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# Order estimates of the uniform approximations by Zygmund sums on the classes of convolutions of periodic functions

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The Zygmund sums of a function $f \in L_1$ are trigonometric polynomials of the form $Z^s_{n-1}(f;t) := \frac{a_0}{2} + \sum_{k=1}^{n-1}\left(1-\left(\frac{k}{n}\right)^s\right)\left(a_k(f)\cos kt + b_k(f)\sin kt\right)$, $s>0$, where $a_k(f)$ and $b_k(f)$ are the Fourier coefficients of $f$. We establish the exact-order estimates of uniform approximations by the Zygmund sums $Z^s_{n-1}$ of $2\pi$-periodic continuous functions from the classes $C^{\psi}_{\beta,p}$. These classes are defined by the convolutions of functions from the unit ball in the space $L_p$, $1 \leq p < \infty$, with generating fixed kernels $\Psi_\beta(t) \sim \sum_{k=1}^{\infty}\psi(k)\cos\left(kt + \frac{\beta\pi}{2}\right)$, $\Psi_\beta \in L_{p'}$, $\beta \in \mathbb{R}$, $\frac{1}{p} + \frac{1}{p'} = 1$. We additionally assume that the product $\psi(k)k^{s+1/p}$ is generally monotonically increasing with the rate of some power function, and, besides, for $1 < p < \infty$ it holds that $\sum_{k=n}^{\infty}\psi^{p'}(k)k^{p'-2} < \infty$, and for $p=1$ the following condition $\sum_{k=n}^{\infty}\psi(k) < \infty$ is true.

It is shown, that under these conditions Zygmund sums $Z^s_{n-1}$ and Fejér sums $\sigma_{n-1} = Z^1_{n-1}$ realize the order of the best uniform approximations by trigonometric polynomials of these classes, namely for $1 < p < \infty$

$$E_n(C^{\psi}_{\beta,p})_C \asymp \mathcal{E}\left(C^{\psi}_{\beta,p}; Z^s_{n-1}\right)_C \asymp \left(\sum_{k=n}^{\infty}\psi^{p'}(k)k^{p'-2}\right)^{1/p'}, \qquad \frac{1}{p} + \frac{1}{p'} = 1,$$

and for $p = 1$

$$E_n(C^{\psi}_{\beta,1})_C \asymp \mathcal{E}\left(C^{\psi}_{\beta,1}; Z^s_{n-1}\right)_C \asymp \begin{cases} \sum\limits_{k=n}^{\infty}\psi(k), & \cos\frac{\beta\pi}{2} \neq 0, \\ \psi(n)n, & \cos\frac{\beta\pi}{2} = 0, \end{cases}$$

where

$$E_n(C^{\psi}_{\beta,p})_C := \sup_{f \in C^{\psi}_{\beta,p}} \inf_{t_{n-1} \in \mathcal{T}_{2n-1}} \|f(\cdot) - t_{n-1}(\cdot)\|_C,$$

and $\mathcal{T}_{2n-1}$ is the subspace of trigonometric polynomials $t_{n-1}$ of order $n-1$ with real coefficients,

$$\mathcal{E}\left(C^{\psi}_{\beta,p}; Z^s_{n-1}\right)_C := \sup_{f \in C^{\psi}_{\beta,p}} \|f(\cdot) - Z^s_{n-1}(f;\cdot)\|_C.$$

*Key words and phrases:* best approximation, Zygmund sum, Fejér sum, subspace of trigonometric polynomials, order estimate.

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## 1 Notations, definitions and auxiliary statements

Denote by $L_p$, $1 \le p \le \infty$, the space of $2\pi$-periodic summable on $[0, 2\pi]$ functions $f$ with the norm

$$\|f\|_p = \begin{cases} \Big( \int_0^{2\pi} |f(t)|^p dt \Big)^{1/p}, & 1 \le p < \infty, \\ \operatorname{ess\,sup}\limits_t |f(t)|, & p = \infty, \end{cases}$$

and by $C$ the space of $2\pi$-periodic continuous functions with the norm defined by the equality $\|f\|_C = \max\limits_t |f(t)|$.

Let $f \in L_1$ and

$$S[f](x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k(f) \cos kx + b_k(f) \sin kx),$$

be the Fourier series of the function $f$.

If for a sequence $\psi(k) \in \mathbb{R}$ and fixed number $\beta \in \mathbb{R}$ the series

$$\sum_{k=1}^{\infty} \frac{1}{\psi(k)} \Big( a_k(f) \cos \Big( kx + \frac{\beta\pi}{2} \Big) + b_k(f) \sin \Big( kx + \frac{\beta\pi}{2} \Big) \Big)$$

is the Fourier series of a summable function $\varphi$, then this function is called a $(\psi, \beta)$-derivative of the function $f$ and is denoted by $f^\psi_\beta$. A set of functions, for which this condition is satisfied, is denoted by $L^\psi_\beta$, and subset all continuous functions from $L^\psi_\beta$ is denoted by $C^\psi_\beta$.

If $f \in L^\psi_\beta$ and furthermore $f^\psi_\beta \in \mathfrak{N}$, where $\mathfrak{N} \subset L_1$, then we write that $f \in L^\psi_\beta \mathfrak{N}$. Let us put $L^\psi_\beta \mathfrak{N} \cap C = C^\psi_\beta \mathfrak{N}$. The concept of $(\psi, \beta)$-derivative is a natural generalization of the concept of $(r, \beta)$-derivative in the Weyl-Nagy sense and coincides almost everywhere with the last one, when $\psi(k) = k^{-r}$, $r > 0$. Namely, in this case $L^\psi_\beta \mathfrak{N} = W^r_\beta \mathfrak{N}$, $f^\psi_\beta = f^r_\beta$, where $f^r_\beta$ is the derivative in the Weyl-Nagy sense, and $W^r_\beta \mathfrak{N}$ are the Weyl-Nagy classes [22], [20]. In the case $\beta = r$, the classes $W^r_\beta \mathfrak{N}$ are the well known Weyl classes $W^r_r \mathfrak{N}$, while the derivatives $f^r_\beta$ coincide almost everywhere with the derivatives in the sense of Weyl $f^r_r$. If, in addition, $\beta = r$, $r \in \mathbb{N}$, then $f^r_\beta$ coincide almost everywhere with the usual derivatives $f^{(r)}$ of the order $r$ of the function $f$ ($f^r_\beta = f^r_r = f^{(r)}$) and at the same time $W^r_\beta \mathfrak{N} = W^r_r \mathfrak{N} = W^r \mathfrak{N}$.

According to [20, Statement 3.8.3], if the series

$$\sum_{k=1}^{\infty} \psi(k) \cos \big( kt - \frac{\beta\pi}{2} \big), \qquad \beta \in \mathbb{R},$$

is the Fourier series of the function $\Psi_\beta \in L_1$, then the elements $f$ of the classes $L^\psi_\beta \mathfrak{N}$ for almost every $x \in \mathbb{R}$ are represented as the convolution

$$f(x) = \frac{a_0}{2} + (\Psi_\beta * \varphi)(x) = \frac{a_0}{2} + \frac{1}{\pi} \int\limits_{-\pi}^{\pi} \Psi_\beta(x - t) \varphi(t) dt, \quad a_0 \in \mathbb{R},\ \varphi \perp 1,\ \varphi \in \mathfrak{N}, \tag{1}$$

where $\varphi$ almost everywhere coincides with $f^\psi_\beta$.

As sets $\mathfrak{N}$ we will consider the unit balls of the spaces $L_p$:

$$U_p = \{ \varphi \in L_p : \|\varphi\|_p \le 1 \}, \qquad 1 \le p \le \infty.$$

Then put: $L^{\psi}_{\beta,p} := L^{\psi}_{\beta}U_p$, $C^{\psi}_{\beta,p} := C^{\psi}_{\beta}U_p$, $W^{r}_{\beta,p} := W^{r}_{\beta}U_p$.

According to [20, Statement 1.2], if the fixed kernel $\Psi_\beta$ of the classes $L^{\psi}_{\beta,p}$ and $C^{\psi}_{\beta,p}$ satisfies the inclusion $\Psi_\beta \in L_{p'}$, $\frac{1}{p}+\frac{1}{p'}=1$, $1\le p\le\infty$, then the convolutions of the form (1) are continuous functions, where $\mathfrak{N}=U_p$. It is clear that in this case for $f\in C^{\psi}_{\beta,p}$ the equality (1) is fulfilled for all $x\in\mathbb{R}$.

We assume that the sequences $\psi(k)$ are traces on the set of natural numbers $\mathbb{N}$ of some positive continuous convex downwards functions $\psi(t)$ of the continuous argument $t\ge 1$, that tends to zero for $t\to\infty$. The set of all such functions $\psi(t)$ is denoted by $\mathfrak{M}$.

To classify functions $\psi$ from $\mathfrak{M}$ on their speed of decreasing to zero it is convenient to use the following characteristic

$$\alpha(t)=\alpha(\psi;t)=\frac{\psi(t)}{t|\psi'(t)|}, \qquad \psi'(t):=\psi'(t+0).$$

With its help we consider the following subsets of the set $\mathfrak{M}$ (see, e.g. [20])

$$\mathfrak{M}_0 := \{\psi\in\mathfrak{M}:\ \ \exists K>0\ \ \forall t\ge 1\ \ 0<K\le\alpha(\psi;t)\},$$

$$\mathfrak{M}_C := \{\psi\in\mathfrak{M}:\ \ \exists K_1,K_2>0\ \ \forall t\ge 1\ \ 0<K_1\le\alpha(\psi;t)\le K_2\}.$$

It is clear that $\mathfrak{M}_C\subset\mathfrak{M}_0$.

Zygmund sums of the order $n-1$ of the function $f\in L_1$ are the trigonometric polynomials of the form

$$Z^s_{n-1}(f;t)=\frac{a_0}{2}+\sum_{k=1}^{n-1}\left(1-\left(\frac{k}{n}\right)^s\right)\big(a_k(f)\cos kt+b_k(f)\sin kt\big), \quad s>0, \tag{2}$$

where $a_k(f)$ and $b_k(f)$ are Fourier coefficients of the function $f$.

In the case $s=1$ polynomials $Z^s_{n-1}$ are Fejér sums

$$Z^1_{n-1}(f;t)=:\sigma_{n-1}(f;t)=\frac{a_0}{2}+\sum_{k=1}^{n-1}\left(1-\frac{k}{n}\right)(a_k(f)\cos kt+b_k(f)\sin kt).$$

In this paper we consider the following approximation characteristics

$$\mathcal{E}\left(C^{\psi}_{\beta,p};Z^s_{n-1}\right)_C=\sup_{f\in C^{\psi}_{\beta,p}}\|f(\cdot)-Z^s_{n-1}(f;\cdot)\|_C, \quad 1\le p\le\infty, \quad \beta\in\mathbb{R}, \tag{3}$$

and solve the problem of establishing the order of decreasing to zero as $n\to\infty$ of the mentioned quantities with respect to relations between parameters $\psi$, $\beta$, $p$ and $s$. It is clear that we can make conclusion about the approximation ability of a linear polynomial approximation method (including Fejér $\sigma_{n-1}$ and Zygmund $Z^s_{n-1}$ methods) on the class $C^{\psi}_{\beta,p}$, after comparison the rate of decreasing of the exact upper bounds of uniform deviations of trigonometric sums, which are generated by this method, on the set $C^{\psi}_{\beta,p}$ with the rate of decreasing of the best uniform approximations of the class $C^{\psi}_{\beta,p}$ by trigonometric polynomials $t_{n-1}$ of order not higher than $n-1$, namely the quantities of the form

$$E_n(C^{\psi}_{\beta,p})_C=\sup_{f\in C^{\psi}_{\beta,p}}\inf_{t_{n-1}}\|f(\cdot)-t_{n-1}(\cdot)\|_C, \qquad 1\le p\le\infty,$$

where $\mathcal{T}_{2n-1}$ is the subspace of trigonometric polynomials $t_{n-1}$ of order $n-1$ with real coefficients. In this case, since always the following estimate holds

$$E_n\left(C^{\psi}_{\beta,p}\right)_C \le \mathcal{E}\left(C^{\psi}_{\beta,p}; Z^s_{n-1}\right)_C, \qquad n \in \mathbb{N}, \tag{4}$$

it is important to know under which restrictions on the parameters $\psi, s, \beta$ and $p$ the following equality takes place

$$E_n\left(C^{\psi}_{\beta,p}\right)_C \asymp \mathcal{E}\left(C^{\psi}_{\beta,p}; Z^s_{n-1}\right)_C. \tag{5}$$

The notation $A(n) \asymp B(n)$ means, that $A(n) = O(B(n))$ and at the same time $B(n) = O(A(n))$, where by the notation $A(n) = O(B(n))$ we mean, that there exists a constant $K > 0$ such that the inequality $A(n) \le K(B(n))$ holds.

In the work [27] A. Zygmund introduced trigonometric polynomials of the form (2) and found exact order estimates of the quantities $\mathcal{E}\left(W^r_\infty; Z^s_{n-1}\right)_C$ at $r \in \mathbb{N}$. B. Nagy investigated in [7] the quantities $\mathcal{E}\left(W^r_{\beta,\infty}; Z^s_{n-1}\right)_C$ at $r > 0$, $\beta \in \mathbb{Z}$, and for $s \le r$ he established the asymptotic equality, and for $s > r$ he found order estimates. Later, S.A. Telyakovsky [23] obtained asymptotically exact equalities for the quantities $\mathcal{E}\left(W^r_{\beta,\infty}; Z^s_{n-1}\right)_C$ for $r > 0$ and $\beta \in \mathbb{R}$ for $n \to \infty$. On the Weyl-Nagy classes, the exact order estimates of the quantities $\mathcal{E}\left(W^r_{\beta,p}; Z^s_{n-1}\right)_C$ for $1 < p < \infty$ and $r > 1/p$ and for $p = 1$ and $r \ge 1$, $\beta \in \mathbb{R}$ are found in the work [6].

Concerning the Fejér sums $\sigma_{n-1}(f;t)$ it should be noticed that the order estimates of quantities $\mathcal{E}\left(W^r_{\beta,\infty}; \sigma_{n-1}\right)_C$, $r > 0$, for $\beta \in \mathbb{Z}$ were found by S.M. Nikol'skii [8]; for the quantities $\mathcal{E}\left(W^r_{r,p}; \sigma_{n-1}\right)_C$ for $1 < p \le \infty$ and $r > 1/p$, and also for $p = 1$ and $r \ge 1$ were found by V.M. Tikhomirov [25] and by A.I. Kamzolov [5].

Approximation properties of Zygmund sums on the classes of $(\psi,\beta)$-differentiable functions were studied in the works [2, 14, 15], (see also [20]). Particularly in the work [2] of D.M. Bushev the asymptotic equalities for the quantities $\mathcal{E}(C^{\psi}_{\beta,\infty}; Z^s_{n-1})_C$ were established for some quite natural constraints on $\psi$ and $s$ as $n \to \infty$. In the case, when the series $\sum_{k=1}^{\infty} \psi^2(k)$ is convergent, the exact values of the quantities $\mathcal{E}\left(C^{\psi}_{\beta,2}; Z^s_{n-1}\right)_C$ were established in the work [15] of A.S. Serdyuk and I.V.Sokolenko.

In the work [14], the authors found the exact order estimatites of uniform approximations by Zygmund sums $Z^s_{n-1}$ on the classes $C^{\psi}_{\beta,p}$, $1 < p < \infty$, when $\psi \in \Theta_p$, and $\Theta_p$, $1 < p < \infty$, is the set of non-increasing functions $\psi(t)$, for which there exists $\alpha > 1/p$ such that the function $t^{\alpha}\psi(t)$ almost decreases, and $\psi(t)t^{s+1/p-\varepsilon}$ increases on $[1,\infty)$ for some $\varepsilon > 0$.

Concerning the estimates of the best uniform approximations of functional compacts, it should be noticed the following. For the Weyl-Nagy classes $W^r_{\beta,p}$, $r > 1/p$, $\beta \in \mathbb{R}$, $1 \le p \le \infty$, the exact order estimates of the best approximations $E_n\left(W^r_{\beta,p}\right)_C$ are known (see, e.g. [24]). Moreover, for $p = \infty$ the exact values of the quantities $E_n\left(W^r_{\beta,\infty}\right)_C$ for all $r > 0$, $\beta \in \mathbb{R}$ and $n \in \mathbb{N}$ are known (see [3]).

The order estimates of the best approximations of the classes $C^{\psi}_{\beta,p}$ under certain restrictions on $\psi$, $\beta$ and $p$ were investigated in the works [4,17,18,20]. In some partial cases (especially for $p = \infty$) the exact or asymptotically exact values of the quantities $E_n\left(C^{\psi}_{\beta,p}\right)_C$ are also known (see [9–13,16,20]).

In this paper, we establish the exact order estimates of the quantities of the form (3) for all $1 \le p < \infty$ and $\beta \in \mathbb{R}$, in case, when $\psi(t)t^{1/p} \in \mathfrak{M}_0$, the product $\psi(k)k^{s+1/p}$ generally monotonically increases, $\psi(k)k^{s+1/p-\varepsilon}$ almost increases (according to Bernstein) for some $\varepsilon > 0$ and for $1 < p < \infty$

$$\sum_{k=n}^{\infty} \psi^{p'}(k)k^{p'-2} < \infty, \qquad \frac{1}{p} + \frac{1}{p'} = 1, \tag{6}$$

and for $p = 1$

$$\sum_{k=n}^{\infty} \psi(k) < \infty. \tag{7}$$

The conditions (6) and (7) and the monotonic decreasing to zero of the sequence $\psi(k)$ ensure the inclusion $\Psi_\beta \in L_{p'}$, $1/p + 1/p' = 1$, $1 \le p < \infty$ (see, e.g. [28, Lemma 12.6.6, p. 193]).

In this paper, it is also shown that for some conditions Zygmund sums (and at $s = 1$ also the Fejér sums) realize the orders of the best uniform approximations on the classes $C^{\psi}_{\beta,p}$, that is the order estimate (5) is true. Previously, this property was proved for Fourier sums [4, 18, 19, 21].

Let us formulate some necessary definitions.

A non-negative sequence $a = \{a_k\}_{k=1}^{\infty}$, $k \in \mathbb{N}$, is said to be generally monotonically increasing (we write $a \in GM^+$), if there exists a constant $A \ge 1$, such that for any natural $n_1$ and $n_2$ such that $n_1 \le n_2$ the inequalities

$$a_{n_1} + \sum_{k=n_1}^{m-1} |a_k - a_{k+1}| \le A a_m, \qquad m = \overline{n_1, n_2}, \tag{8}$$

hold (see, e.g. [1, p. 811]). It is easy to see that if the positive sequence $a = \{a_k\}_{k=1}^{\infty}$ increases, starting from some number, then it generally monotonically increasing.

A non-negative sequence $a = \{a_k\}_{k=1}^{\infty}$, $k \in \mathbb{N}$, is said to be almost increasing (according to Bernstein, see, e.g. [26, p. 730]) if there exists a constant $K$, such that for all $n_1 \le n_2$ we have

$$a_{n_1} \le K a_{n_2}. \tag{9}$$

In this case, if for the sequence $a = \{a_k\}_{k=1}^{\infty}$ there exists a constant $\varepsilon > 0$, such that $\{a_k k^{-\varepsilon}\}$ almost increases, then we write $a \in GA^+$. It is clear that if the sequence $a$ belongs to $GM^+$, then it is almost increasing according to Bernstein.

Let us put further $g_\delta(t) := \psi(t)t^\delta$, $t \in [1, \infty)$ with $\delta > 0$.

## 2 Order estimates of the approximations by Zygmund sums on the classes of convolutions

**Theorem 1.** *Let $s > 0$, $1 \le p < \infty$, $g_{1/p} \in \mathfrak{M}_0$, $g_{s+1/p} \in GM^+ \cap GA^+$, $\beta \in \mathbb{R}$ and $n \in \mathbb{N}$. In the case $1 < p < \infty$, if the condition (6) holds and the inequality*

$$\inf_{t \ge 1} \alpha(g_{1/p}; t) > \frac{p'}{2} \tag{10}$$

*holds, then the following order estimates take place*

$$E_n\left(C^{\psi}_{\beta,p}\right)_C \asymp \mathcal{E}\left(C^{\psi}_{\beta,p}; Z^s_{n-1}\right)_C \asymp \left(\sum_{k=n}^{\infty} \psi^{p'}(k)k^{p'-2}\right)^{1/p'}, \qquad \frac{1}{p} + \frac{1}{p'} = 1; \tag{11}$$

*in the case $p = 1$, if the condition (7) holds and the inequality*

$$\inf_{t\geq 1} \alpha(g_1; t) > 1 \tag{12}$$

*holds, then the following order estimates take place*

$$E_n\left(C^{\psi}_{\beta,1}\right)_C \asymp \mathcal{E}\left(C^{\psi}_{\beta,1}; Z^s_{n-1}\right)_C \asymp \begin{cases} \sum\limits_{k=n}^{\infty} \psi(k), & \cos\frac{\beta\pi}{2} \neq 0, \\ \psi(n)n, & \cos\frac{\beta\pi}{2} = 0. \end{cases} \tag{13}$$

*Proof.* Since the operator $Z^s_{n-1} : f(t) \to Z^s_{n-1}(f,t)$ is linear polynomial operator, which is invariant under the shift, i.e.

$$Z^s_{n-1}(f_h, t) = Z^s_{n-1}(f, t+h), \quad f_h(t) = f(t+h), \qquad h \in \mathbb{R},$$

and norm in $C$ and classes $C^{\psi}_{\beta,p}$ also are invariant under the shift, that is

$$\|f_h\|_C = \|f\|_C; \quad f(t) \in C^{\psi}_{\beta,p} \Rightarrow f_h(t) \in C^{\psi}_{\beta,p},$$

then

$$\mathcal{E}\left(C^{\psi}_{\beta,p}; Z^s_{n-1}\right)_C = \sup_{f\in C^{\psi}_{\beta,p}} |f(0) - Z^s_{n-1}(f;0)|. \tag{14}$$

By virtue of (1) and (2) for any function $f \in C^{\psi}_{\beta,p}$, $1 \leq p < \infty$, $\beta \in \mathbb{R}$, $s > 0$, the following equality holds

$$f(0) - Z^s_{n-1}(f;0) = \frac{1}{\pi}\int_{-\pi}^{\pi} \Big(\frac{1}{n^s}\sum_{k=1}^{n-1} \psi(k)k^s \cos\Big(kt + \frac{\beta\pi}{2}\Big) + \Psi_{-\beta,n}(t)\Big)\varphi(t)\,dt, \tag{15}$$

where $\Psi_{-\beta,n}(t) = \sum_{k=n}^{\infty} \psi(k)\cos\left(kt + \frac{\beta\pi}{2}\right)$, $\|\varphi\|_p \leq 1$, $n \in \mathbb{N}$.

Relations (14) and (15), Hölder's inequality and triangle inequality imply that for $1 \leq p < \infty$

$$\begin{aligned} \mathcal{E}\left(C^{\psi}_{\beta,p}; Z^s_{n-1}\right)_C &\leq \frac{1}{\pi}\left\| \frac{1}{n^s}\sum_{k=1}^{n-1} \psi(k)k^s \cos\left(kt + \frac{\beta\pi}{2}\right) + \Psi_{-\beta,n}(t)\right\|_{p'} \\ &\leq \frac{1}{\pi n^s}\left\|\sum_{k=1}^{n-1} \psi(k)k^s \cos\left(kt + \frac{\beta\pi}{2}\right)\right\|_{p'} + \frac{1}{\pi}\left\|\Psi_{-\beta,n}(t)\right\|_{p'}, \quad \frac{1}{p} + \frac{1}{p'} = 1. \end{aligned} \tag{16}$$

Let us show that, if $g_{s+1/p} \in GM^+ \cap GA^+$, where $g_{s+1/p} = \left\{\psi(k)k^{s+1/p}\right\}_{k=1}^{\infty}$, then

$$\left\|\sum_{k=1}^{n-1} \psi(k)k^s \cos\left(kt + \frac{\beta\pi}{2}\right)\right\|_{p'} = O\big(\psi(n)n^{s+\frac{1}{p}}\big), \qquad 1 \leq p < \infty. \tag{17}$$

Applying Abel transformation to the function $\sum_{k=1}^{n-1} \psi(k)k^s \cos\left(kt + \frac{\beta\pi}{2}\right)$, we have

$$\begin{aligned} \sum_{k=1}^{n-1} \psi(k)k^s \cos\left(kt + \frac{\beta\pi}{2}\right) &= \sum_{k=1}^{n-2}\Big(\psi(k)k^s - \psi(k+1)(k+1)^s\Big)D_{k,\beta}(t) \\ &\quad + \psi(n-1)(n-1)^s D_{n-1,\beta}(t) - \frac{1}{2}\cos\frac{\beta\pi}{2}, \end{aligned} \tag{18}$$

where

$$D_{k,\beta}(t) := \frac{1}{2}\cos\frac{\beta\pi}{2} + \sum_{\nu=1}^{k}\cos\Big(\nu t - \frac{\beta\pi}{2}\Big).$$

Then, in view of $\|D_{k,\beta}(\cdot)\|_{p'} = O(k^{1-\frac{1}{p'}}) = O(k^{\frac{1}{p}})$, $1 \le p < \infty$, $k \in \mathbb{N}$, $\beta \in \mathbb{R}$, (see, e.g. [4]) from (18) we get

$$\begin{aligned}\Big\|\sum_{k=1}^{n-1}\psi(k)k^s\cos\Big(kt+\frac{\beta\pi}{2}\Big)\Big\|_{p'} &= O(1) + O\bigg(\sum_{k=1}^{n-2}\big|\psi(k)k^s - \psi(k+1)(k+1)^s\big|k^{\frac{1}{p}}\bigg) \\ &\quad + O\Big(\psi(n-1)(n-1)^{s+\frac{1}{p}}\Big).\end{aligned} \tag{19}$$

Since $g_{s+1/p} \in GM^{+}$, then, by using the triangle inequality, inequality (8) and Lagrange theorem, we have

$$\begin{aligned}\sum_{k=1}^{n-2}\big|\psi(k)k^s - \psi(k+1)(k+1)^s\big|k^{\frac{1}{p}} &\le \sum_{k=1}^{n-2}\big|\psi(k)k^{s+\frac{1}{p}} - \psi(k+1)(k+1)^{s+\frac{1}{p}}\big| \\ &\quad + \sum_{k=1}^{n-2}\big|\psi(k+1)(k+1)^{s+\frac{1}{p}} - \psi(k+1)(k+1)^s k^{\frac{1}{p}}\big| \\ &\le A\psi(n-1)(n-1)^{s+\frac{1}{p}} + \frac{1}{p}\sum_{k=1}^{n-2}\psi(k+1)(k+1)^s k^{\frac{1}{p}-1} \\ &\le A\psi(n-1)(n-1)^{s+\frac{1}{p}} + 2\sum_{k=2}^{n-1}\frac{\psi(k)k^{s+\frac{1}{p}}}{k}.\end{aligned} \tag{20}$$

According to the condition $g_{s+1/p} \in GA^{+}$, there exits $\varepsilon > 0$ such that the sequence $\{g_{s+1/p}(k)k^{-\varepsilon}\} = \{\psi(k)k^{s+1/p-\varepsilon}\}$ almost increases, and hence taking into account (9), we obtain

$$\begin{aligned}\sum_{k=2}^{n-1}\frac{\psi(k)k^{s+1/p}}{k} &= \sum_{k=2}^{n-1}\frac{\psi(k)k^{s+1/p-\varepsilon}}{k^{1-\varepsilon}} \le K\psi(n-1)(n-1)^{s+1/p-\varepsilon}\sum_{k=2}^{n-1}\frac{1}{k^{1-\varepsilon}} \\ &< K\psi(n-1)(n-1)^{s+1/p-\varepsilon}\int_1^{n-1}\frac{dt}{t^{1-\varepsilon}} < \frac{K}{\varepsilon}\psi(n-1)(n-1)^{s+1/p}.\end{aligned} \tag{21}$$

From (20) and (21) we get the following inequality

$$\big|\psi(k)k^s - \psi(k+1)(k+1)^s\big|k^{\frac{1}{p}} \le \Big(A + \frac{2K}{\varepsilon}\Big)\psi(n-1)(n-1)^{s+1/p}. \tag{22}$$

From (19) and (22) we obtain the estimation (17).

To estimate the norm $\|\Psi_{-\beta,n}(\cdot)\|_{p'}$ for $1 < p' < \infty$ we use the statement, which was established in [18], and according to which in the case when $\{a_k\}_{k=1}^{\infty}$ is the monotonically non-increasing sequence of positive numbers such that $\sum_{k=1}^{\infty} a_k^{p'} k^{p'-2} < \infty$, then for arbitrary $n \in \mathbb{N}$ and $\gamma \in \mathbb{R}$ the following estimate holds

$$\Big\|\sum_{k=n}^{\infty} a_k\cos(kx+\gamma)\Big\|_{p'} = O\Big(\sum_{k=n}^{\infty} a_k^{p'}k^{p'-2} + a_n^{p'}n^{p'-1}\Big)^{1/p'}. \tag{23}$$

Putting in (23) $a_k = \psi(k)$, $\gamma = \frac{\beta\pi}{2}$ we obtain that for $1 < p < \infty$, $\beta \in \mathbb{R}$ and $n \in \mathbb{N}$

$$\|\Psi_{-\beta,n}(\cdot)\|_{p'} = O\Big( \sum_{k=n}^{\infty} \psi^{p'}(k)k^{p'-2} + \psi^{p'}(n)n^{p'-1}\Big)^{1/p'}. \tag{24}$$

Then, using [18, Lemma 3], we conclude that for $1 < p' < \infty$, $n \in \mathbb{N}$, under condition (6) and imbedding $g_{1/p} \in \mathfrak{M}_0$ the following estimate holds

$$\psi^{p'}(n)n^{p'-1} = O\left( \sum_{k=n}^{\infty} \psi^{p'}(k)k^{p'-2} \right). \tag{25}$$

According to the conditions of Theorem 1 we have that $g_{1/p} \in \mathfrak{M}_0$, so taking into account (25), from (24), we obtain

$$\|\Psi_{-\beta,n}(\cdot)\|_{p'} = O\left( \sum_{k=n}^{\infty} \psi^{p'}(k)k^{p'-2} \right)^{1/p'}, \qquad 1 < p' < \infty, \quad \beta \in \mathbb{R}, \quad n \in \mathbb{N}. \tag{26}$$

Combining (16), (17) and (26) in the case when $g_{1/p} \in \mathfrak{M}_0$, and $g_{s+1/p} \in GM^+ \cap GA^+$, we arrive at the estimate

$$\mathcal{E}\left(C^{\psi}_{\beta,p}; Z^s_{n-1}\right)_C = O\left( \sum_{k=n}^{\infty} \psi^{p'}(k)k^{p'-2} \right)^{1/p'}, \qquad 1 < p < \infty, \quad \frac{1}{p} + \frac{1}{p'} = 1. \tag{27}$$

As follows from [18, Corollary 1 and 2], for $1 < p < \infty$, $1/p + 1/p' = 1$, $n \in \mathbb{N}$ and $\beta \in \mathbb{R}$, under conditions (6) and (10) and imbedding $g_{1/p} \in \mathfrak{M}_0$ for $E_n\left(C^{\psi}_{\beta,p}\right)_C$ we arrive at the following order estimates

$$E_n\left(C^{\psi}_{\beta,p}\right)_C \asymp \Big( \sum_{k=n}^{\infty} \psi^{p'}(k)k^{p'-2}\Big)^{1/p'}. \tag{28}$$

Therefore, by virtue of inequality (4) and relations (27) and (28) we obtain order equality (11).

Further, let us consider the case $p = 1$. Let us establish the estimate of the norm $\|\Psi_{-\beta,n}(\cdot)\|_{p'} = \|\Psi_{-\beta,n}(\cdot)\|_{\infty}$. It is obvious that for any $\beta \in \mathbb{R}$ the following inequality holds

$$\|\Psi_{-\beta,n}(\cdot)\|_{\infty} = \left\| \sum_{k=n}^{\infty} \psi(k) \cos\left(kt + \frac{\beta\pi}{2}\right) \right\|_{\infty} \le \sum_{k=n}^{\infty} \psi(k). \tag{29}$$

If $\beta = 2k+1$, $k \in \mathbb{Z}$, then following estimate takes place

$$\|\Psi_{-\beta,n}(\cdot)\|_{\infty} = \left\| \sum_{k=n}^{\infty} \psi(k) \sin kt \right\|_{\infty} \le (\pi + 2)\psi(n)n \tag{30}$$

(see, e.g. [21, relation (82)]).

According to [21, Lemma 3], if $g_1 \in \mathfrak{M}_0$, where $g_1 = \{\psi(k)k\}_{k=1}^{\infty}$ and the condition (7) holds, then the following estimates are true

$$\psi(n)n = O\Big( \sum_{k=n}^{\infty} \psi(k) \Big). \tag{31}$$

If $g_1 \in \mathfrak{M}_0$ and the conditions (7) hold, then combining (16), (17), (29) – (31), we obtain the following estimates

$$\mathcal{E}\left(C^{\psi}_{\beta,1}; Z^s_{n-1}\right)_C = \begin{cases} O\Big(\sum\limits_{k=n}^{\infty}\psi(k)\Big), & \cos\frac{\beta\pi}{2} \neq 0, \\ O\big(\psi(n)n\big), & \cos\frac{\beta\pi}{2} = 0. \end{cases} \tag{32}$$

To estimate the quantity $\mathcal{E}\left(C^{\psi}_{\beta,1}; Z^s_{n-1}\right)_C$ from below, we use [21, Theorems 3 and 4], according to which, if $g_1 \in \mathfrak{M}_0$ and the conditions (7) and (12) are true, then for $n \in \mathbb{N}$ and $\beta \in \mathbb{R}$ the following the order equalities take place

$$E_n\left(C^{\psi}_{\beta,1}\right)_C \asymp \begin{cases} \sum\limits_{k=n}^{\infty}\psi(k), & \cos\frac{\beta\pi}{2} \neq 0, \\ \psi(n)n, & \cos\frac{\beta\pi}{2} = 0. \end{cases} \tag{33}$$

The estimate (13) follows from the inequality (4), estimates (32) and (33). □

Assume that the conditions of Theorem 1 take place, moreover, more stronger imbedding $g_{1/p} \in \mathfrak{M}_C$ holds. As it follows from [18, Lemma 3], if $g_{1/p} \in \mathfrak{M}_C$ and the condition (6) holds, then for $1 < p < \infty$ the following estimates take place

$$\sum_{k=n}^{\infty}\psi^{p'}(k)k^{p'-2} \asymp \psi^{p'}(n)n^{p'-1}. \tag{34}$$

In addition, as it was shown in [21, Lemma 3], if $g_1 \in \mathfrak{M}_C$ and the condition (7) holds, then the following order estimates are true

$$\sum_{k=n}^{\infty}\psi(k) \asymp \psi(n)n. \tag{35}$$

Formulas (34) and (35), and Theorem 1 allow us to write the following statement.

**Theorem 2.** *Let* $s > 0$, $1 \leq p < \infty$, $g_{1/p} \in \mathfrak{M}_C$, $g_{s+1/p} \in GM^+ \cap GA^+$, $\beta \in \mathbb{R}$ *and* $n \in \mathbb{N}$.

*In the case* $1 < p < \infty$, *if the conditions* (6) *and* (10) *hold, then the following order estimates take place*

$$E_n(C^{\psi}_{\beta,p})_C \asymp \mathcal{E}\left(C^{\psi}_{\beta,p}; Z^s_{n-1}\right)_C \asymp \psi(n)n^{1/p}, \tag{36}$$

*and in the case* $p = 1$ *if the conditions* (7) *and* (12) *hold, then the following order estimates take place*

$$E_n(C^{\psi}_{\beta,1})_C \asymp \mathcal{E}\left(C^{\psi}_{\beta,1}; Z^s_{n-1}\right)_C \asymp \psi(n)n. \tag{37}$$

*Proof.* Order estimates (36) were established in [14]. Note, that when $1 < p < \infty$, $g_{1/p} \in \mathfrak{M}_0$ and

$$\lim_{t\to\infty}\alpha\big(g_{1/p}; t\big) = \infty, \tag{38}$$

then the order estimates (36) do not take place, since in this case we have the following (see [18])

$$\psi(n)n^{\frac{1}{p}} = o\Big(\big(\sum_{k=n}^{\infty}\psi^{p'}(k)k^{p'-2}\big)^{1/p'}\Big), \qquad n \to \infty.$$

Similarly, when $p=1$, $g_{1/p}=g_1\in\mathfrak{M}_0$ and

$$\lim_{t\to\infty}\alpha\big(g_1;t\big)=\infty, \tag{39}$$

then as follows from [21, Lemma 3]

$$\psi(n)n=o\Big(\sum_{k=n}^{\infty}\psi(k)\Big),$$

in this case, for $\beta$ such that $\cos\frac{\beta\pi}{2}\neq 0$ order estimates (37) do not take place.

As example of the function $\psi(t)$, for which the conditions of Theorem 1 and the equalities (38) and (39) take place, we can use the function

$$\psi(t)=t^{-1/p}\ln^{-\gamma}(t+K),\quad \gamma>\begin{cases}\frac{1}{p'}, & 1<p<\infty,\\ 1, & p=1,\end{cases}\qquad K>\begin{cases}e^{\gamma p'/2}, & 1<p<\infty,\\ e^{\gamma}, & p=1,\end{cases} \tag{40}$$

(see [18,21]). Let us write the order estimates for the quantities $E_n\left(C^{\psi}_{\beta,p}\right)_C$ and $\mathcal{E}\left(C^{\psi}_{\beta,p};Z^s_{n-1}\right)_C$ in the case, when $\psi(t)$ has the form (40). $\square$

**Theorem 3.** *Let $\psi(t)=t^{-1/p}\ln^{-\gamma}(t+K)$, $\beta\in\mathbb{R}$ and $n\in\mathbb{N}$. If $1<p<\infty$, $\gamma>1/p'$, $K>e^{\gamma p'/2}$, $1/p+1/p'=1$, then*

$$E_n(C^{\psi}_{\beta,p})_C\asymp\mathcal{E}\left(C^{\psi}_{\beta,p};Z^s_{n-1}\right)_C\asymp\psi(n)n^{1/p}\ln^{1/p'}n,\qquad n\geq 2; \tag{41}$$

*if $p=1$, $\gamma>1$, $K>e^{\gamma}$, then*

$$E_n(C^{\psi}_{\beta,1})_C\asymp\mathcal{E}\left(C^{\psi}_{\beta,1};Z^s_{n-1}\right)_C\asymp\begin{cases}\psi(n)n\ln n, & \cos\frac{\beta\pi}{2}\neq 0,\\ \psi(n)n, & \cos\frac{\beta\pi}{2}=0,\end{cases}\qquad n\geq 2. \tag{42}$$

*Proof.* We show that for the indicated function $\psi$ of the form (40) all conditions of the Theorem 1 are true. Indeed, for $1<p<\infty$, $\gamma>1/p'$, $K>e^{\gamma p'/2}$ we have

$$\sum_{k=n}^{\infty}\psi^{p'}(k)k^{p'-2}=\sum_{k=n}^{\infty}\frac{1}{k\ln^{\gamma p'}(k+K)}<\infty,\qquad \alpha\big(g_{1/p};t\big)=\frac{(t+K)\ln(t+K)}{\gamma t}>\frac{\ln(t+e^{\gamma p'/2})}{\gamma},$$

and hence $\lim\limits_{t\to\infty}\alpha\big(g_{1/p};t\big)=\infty$ and $\alpha\big(g_{1/p};t\big)>\frac{p'}{2}$.

For $p=1$, $\gamma>1$, $K\geq e^{\gamma}$, we have

$$\sum_{k=n}^{\infty}\psi(k)\leq\sum_{k=n}^{\infty}\frac{1}{k\ln^{\gamma}(k+e^{\gamma})}<\infty,\qquad \alpha\big(g_1;t\big)>\frac{\ln(t+e^{\gamma})}{\gamma},$$

and hence $\lim\limits_{t\to\infty}\alpha\big(g_1;t\big)=\infty$ and $\alpha\big(g_1;t\big)>1$.

It is obvious that for any $s>0$ and $1\leq p<\infty$ the functions $g_{s+1/p}(t)=t^s\ln^{-\gamma}(t+K)$ increase monotonically, starting from some point $t_0$. Therefore, it is not difficult to be convinced that the sequence $g_{s+1/p}(k)$ belongs to the set $GM^+\cap GA^+$.

Therefore, the function $\psi$ of the form (40) satisfies the conditions of Theorem 1.

Further, using [18, formula (79)], we obtain

$$\Big(\sum_{k=n}^{\infty}\psi^{p'}(k)k^{p'-2}\Big)^{1/p'} \asymp \Big(\int\limits_{n}^{\infty}\psi^{p'}(t)t^{p'-2}dt\Big)^{1/p'} = \Big(\int\limits_{n}^{\infty}\frac{dt}{t\ln^{\gamma p'}(t+K)}\Big)^{1/p'} \asymp \ln^{1/p'-\gamma} n$$
$$= \psi(n)n^{1/p}\ln^{1/p'} n\frac{\ln^{-\gamma} n}{\ln^{-\gamma}(n+K)} \asymp \psi(n)n^{1/p}\ln^{1/p'} n, \qquad n\geq 2.$$

Then formula (41) follows from the estimate (11) and the above relations.

Similarly, by virtue of [21, inequality (87)] we get

$$\sum_{k=n}^{\infty}\psi(k) \asymp \int_{n}^{\infty}\psi(t)dt = \int_{n}^{\infty}\frac{dt}{t\ln^{\gamma}(t+K)} \asymp \ln^{1-\gamma} n \asymp \psi(n)n\ln n, \qquad n>2. \tag{43}$$

Formula (42) follows from the estimates (13) and relations (43), in the case where $\beta$ is such that $\cos\frac{\beta\pi}{2}\neq 0$. □

As it was already mentioned, for $s=1$ the Zygmund sums $Z^s_{n-1}$ coincide with the known Fejér sums $\sigma_{n-1}$. Therefore, Theorem 1 and 2 imply the following statements.

**Proposition 1.** *Let* $1\leq p<\infty$, $g_{1/p}\in\mathfrak{M}_0$, $g_{1+1/p}\in GM^+\cap GA^+$, $\beta\in\mathbb{R}$ *and* $n\in\mathbb{N}$.

*In the case* $1<p<\infty$, *if the conditions* (6) *and* (10) *hold, then the following order estimates take place*

$$E_n(C^{\psi}_{\beta,p})_C \asymp \mathcal{E}\left(C^{\psi}_{\beta,p};\sigma_{n-1}\right)_C \asymp \big(\sum_{k=n}^{\infty}\psi^{p'}(k)k^{p'-2}\big)^{1/p'};$$

*in the case* $p=1$, *if the conditions* (7) *and* (12) *hold, then the following order equlaities take place*

$$E_n(C^{\psi}_{\beta,1})_C \asymp \mathcal{E}\left(C^{\psi}_{\beta,1};\sigma_{n-1}\right)_C \asymp \begin{cases}\sum\limits_{k=n}^{\infty}\psi(k), & \cos\frac{\beta\pi}{2}\neq 0,\\ \psi(n)n, & \cos\frac{\beta\pi}{2}=0.\end{cases}$$

**Proposition 2.** *Let* $1\leq p<\infty$, $g_{1/p}\in\mathfrak{M}_C$, $g_{1+1/p}\in GM^+\cap GA^+$, $\beta\in\mathbb{R}$ *and* $n\in\mathbb{N}$.

*In the case* $1<p<\infty$, *if the conditions* (6) *and* (10) *hold, then the following order estimates take place*

$$E_n(C^{\psi}_{\beta,p})_C \asymp \mathcal{E}\left(C^{\psi}_{\beta,p};\sigma_{n-1}\right)_C \asymp \psi(n)n^{1/p};$$

*in the case* $p=1$, *if the conditions* (7) *and* (12) *hold, then the following order estimates take place*

$$E_n(C^{\psi}_{\beta,1})_C \asymp \mathcal{E}\left(C^{\psi}_{\beta,1};\sigma_{n-1}\right)_C \asymp \psi(n)n.$$

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Сердюк А.С., Грабова У.З. *Порядкові оцінки рівномірних наближень сумами Зиґмунда на класах згорток періодичних функцій* // Карпатські матем. публ. — 2021. — Т.13, №1. — С. 68–80.

Суми Зиґмунда $Z^s_{n-1}(f;t)$ функції $f \in L_1$ — це тригонометричні поліноми вигляду $Z^s_{n-1}(f;t) := \frac{a_0}{2} + \sum_{k=1}^{n-1}\left(1-\left(\frac{k}{n}\right)^s\right)(a_k(f)\cos kt + b_k(f)\sin kt)$, $s > 0$, де $a_k(f)$ і $b_k(f)$ — коефіцієнти Фур'є функції $f$. Отримано точні порядкові оцінки рівномірних наближень сумами Зиґмунда $Z^s_{n-1}$ на класах $C^\psi_{\beta,p}$. Ці класи складаються з $2\pi$-періодичних неперервних функцій $f$, які зображаються у вигляді згортки функцій, що належать одиничним кулям просторів $L_p$, $1 \le p < \infty$, з фіксованими твірними ядрами $\Psi_\beta(t) \sim \sum_{k=1}^\infty \psi(k)\cos\left(kt + \frac{\beta\pi}{2}\right)$, $\Psi_\beta \in L_{p'}$, $\beta \in \mathbb{R}$, $\frac{1}{p}+\frac{1}{p'}=1$, у випадку, коли добуток $\psi(k)k^{s+1/p}$ узагальнено монотонно зростає з деякою степеневою швидкістю, і, крім того, при $1 < p < \infty$ виконується нерівність $\sum_{k=n}^\infty \psi^{p'}(k)k^{p'-2} < \infty$, а при $p = 1$ — нерівність $\sum_{k=n}^\infty \psi(k) < \infty$. Показано, що при виконанні зазначених умов суми Зиґмунда $Z^s_{n-1}$, а також суми Фейєра $\sigma_{n-1} = Z^1_{n-1}$ реалізують порядки найкращих рівномірних наближень тригонометричними поліномами на вказаних функціональних класах, а саме при $1 < p < \infty$

$$E_n(C^\psi_{\beta,p})_C \asymp \mathcal{E}\left(C^\psi_{\beta,p}; Z^s_{n-1}\right)_C \asymp \Big(\sum_{k=n}^\infty \psi^{p'}(k)k^{p'-2}\Big)^{1/p'}, \qquad \frac{1}{p}+\frac{1}{p'}=1,$$

а при $p = 1$

$$E_n(C^\psi_{\beta,1})_C \asymp \mathcal{E}\left(C^\psi_{\beta,1}; Z^s_{n-1}\right)_C \asymp \sum_{k=n}^\infty \psi(k), \qquad \cos\frac{\beta\pi}{2} \neq 0,$$

$$E_n(C^\psi_{\beta,p})_C \asymp \mathcal{E}\left(C^\psi_{\beta,p}; Z^s_{n-1}\right)_C \asymp \psi(n)n, \qquad \cos\frac{\beta\pi}{2} = 0,$$

де

$$E_n(C^\psi_{\beta,p})_C := \sup_{f\in C^\psi_{\beta,p}} \inf_{t_{n-1}\in\mathcal{T}_{2n-1}} \|f(\cdot) - t_{n-1}(\cdot)\|_C,$$

$\mathcal{T}_{2n-1}$ — підпростір тригонометричних поліномів $t_{n-1}$ порядку $n-1$ з дійсними коефіцієнтами,

$$\mathcal{E}\left(C^\psi_{\beta,p}; Z^s_{n-1}\right)_C := \sup_{f\in C^\psi_{\beta,p}} \|f(\cdot) - Z^s_{n-1}(f;\cdot)\|_C.$$

*Ключові слова і фрази:* найкраще наближення, сума Зиґмунда, сума Фейєра, підпростір тригонометричних поліномів, порядкова оцінка.