 1 > 0$. Using the fact that $x \leq e^x - 1$ for all $x \in \mathbb{R}$, from (22) we obtain $0 < \Lambda_1 < 3.9\delta^{-m}$. Now, if $\Lambda_1 < 0$, it is easy to see that $3.9\delta^{-m} < 1/2$ holds for all $m \geq 4$. Thus, from (22) we have that $\left|e^{\Lambda_1} - 1\right| < 1/2$ and therefore $e^{|\Lambda_1|} < 2$. Since $\Lambda_1 < 0$, we have

$$0 < |\Lambda_1| \leq e^{|\Lambda_1|} - 1 = e^{|\Lambda_1|}\left|e^{\Lambda_1} - 1\right| < 7.8\delta^{-m}.$$

Hence, in both cases one has

$$0 < |\Lambda_1| < 7.8\delta^{-m}. \tag{23}$$

In order to apply Lemma 3, we fix

$$c := 7.8, \quad \rho := 1.76, \quad \psi := \frac{\log(4\sqrt{2}f_k(\alpha))}{\log\delta},$$

$$\vartheta := \frac{\log\delta}{\log\alpha}, \quad \vartheta_1 := -\log\delta, \quad \vartheta_2 := \log\alpha, \quad \beta := \log(4\sqrt{2}f_k(\alpha)).$$

For each $k \in [3, 220]$, we find a good approximation of α and a convergent p_ℓ/q_ℓ of the continued fraction of ϑ such that $q_\ell > X_0$, where $X_0 = \lfloor 6.8\cdot 10^{15}k^4\log^3 k\rfloor$, which is an upper bound of $\max\{n-1, m\}$ from Lemma 4. After doing this, we use Lemma 3 on inequality (23). A computer search with Mathematica revealed that the maximum value of $\left\lfloor \frac{1}{\delta}\log(q^2c/|\vartheta_2|\,X_0)\right\rfloor$ over all $k \in [3, 220]$ is $45.6224\ldots$, which according to Lemma 3, is an upper bound on m. Hence, we deduce that the possible solutions (m, n, k) of the equation (1) for which $k \in [3, 220]$ have $m \leq 45$, therefore we use inequalities (17) to obtain $n \leq 151$.

Finally, we used Mathematica to compare $F_n^{(k)}$ and B_m for the range $5 \leq n \leq 151$ and $2 \leq m \leq 45$, with $m < n/2.4$ and checked that the only solution of the equation (1) is $6930 = B_6 = F_{15}^{(5)}$. □

3.3 The case $k > 220$

In this subsection, we analyze the case $k > 220$.

Lemma 6. *The Diophantine equation* (1) *has no solution when* $k > 220$ *and* $n \geq k+2$.

Proof. For $k > 220$ we have $2.4m < n < 6.8 \cdot 10^{15} k^4 \log^3 k < 2^{k/2}$. Using (8) and (16), we express the equation (1) as

$$2^{n-2} - \frac{\delta^m}{4\sqrt{2}} = 2^{n-2}\zeta - \frac{\overline{\delta}^m}{4\sqrt{2}},$$

by taking absolute value we obtain

$$\left|2^{n-2} - \frac{\delta^m}{4\sqrt{2}}\right| < \frac{2^{n-2}}{2^{k/2}} + \frac{1}{4\sqrt{2}},$$

which gives

$$\left|1 - (\sqrt{2})^{-1}2^{-n}\delta^m\right| < \frac{1.1}{2^{k/2}}, \tag{24}$$

where we have used the fact $1/(\sqrt{2}\cdot 2^n) < 0.1/2^{k/2}$, because $n \geq k+2$. We will apply Theorem 3 to obtain a lower bound to the left-hand side of inequality (24). Choose

$$t := 3, \quad (\eta_1, b_1) := (\sqrt{2}, -1), \quad (\eta_2, b_2) := (2, -n), \quad (\eta_3, b_3) := (\delta, m).$$

Since $\eta_1, \eta_2, \eta_3 \in \mathbb{K} := \mathbb{Q}(\sqrt{2})$, then $d_{\mathbb{K}} = 2$. The left-hand side of (24) is not zero. Indeed, if this is zero, we would then get that δ^{2m} is a rational numbers, which is impossible for all positive integers m.

We can choose $B := n$, because $m \leq n$. On the other hand, since

$$h(\eta_1) = \log(\sqrt{2}), \quad h(\eta_2) = \log 2, \quad h(\eta_3) = (\log\delta)/2,$$

we deduce that

$$\max\{2h(\eta_1), |\log\eta_1|, 0.16\} = \log 2 := A_1, \qquad \max\{2h(\eta_2), |\log\eta_2|, 0.16\} = 2\log 2 := A_2$$

and

$$\max\{2h(\eta_3), |\log\eta_3|, 0.16\} = \log\delta := A_3$$

Therefore, according to Theorem 3 we have

$$\left|1 - (\sqrt{2})^{-1}2^{-n}\delta^m\right| > \exp\left(-2.81\cdot 10^{12}\log n\right), \tag{25}$$

where we have used the fact that $1 + \log n < 1.7\log n$ for all $n \geq 5$. Comparing of (24) and (25) gives $k < 8.2 \cdot 10^{12} \log n$.

From Lemma 4 and the fact that $36.5 + 4\log k + 3\log\log k < 11.8\log k$ for all $k > 220$, we obtain

$$\begin{aligned} k &< 8.2\cdot 10^{12}\log(6.8\cdot 10^{15}k^4\log^3 k) \\ &< 8.2\cdot 10^{12}\log(36.5 + 4\log k + 3\log\log k) < 9.7\cdot 10^{13}\log k. \end{aligned}$$

Hence, we obtain $k < 3.5\cdot 10^{15}$, and so again from Lemma 4 we get

$$n < 4.7\cdot 10^{82} \quad \text{and} \quad m < 2.1\cdot 10^{82}. \tag{26}$$

Let $\Lambda_2 := m\log\delta - n\log 2 - \log(\sqrt{2})$. By a similar method to show the inequality (23), one can see that $0 < |\Lambda_2| < \frac{2.2}{2^{k/2}} < 2.2\exp(-0.34k)$ holds for all $k > 220$.

Now, we will apply Lemma 3. The inequality (26) implies that we can take $X_0 := 4.7 \cdot 10^{82}$. Further, we can choose

$$c := 2.2, \quad \rho := 0.34, \quad \psi := -\frac{\log(\sqrt{2})}{\log\delta},$$

$$\vartheta := \frac{\log 2}{\log\delta}, \quad \vartheta_1 := \log 2, \quad \vartheta_2 := -\log\delta, \quad \beta := \log(\sqrt{2}).$$

With the help of Maple, we find that $q_{163} \approx 4.14 \cdot 10^{83}$ satisfies the hypotheses of Lemma 3. Furthermore, according to Lemma 3 we obtain $k < 618$.

With this new upper bound on k, we get from Lemma 4

$$n < 2 \cdot 10^{29} \quad \text{and} \quad m < 8.4 \cdot 10^{28}.$$

Applying again Lemma 3 with $X_0 := 2 \cdot 10^{29}$ and

$$q_{60} := 208903764897193259964937500 1624$$

in this time, we obtain $k < 216$, which contradicts our assumption that $k > 220$. Hence, we have shown that there are no solutions (n, k, m) to equation (1) with $k > 220$. □

Thus, the Theorem 1 is proved.

4 k-Fibonacci Lucas-balancing numbers

This section is devoted to prove Theorem 2. The proof of Theorem 2 is similar to that of Theorem 1. For the sake of completeness, we will give some details.

4.1 An inequality for n and m in terms of k

Since $F_1^{(k)} = F_2^{(k)} = 1 = C_0$, then we may assume that $n \geq 3$. For $3 \leq n \leq k+1$, we have $F_n^{(k)} = 2^{n-2}$, but C_m is an odd number for all $m \geq 0$, thus we deduce that the Diophantine equation (2) has no solution when $3 \leq n \leq k+1$. From now, we suppose that $n \geq k+2$.

By relations (14), (11) and equation (2) we have

$$\alpha^{n-2} \leq \delta^m \quad \text{and} \quad \delta^{m-1} \leq \alpha^{n-1},$$

hence we get

$$(n-2)\left(\frac{\log\alpha}{\log\delta}\right) \leq m \leq (n-1)\left(\frac{\log\alpha}{\log\delta}\right) + 1.$$

Using the fact that $3/2 < \alpha < 2$ for all $k \geq 2$ (see (12)), we deduce that

$$0.2n - 0.5 < m < 0.4n + 0.7. \tag{27}$$

Lemma 7. *If (n, k, m) is a solution in integers of equation (2) with $k \geq 2$ and $n \geq k+2$, then the inequalities*

$$2.4m < n < 2.4 \cdot 10^{16} k^4 \log^3 k \tag{28}$$

hold.

Proof. By combining (2) with (9) and (13), we obtain

$$\left| f_k(\alpha)\alpha^{n-1} - \frac{\delta^m}{2} \right| < \frac{1}{2} + \frac{|\beta|^m}{2} < 2.$$

Multiplying both sides by $2\delta^{-m}$ we get

$$\left| 2f_k(\alpha)\alpha^{n-1}\delta^{-m} - 1 \right| < 2\delta^{-m}. \tag{29}$$

In order to show inequality (28), we will apply Theorem 3 with the parameters $t := 3$, $(\eta_1, b_1) := (2f_k(\alpha), 1)$, $(\eta_2, b_2) := (\alpha, n-1)$, $(\eta_3, b_3) := (\delta, -m)$, and $\Gamma_3 := 2f_k(\alpha)\alpha^{n-1}\delta^{-m} - 1$.

From (29), we have that

$$|\Gamma_3| < 2\delta^{-m}. \tag{30}$$

For this choice, the field $\mathbb{K} := \mathbb{Q}(\alpha, \sqrt{2})$ contains η_1, η_2, η_3 and has $d_{\mathbb{K}} \leq 2k$. As calculated before, we can choose $A_2 := 2\log 2$ and $A_3 := k \log \delta$.

On the other hand, using (15) and the proprieties (3) together with (4), we deduce

$$h(\eta_1) \leq h(2) + h(f_k(\alpha)) < \log 2 + \log(k+1) + \log 4 < 4.6 \log k$$

for all $k \geq 2$. Thus, we obtain $\max\{2kh(\eta_1), |\log \eta_1|, 0.16\} = 9.2k \log k := A_1$. The fact that $0.4n + 0.7 < n$ hold for all $n \geq 4$ and the inequality (17) imply that we may take $B := n$.

To apply Theorem 3, we need to show that $\Gamma_3 \neq 0$, if it were, then

$$f_k(\alpha) = \frac{\delta^m}{2} \alpha^{-n+1}.$$

Hence $f_k(\alpha)$ is an algebraic integer, which is impossible. Thus, $\Gamma_3 \neq 0$. Therefore, after applying Theorem 3 and comparing the resulting inequality with inequality (30), we obtain

$$m \log \delta - \log 2 < 1.3 \cdot 10^{13} k^4 \log k (1 + \log 2k)(1 + \log n).$$

Taking into account the facts $1 + \log 2k < 3.5 \log k$ and $1 + \log n < 1.8 \log n$, which hold for $k \geq 2$ and $n \geq 4$, we deduce that

$$m < 4.65 \cdot 10^{13} k^4 \log^2 k \log n.$$

From the above inequality together with (27), it comes

$$\frac{n}{\log n} < 2.33 \cdot 10^{14} k^4 \log^2 k. \tag{31}$$

Using (31) and (21) with $T := 2.33 \cdot 10^{14} k^4 \log^2 k$ we get

$$\begin{aligned} n &< 2(2.33 \cdot 10^{14} k^4 \log^2 k) \log(2.33 \cdot 10^{14} k^4 \log^2 k) \\ &< (4.66 \cdot 10^{14} k^4 \log^2 k)(33.1 + 4 \log k + 2 \log \log k) < 2.4 \cdot 10^{16} k^4 \log^3 k, \end{aligned}$$

where we have used that $33.1 + 4 \log k + 2 \log \log k < 51 \log k$, which holds for all $k \geq 2$. $\square$

4.2 The case $2 \leq k \leq 230$

In this subsection, we study the case $k \in [2, 230]$. We prove the following assertion.

Lemma 8. *The Diophantine equation* (2) *has no solution when* $k \in [2, 230]$ *and* $n \geq k+2$.

Proof. Put $\Lambda_3 = \log(\Gamma_3 + 1) = (n-1)\log\alpha - m\log\delta + \log(2f_k(\alpha))$.

Using a similar method to prove the inequality (23), we prove that

$$0 < |\Lambda_3| < 4\delta^{-m} < 4\exp(-1.76m).$$

In Lemma 3, we fix

$$c := 4, \quad \delta := 1.76, \quad \psi := \frac{\log(2f_k(\alpha))}{\log\delta},$$

$$\vartheta := \frac{\log\delta}{\log\alpha}, \quad \vartheta_1 := -\log\delta, \quad \vartheta_2 := \log\alpha, \quad \beta := \log(2f_k(\alpha)).$$

For each $k \in [2, 230]$, we find a good approximation of α and a convergent p_ℓ/q_ℓ of the continued fraction of ϑ such that $q_\ell > X_0$, where $X_0 = \lfloor 2.4 \cdot 10^{16} k^4 \log^3 k \rfloor$, which is an upper bound of $\max\{n-1, m\}$ from Lemma 7. After doing this, we use Lemma 3 on inequality (23). A computer search with Mathematica revealed that the maximum value of $\left\lfloor \frac{1}{\delta}\log(q^2 c/|\vartheta_2|\, X_0) \right\rfloor$ over all $k \in [2, 230]$ is $91.40\ldots$, which according to Lemma 3, is an upper bound on m. Hence, we deduce that the possible solutions (m, n, k) of the equation (1) for which $k \in [2, 230]$ have $m \leq 91$, therefore we use inequalities (17) to obtain $n \leq 457$.

Finally, we used Mathematica to compare $F_n^{(k)}$ and C_m for the range $4 \leq n \leq 222$ and $2 \leq m \leq 44$, with $m < n/2.4$ and checked that the only solution of the equation (1) is $3 = C_1 = F_4^{(2)}$. □

4.3 The case $k > 230$

In this subsection, we analyze the case $k > 230$.

Lemma 9. *The Diophantine equation* (1) *has no solution when* $k > 230$ *and* $n \geq k+2$.

Proof. For $k > 230$ we have $2.4m < n < 2.4 \cdot 10^{16} k^4 \log^3 k < 2^{k/2}$. By (2), (9) and (16) we obtain

$$\left| 2^{n-2} - \frac{\delta^m}{2} \right| < \frac{2^{n-2}}{2^{k/2}} + \frac{1}{2},$$

which leads to

$$\left| 1 - 2^{-(n-1)}\delta^m \right| < \frac{1.3}{2^{k/2}}, \tag{32}$$

where we have used the fact $1/2^{n-1} < 0.3/2^{k/2}$, because $n \geq k+2$. We will give a lower bound to the left-hand side of inequality (32) by using Theorem 3. We choose $t := 2$, $(\eta_1, b_1) := (2, -n+1)$, $(\eta_2, b_2) := (\delta, m)$. We have $\eta_1, \eta_2 \in \mathbb{K} := \mathbb{Q}(\sqrt{2})$, so $d_{\mathbb{K}} = 2$. If the left-hand side of (32) is zero, then we get that $\delta^{2m} \in \mathbb{Q}$, which is a contradiction. Thus, the left-hand side of (32) is not zero.

The fact that $m \leq n$ imply that we can choose $B := n$. On the other hand, since $h(\eta_1) = \log 2$, $h(\eta_2) = (\log\delta)/2$, it follows that

$$\max\{2h(\eta_1), |\log\eta_1|, 0.16\} = 2\log 2 := A_1 \quad \text{and} \quad \max\{2h(\eta_2), |\log\eta_2|, 0.16\} = \log\delta := A_2.$$

So, Theorem 3 tell us that

$$\left|1 - 2^{-(n-1)}\delta^m\right| > \exp\left(-2.3 \cdot 10^{10} \log n\right), \tag{33}$$

where we have used the fact that $1 + \log n < 1.8 \log n$ for all $n \geq 4$. Comparing (32) and (33), we obtain $k < 6.7 \cdot 10^{10} \log n$.

By Lemma 7 and using the fact $37.8 + 4\log k + 3\log\log k < 12\log k$ for all $k > 220$, we get

$$\begin{aligned} k &< 6.7 \cdot 10^{10} \log(2.4 \cdot 10^{16} k^4 \log^3 k) \\ &< 6.7 \cdot 10^{10} \log(37.8 + 4\log k + 3\log\log k) < 8.1 \cdot 10^{11} \log k \end{aligned}$$

Hence, we obtain $k < 2.5 \cdot 10^{13}$. Lemma 7 imply

$$n < 2.8 \cdot 10^{74} \quad \text{and} \quad m < 1.2 \cdot 10^{74}. \tag{34}$$

Put $\Lambda_4 = m\log\delta - (n-1)\log 2$. Using a similar method to prove the inequality (23), we show that $0 < |\Lambda_4| < \dfrac{2.6}{2^{k/2}} < 2.6\exp(-0.34\,k)$ holds for all $k > 210$.

We apply Lemma 1 with $c = 2.6$, $\rho = 0.34$ and $X_0 := 2.8 \cdot 10^{74}$, which is an upper bound on m by (34). Thus, from Lemma 1 we get $Y_0 := 356.899840124\ldots$. Let

$$[a_0, a_1, a_2, \ldots] := [0, 2, 1, 1, 5, 3, 2, 1, 22, 1, 5, 38, 1, 1, 1, 8, 1, 3, 7, 1, 5, 2, 5, 2, 2, 200, \ldots]$$

be the continued fraction expansion of $\log 2/\log\delta$. Since $A = \max_{0\leq 356} a_k = 4008$, then according to Lemma 2 we have

$$k < \frac{1}{0.34} \cdot \left(\frac{2.6 \cdot 4010 \cdot 2.8 \cdot 10^{74}}{\log\delta}\right) < 530.$$

With this new upper bound on k we get by Lemma 7 that $n < 4.7 \cdot 10^{29}$ and $m < 2 \cdot 10^{29}$.

We apply again Lemma 2 with $X_0 := 4.7 \cdot 10^{29}$. Hence by Lemma 1, we obtain $Y_0 = 142.65243\ldots$ and $A = 1014$ in this time. According to Lemma 2 it comes

$$k < \frac{1}{0.34} \cdot \left(\frac{2.6 \cdot 1016 \cdot 4.7 \cdot 10^{29}}{\log\delta}\right) < 223,$$

which contradicts our assumption that $k > 230$. Thus, we have shown that there are no solutions (n, k, m) to equation (1) with $k > 230$. □

Thus, the Theorem 2 is proved.

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Received 03.11.2020
Revised 21.12.2020

Райяне С.Е. *Про збалансовані та Люка-збалансовані числа, що є елементами k-узагальненої послідовності Фібоначчі* // Карпатські матем. публ. — 2021. — Т.13, №1. — С. 259–271.

Збалансове число n і балансир r є розв'язками діофантового рівняння

$$1 + 2 + \cdots + (n-1) = (n+1) + (n+2) + \cdots + (n+r).$$

Відомо, що якщо число n є збалансованим, то $8n^2 + 1$ є повним квадратом, квадратний корінь з якого називають Люка-збалансованим числом. Для цілого $k \geq 2$ символом $(F_n^{(k)})_n$ позначимо k-узагальнену послідовність Фібоначчі, яка починається з $0, \ldots, 0, 1, 1$ (k чисел), а кожне наступне число є сумою k попередніх. Ми довели, що серед елементів k-узагальненої послідовності Фібоначчі єдиними збалансованими числами є 1 і 6930, а Люка-збалансованими – числа 1 і 3. Отримані нами результати узагальнюють результати з [Fibonacci Quart. 2004, **42** (4), 330–340].

Ключові слова і фрази: k-узагальнені числа Фібоначчі, збалансовані числа, Люка-збалансовані числа, лінійна форма в логарифмах, метод редукції.