ISSN 2075-9827 e-ISSN 2313-0210
Carpathian Math. Publ. 2021, **13** (1), 160–179
doi:10.15330/cmp.13.1.160-179

https://journals.pnu.edu.ua/index.php/cmp
Карпатські матем. публ. 2021, Т.13, №1, С.160–179

Evolution pseudodifferential equations with analytic symbols in spaces of S type

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A nonlocal multipoint by time problem for an evolution equation with a pseudodifferential operator is studied. This operator is treated as an infinite order differentiation operator in generalized spaces of S type. We consider the case when the initial condition of the problem is an element of an ultradistributions type space and the nonlocial condition contains pseudodifferential operators. The solvability of such problem is established, the properties of the fundamental solution are investigated, the analytical representation of the solution is found.

Key words and phrases: nonlocal multipoint problem, pseudodifferential operator, generalized function.

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Introduction

Pseudodifferential operators (PDO) and equations with pseudodifferential operators are closely related to important problems of analysis, modern mathematical physics, probability theory, fractal theory, quantum field theory etc. The class of pseudodifferential operators includes differential operators, fractional differentiation and integration operators, convolutions and so on.

A wide class of PDO can be formally represented as $A = I^{-1}_{\sigma\to x}[a(t,x;\sigma)I_{x\to\sigma}]$, $\{x,\sigma\} \subset \mathbb{R}$, $t > 0$, where a is the symbol of the operator A that satisfies certain conditions, I, I^{-1} are direct and inverse Fourier or Bessel transform respectively. If the symbol a is an entire even function of the argument σ, then the evolution equations with the operator A also contain singular differential equations, in particular, the equations with the Bessel operator $B_\nu = d^2/dx^2 + (2\nu+1)x^{-1}d/dx$, $\nu > -1/2$, which in its structure contains the expression $1/x$ and is formally represented as $B_\nu = F^{-1}_{B_\nu}[-\sigma^2 F_{B_\nu}]$, where F_{B_ν} is the Bessel integral transformation. If $a(t,x;\sigma) \equiv P(t,x;\sigma)$, where P is a polynomial of the variable σ for fixed t,x, which satisfies the condition of "parabolicity", such equations belong to parabolic equations if $I_{x\to\sigma} = F$ is a Fourier transform, or to B-parabolic equations if $I_{x\to\sigma} = F_{B_\nu}$. B-parabolic equations are degenerated at the boundary and are close in their internal properties to uniformly parabolic equations.

The theory of linear parabolic and B-parabolic equations with partial derivatives originates from the study of the thermal conductivity equation. The classical theory of the Cauchy problem and boundary value problems for such equations and systems of equations is constructed in the works of I.G. Petrovsky, S.D. Eidelman, S.D. Ivasyshen, M.I. Matiychuk,

УДК 517.98
2010 *Mathematics Subject Classification:* 35S16.

M.V. Zhitarashu, A. Friedman, S. Teklind, V.O. Solonnikov, I.A. Kipriyanov, V.V. Krekhivskyi and others. The Cauchy problem with initial data in the spaces of generalized functions such as distributions and ultradistributions was studied by G.Ye. Shilov, B.L. Gurevich, M.L. Gorbachuk, V.I. Gorbachuk, O.I. Kashpirovsky, S.D. Ivasyshen, Ya.I. Zhytomyrskyi, V.V Gorodetskyi, V.A. Litovchenko and others.

Many mathematicians have studied the Cauchy problem for evolution equations with PDO, using different methods and approaches, e.g. M. Nagase, R. Shinkai, C. Tsutsumi, M.A. Shubin, M. Taylor, L. Hermander, A.N. Kochubey, S.D. Eidelman, Y.A. Dubinsky, B.Y. Ptashnyk, M.I. Matiychuk, M.I. Konarovska etc. Important results on the solvability of the Cauchy problem in different functional spaces are obtained. In this case, the initial functions often have features at one or more points and allow regularization in certain spaces of generalized functions such as Sobolev-Schwartz distributions, ultradistributions, hyperfunctions and others. Thus, the Cauchy problem for these equations has a natural formulation in the classes of generalized functions of finite and infinite orders.

In this paper, we investigate the problem, which can be understood as a generalization of the Cauchy problem, when the initial condition $u(t,\cdot)|_{t=0} = f$ is replaced by the condition

$$\sum_{k=0}^{m} \alpha_k B_k u(t,\cdot)\big|_{t=t_k} = f,$$

where $t_0 = 0$, $\{t_1, \ldots, t_m\} \subset (0, T]$, $0 < t_1 < t_2 < \cdots < t_m \leq T$, $\{\alpha_0, \alpha_1, \ldots, \alpha_m\} \subset \mathbb{R}$, $m \in \mathbb{N}$ are fixed numbers, $B_0, B_1 \ldots, B_m$ are pseudodifferential operators built on certain functions (symbols) $g_0, g_1, \ldots, g_m$ (if $\alpha_0 = 1$, $\alpha_1 = \ldots = \alpha_m = 0$, $B_0 = I$ is the identity operator, then we obviously have a Cauchy problem). This condition is interpreted in the classical sense or in the weak sense if f is a generalized function, i.e. as a limit relation

$$\sum_{k=0}^{m} \alpha_k \lim_{t \to t_k} \langle B_k u(t,\cdot), \varphi \rangle = \langle f, \varphi \rangle$$

for an arbitrary function φ from the test space (here $\langle f, \cdot \rangle$ denotes the action of the functional f on the test function). This problem refers to nonlocal multipoint by time problems for partial differential equations. A detailed review of works on nonlocal problems for differential-operator equations and partial differential equations is given in [9]. Nonlocal by time problems, in turn, refer to nonlocal boundary value problems that arise when modeling many processes and problems of practice (see, for example, [1, 2, 14, 18]). Such problems include problems that are studied, for example, in the papers [16, 17].

At present, the nonlocal multipoint time problem has not been studied in the case of evolution equations with PDO operating in spaces of type S, the symbols of which are functions that allow analytic extension into the whole complex plane and satisfy a certain condition of "parabolic", and the function f in the corresponding condition is an element of space of type S' that is topologically dual of the space of type S.

Note that spaces of type S are often used in the study of the problems of uniqueness classes and classes of correctness of a Cauchy problem for partial differential equations and consist of infinitely differentiable on $\mathbb{R}$ functions, whose behavior on the real axis are characterized by $m_{kn} = \sup_{x \in \mathbb{R}} |x^k \varphi^{(n)}(x)|$, $\{k, n\} \subset \mathbb{Z}_+$, where the double sequence $\{m_{kn}\}$ satisfies certain conditions. I.M. Gelfand and G.E. Shilov (see [4]) investigated the case $m_{kn} = k^{k\alpha} n^{n\beta}$, $\alpha, \beta > 0$;

spaces of type S in this case are denoted by the symbol S_α^β and consist of infinitely differentiable functions on $\mathbb{R}$, which together with all their derivatives decrease as $|x| \to +\infty$ faster than $\exp\{-a|x|^{1/\alpha}\}, a > 0, x \in \mathbb{R}$.

In the works [5,6,10–12,15], it is established that spaces of type S and S' are natural sets of initial data of a Cauchy problem for wide classes of equations with partial derivatives of finite and infinite orders, where the solutions are entire functions by space variables. For example, for the heat equation $\partial u/\partial t = \partial^2 u/\partial x^2$ the fundamental solution of Cauchy problem is the function $G(t,x) = (2\sqrt{\pi t})^{-1}\exp\{-x^2/(ut)\}$, $t > 0$, which as a function of x is an element of space $S_{1/2}^{1/2}$ (see [12, p. 46]), that refers to spaces of S type.

In this article, we investigate the nonlocal multipoint by time problem for the equation $\partial u/\partial t = Agu$, $(t,x) \in (0,T] \times \mathbb{R}$ in S type spaces, which is constructed by sequences $m_{kn} = a_k b_n$, that defined by certain conditions. Here Ag is pseudodifferential operator in S type spaces with analytical symbol g, which can also be understood as an operator of differentiation of "infinite order":

$$Ag = F^{-1}_{\sigma\to x}[g(\sigma)F_{x\to\sigma}] = \sum_{k=0}^{\infty} c_k (id/dx)^k,$$

function g is a symbol of operator Ag, which satisfyies certain conditions that generalize the known condition of "parabolic" for parabolic pseudodifferential equations.

In Section 1, we define the spaces of type S and S', multiplier and convolutor in spaces of type S. In Section 2, the correctness of the definition of the operator Ag in generalized spaces of S type as an operator of differentiation of finite order is proved and its continuity is proved. In Section 3, a property of the fundamental solution of a nonlocal multipoint by time problem for the specified equation is established, the solvability of the problem is proved; the representation of the solution in the form of a convolution of the fundamental solution of the problem with the initial generalized function is found.

1 Generalized spaces of S and S' type

I.M. Gelfand and G.E. Shilov in the well-known monograph [4] proposed a method of constructing functional spaces of infinitely differentiable functions on $\mathbb{R}$, which impose certain conditions for decreasing at infinity and increasing of derivatives when the order is increasing. These conditions are given by the inequalities $|x^k\varphi^{(n)}(x)| \leq c_{kn}$, $\{k,n\} \subset \mathbb{Z}_+$, where $\{c_{kn}\}$ is a double sequence of positive numbers. If these numbers change randomly together with the function φ, then we have Schwartz's space $S = S(\mathbb{R})$ of rapidly decreasing on $\mathbb{R}$ functions. If $c_{kn} = a_k b_n$, where $\{a_k : k \in \mathbb{Z}_+\}$, $\{b_n : n \in \mathbb{Z}_+\}$ are some sequences of positive numbers, then we have generalized S type spaces, which are denoted by $S_{a_k}^{b_n}$. In the monograph [4], the case $a_k = k^{k\alpha}$, $\alpha > 0$, $b_n = n^{n\beta}$, $\beta > 0$, was studied in detail; the corresponding spaces are called S spaces and are denoted by S_α^β. In [7], topological structure of the spaces $S_{a_k}^{b_n}$, properties of functions, basic operations in such spaces were studied. Known spaces of the W type, introduced by B.L. Gurevich [8] (see also [3]), in which convex functions are used instead of power functions to characterize the behavior of functions at infinity, are also included in the spaces $S_{a_k}^{b_n}$ with the specific choice of sequences $\{a_k\}$ and $\{b_n\}$ (see [13]).

We will focus on the spaces $S_{a_k}^{b_n}$, which are constructed by the sequences of the form $\{b_n = n!\rho_n : n \in \mathbb{Z}_+\}$, $\{a_k = k!d_k : k \in \mathbb{Z}_+\}$. Here $\{\rho_n : n \in \mathbb{Z}_+\}$, $\rho_0 = 1$, is a sequence of positive numbers having the properties: a) it is monotonically increasing; b) $\exists\, c_b > 0\ \exists\, \gamma_1 \in (0,1)\ \forall\, n \in \mathbb{N} : \rho_{n-1}/\rho_n \le c_b n^{\gamma_1}$; c) $\lim_{n\to\infty} \sqrt[n]{\rho_n} = 0$; d) $\forall\, \varepsilon > 0\ \exists\, c_\varepsilon > 0\ \forall\, n \in \mathbb{N} :\ \rho_n \ge c_\varepsilon \varepsilon^n / n^n$. The above sequence $\{d_k : k \in \mathbb{Z}_+\}$, $d_0 = 1$, also has properties like a)–d), e.g. condition b) have the form: $\exists\, c_a > 0\ \exists\, \gamma_2 \in (0,1)\ \forall\, k \in \mathbb{N} :\ d_{k-1}/d_k \le c_a k^{\gamma_2}$.

The sequence $\rho_n = (n\beta)^{-n\beta} e^{n\beta}$, where $\beta \in (0,1)$ is a fixed parameter, is the example of a sequence $\{\rho_n\}$ with properties a)–d). Let us check, for example, the property d). We have

$$\begin{aligned}\rho_n &= \frac{e^{n\beta}}{(n\beta)^{n\beta}} = \frac{e^{n\beta}}{(n\beta)^{n\beta}} \frac{[n(1-\beta)]^{n(1-\beta)}}{[n(1-\beta)]^{n(1-\beta)}} \frac{e^{n(1-\beta)}}{e^{n(1-\beta)}} \\ &= \frac{e^n}{n^n} \frac{1}{[\beta^\beta (1-\beta)^{1-\beta}]^n} \frac{[n(1-\beta)]^{n(1-\beta)}}{e^{n(1-\beta)}} = \frac{e^n}{n^n} \frac{1}{\omega^n} \sup_{\lambda \ge 0} \frac{\lambda^n}{\exp\{\lambda^{1/(1-\beta)}\}},\end{aligned}$$

where $\omega = \beta^\beta (1-\beta)^{1-\beta} < 1$. If we take an arbitrary $\varepsilon > 0$ and put $\lambda = \varepsilon$, then we get the inequality $\rho_n \ge c_\varepsilon \varepsilon^n / n^n$, where $c_\varepsilon = \exp\{-\varepsilon^{1/(1-\beta)}\}$. Note that condition b) for this sequence is satisfied with the parameter $\gamma_1 = \beta$.

We denote by $S_{a_k}^{b_n}$ the set of functions $\varphi \in C^\infty(\mathbb{R})$ that satisfy the condition

$$\exists\, c, A, B > 0 \quad \forall\, \{k, n\} \subset \mathbb{Z}_+ \quad \forall\, x \in \mathbb{R} : \quad |x^k \varphi^{(n)}(x)| \le c A^k B^n a_k b_n.$$

The set $S_{a_k}^{b_n}$ coincides with the union of countable normed spaces $S_{a_k,\,A}^{b_n,\,B}$ over all indices $\{A, B\} \subset \mathbb{N}$, where $S_{a_k,\,A}^{b_n,\,B}$ denotes the set of functions $\varphi \in S_{a_k}^{b_n}$ that for arbitrary $\delta, \rho > 0$ satisfies inequalities

$$|x^k \varphi^{(n)}(x)| \le c_{\delta\rho} (A+\delta)^k (B+\rho)^n a_k b_n, \quad \{k,n\} \subset \mathbb{Z}_+, \quad x \in \mathbb{R};$$

the system of norms in $S_{a_k,\,A}^{b_n,\,B}$ is determined by the formulas

$$\|\varphi\|_{\delta\rho} = \sup_{x,k,n} \frac{|x^k \varphi^{(n)}(x)|}{(A+\delta)^k (B+\rho)^n a_k b_n}, \quad \{\delta, \rho\} \subset \{1, 1/2, 1/3, \ldots\}.$$

In [7], it was established that a function $\varphi \in C^\infty(\mathbb{R})$ is an element of the space $S_{a_k}^{b_n}$ with $a_k = k!d_k$, $b_n = n!\rho_n$ if and only if it analytically extends into the complex plane to the entire function $\varphi(z)$, $z \in \mathbb{C}$, which satisfies the condition

$$\exists\, a, b, c > 0 \quad \forall\, z = x + iy \in \mathbb{C} : \quad |\varphi(z)| \le c\gamma(ax)\rho(by), \tag{1}$$

where

$$\gamma(x) = \begin{cases} 1, & |x| < 1, \\ \inf\limits_k (a_k / |x|^k), & |x| \ge 1, \end{cases} \qquad \rho(y) = \begin{cases} 1, & |y| < 1, \\ \sup\limits_n (|y|^n / b_n), & |y| \ge 1. \end{cases}$$

Note that ρ is a continuously differentiable even function on $\mathbb{R}$ that is monotonically increasing on the interval $[1, +\infty)$. It follows from property d) (see [7]) that

$$\exists\, c_0, c > 0 \quad \forall\, y \in \mathbb{R} : \quad \rho(y) \ge c_0 \exp(c|y|).$$

For example, if $b_n = n^{n\beta}$, $0 < \beta < 1$, then $\rho(y) \sim \exp\{|y|^{1/\beta}\}$. In addition, as proved in [7], $\ln \rho$ is a convex function on $(0, +\infty)$ in the sense

$$\forall\, \{y_1, y_2\} \subset (0, +\infty) : \quad \ln \rho(y_1) + \ln \rho(y_2) \le \ln \rho(y_1 + y_2). \tag{2}$$

Inequality $\ln\rho(y_1) - \ln\rho(y_1+y_2) \le -\ln\rho(y_2)$ also follows from (2).

The function ρ in (1) is related to the sequence $\{\rho_n\}$, which generates the sequence $\{b_n\} = \{n!\rho_n\}$, as follows [7]

$$\rho_n = \inf_{|\omega|\ge 1}(\rho(\omega)/|\omega|^n) = \nu_n^{-n}\rho(\nu_n),$$

where ν_n is the solution of the equation $\omega\mu(\omega) = n$, $n \in \mathbb{N}$, $\mu(\omega) = \rho'(\omega)/\rho(\omega)$; the sequence $\{\nu_n\}$ is monotonically increasing and unbounded, $\nu_n < n$, $n \in \mathbb{N}$. Accordingly, the function γ in (1) is related to the sequence $\{d_k\}$, which generates the sequence $\{a_k\} = \{k!d_k\}$, as follows [7]

$$d_k = \sup_{|\omega|\ge 1}(\gamma(\omega)/|\omega|^k) = \mu_k^k\gamma(\mu_k),$$

where μ_k is the solution of the equation $\omega\alpha(\omega) = k$, $k \in \mathbb{N}$, $\alpha(\omega) = \gamma'(\omega)/\gamma(\omega)$; the sequence $\{\mu_k\}$ is monotonically increasing and unbounded, $\mu_k < k$, $k \in \mathbb{N}$.

Since $\gamma(x) = 1/\tilde{\gamma}(x)$, where $\tilde{\gamma}(x) = 1$, $|x| < 1$ and $\tilde{\gamma}(x) = \sup_k(|x|^k/a_k)$, $|x| \ge 1$, then γ is a continuously differentiable, even function on $\mathbb{R}$ that monotonically decreases on $[1, +\infty]$, $0 < \gamma(x) \le 1$, $x \in \mathbb{R}$. For example, if $a_k = k^{k\alpha}$, $\alpha \in (0,1)$, $k \in \mathbb{N}$, then the following inequalities hold (see [4])

$$\exp\left(-\frac{\alpha}{e}|x|^{1/\alpha}\right) \le \gamma(x) \le c\exp\left(-\frac{\alpha}{e}|x|^{1/\alpha}\right), \quad c = \exp(\alpha e/2).$$

The function $\ln\gamma$ satisfies on $(0, +\infty)$ the inequality (see [7])

$$\ln\gamma(x_1) + \ln\gamma(x_2) \ge \ln\gamma(x_1+x_2), \quad \{x_1, x_2\} \subset (0, +\infty). \tag{3}$$

From the results given in [7] it follows that the sequence $\{\varphi_\nu : \nu \ge 1\} \subset S_{a_k}^{b_n}$ converges to zero in this space if the functions φ_ν and their derivatives of arbitrary order uniformly converge to zero on each segment $[a,b] \subset \mathbb{R}$ and the following inequalities are satisfied

$$|x^k\varphi_\nu^{(n)}(x)| \le cA^kB^na_kb_n, \quad \{k,n\} \subset \mathbb{Z}_+, \quad x \in \mathbb{R}$$

with some constants $c, A, B > 0$ independent of ν.

The function g is called the multiplier in the space $S_{a_k}^{b_n}$ if $g\psi \in S_{a_k}^{b_n}$ for an arbitrary function $\psi \in S_{a_k}^{b_n}$ and the mapping $\psi \to g\psi$ is a linear and continuous operator from $S_{a_k}^{b_n}$ to $S_{a_k}^{b_n}$. Multiplier in the space $S_{a_k}^{b_n}$, $a_k = k!d_k$, $b_n = n!\rho_n$, is a function $g \in C^\infty(\mathbb{R})$, which may be analytically continued onto the whole complex plane and which satisfies the condition (see [7])

$$\forall\, \varepsilon > 0 \quad \exists\, c_\varepsilon > 0 : \quad |g(z)| \le c_\varepsilon(\gamma(\varepsilon x))^{-1}\rho(\varepsilon y), \quad z = x + iy \in \mathbb{C}.$$

In the introduced spaces $S_{a_k}^{b_n}$, $a_k = k!d_k$, $b_n = n!\rho_n$, there are defined continuous operators that are important for analysis. First of all, this is multiplication operators by x and by all polynomials; the differentiation, shift and stretching operators [7]. In particular, the operation of argument shifting $T_x : \varphi(\xi) \to \varphi(\xi + x)$ is differentiable in the spaces $S_{a_k}^{b_n}$ (even infinitely differentiable) in the sense that the limit relations of the form

$$(\varphi(x+h) - \varphi(x))/h^{-1} \to \varphi'(x), \quad h \to 0,$$

hold for each function $\varphi \in S_{a_k}^{b_n}$ with respect to convergence in the topology of $S_{a_k}^{b_n}$. The spaces $S_{a_k}^{b_n}$ are perfect [7] (that is, spaces whose bounded sets are compact); they are interconnected by the Fourier transform, namely, the formula $F[S_{a_k}^{b_n}] = S_{b_k}^{a_n}$ is correct (see [4]), where

$$F\left[S_{a_k}^{b_n}\right] = \left\{\psi : \ \psi(\sigma) = \int_{\mathbb{R}} \varphi(x) e^{i\sigma x} dx, \ \varphi \in S_{a_k}^{b_n}\right\}.$$

In particular, $F[S_{k^{k\alpha}}^{n^{n\beta}}] = S_{k^{k\beta}}^{n^{n\alpha}}$ or $F[S_\alpha^\beta] = S_\beta^\alpha$.

The set of functions that are extensions onto $\mathbb{C}$ of functions from the space $S_{a_k}^{b_n}$, $\{a_k\} = \{k!d_k\}$, $\{b_n\} = \{n!\rho_n\}$, is denoted by symbol $S_{a_k}^{b_n}(\mathbb{C})$. In the spaces $S_{a_k}^{b_n}(\mathbb{C})$ we can introduce the topology of the inductive limit of countably normed spaces. The sequence of functions $\{\varphi_\nu : \nu \geq 1\} \subset S_{a_k}^{b_n}$ converges to zero if and only if the sequence of functions $\{\varphi_\nu(z) : \nu \geq 1\}$, $z \in \mathbb{C}$, uniformly converges to zero in each bounded area of the complex plane $\mathbb{C}$, and the following inequality holds

$$|\varphi_\nu(z)| \leq c\gamma(ax)\rho(by), \quad z = x + iy \in \mathbb{C},$$

with constants $c, a, b > 0$ independent of ν (see [7]). Moreover, the sequence $\{\varphi_\nu(x) : \nu \geq 1\}$, $x \in \mathbb{R}$, converges to zero in space $S_{a_k}^{b_n}$ if and only if the sequence $\{\varphi_\nu(z) : \nu \geq 1\}$, $z \in \mathbb{C}$, converges to zero in space $S_{a_k}^{b_n}(\mathbb{C})$ (see [7]). The multiplier in the space $S_{a_k}^{b_n}(\mathbb{C})$ is every entire function $g(z)$, $z \in \mathbb{C}$, that satisfies the condition

$$\forall\, \varepsilon > 0 \quad \exists\, c_\varepsilon > 0 : \quad |g(z)| \leq c_\varepsilon(\gamma(\varepsilon x))^{-1}\rho(\varepsilon y), \quad z = x + iy \in \mathbb{C}.$$

Respectively, function $g(x)$, $x \in \mathbb{R}$, is a multiplier in the space $S_{a_k}^{b_n}$.

The symbol $(S_{a_k}^{b_n})'$ denotes the space of all linear continuous functionals over the corresponding space of test functions with weak convergence, and its elements will be called generalized functions.

Since the operation of argument shift $T_x : \psi(\xi) \to \psi(\xi + x)$ is defined in the test space $S_{a_k}^{b_n}$, the convolution of a generalized function $f \in (S_{a_k}^{b_n})'$ with a test function may be defined by the formula

$$(f * \psi)(x) := \langle f_\xi, T_{-x}\check{\psi}(\xi)\rangle \equiv \langle f_\xi, \psi(x - \xi)\rangle, \quad \check{\psi}(\xi) = \psi(-\xi),$$

(the index ξ in f_ξ means that the functional f acts on ψ as a function of the argument ξ). The convolution $f * \psi$ is an infinitely differentiable function. The functional f is called a convolutor in the space $S_{a_k}^{b_n}$ if $f * \psi \in S_{a_k}^{b_n}$ for any $\psi \in S_{a_k}^{b_n}$ and the relation $\psi_\nu \to 0$ as $\nu \to +\infty$ implies $f * \psi_\nu \to 0$ as $\nu \to +\infty$ in the topology of $S_{a_k}^{b_n}$.

Since each space of type S together with a function $\psi(x)$ also contains the function $\psi(-x)$ and $F^{-1}[\psi] = (2\pi)^{-1}F[\psi(-\xi)]$, the Fourier transform of the generalized function $f \in (S_{a_k}^{b_n})'$ is determined by the relation

$$\langle F[f], \psi\rangle = \langle f, F[\psi]\rangle, \quad \forall\, \psi \in S_{b_k}^{a_n},$$

while $F[f] \in (S_{b_k}^{a_n})'$. If $f \in (S_{a_k}^{b_n})'$ is a convolutor in the space $S_{a_k}^{b_n}$, then for an arbitrary function $\psi \in S_{a_k}^{b_n}$ the formula $F[f * \psi] = F[f]F[\psi]$ is correct [7].

2 Infinite order differentiation operators

From the properties of the Fourier transform in spaces of S type it follows that in the space $S_{a_k}^{b_n}$ the pseudodifferential operator $A_g\varphi = F^{-1}_{\sigma\to x}[g(\sigma)F_{x\to\sigma}[\varphi]]$, $\forall\ \varphi \in S_{a_k}^{b_n}$, is well defined and it is continuous. This operator is built by the function (symbol) g, which is a multiplier in the space $S_{b_k}^{a_n}$. If the operator A_g acts in the space $S_{b_k}^{b_n}$, $b_n = n!\rho_n$, the operator A_g can be understood as a differentiation operator of "infinite order". If $g(z) = \sum_{n=0}^{\infty} c_n z^n$, $z \in \mathbb{C}$, is some entire function, then we say that in space $S_{b_k}^{b_n}(\mathbb{C})$ an infinite order differentiation operator $g(D) := \sum_{n=0}^{\infty} c_n (iD)^n$, $D = d/dz$, is specified if for an arbitrary function $\varphi \in S_{b_k}^{b_n}(\mathbb{C})$ series

$$\psi(z) \equiv (g(D)\varphi)(z) := \sum_{n=0}^{\infty} c_n (iD)^n \varphi(z), \quad z \in \mathbb{C},$$

represents the test function from the space $S_{b_k}^{b_n}(\mathbb{C})$. The restriction of the operator $g(D)$ to the space $S_{b_k}^{b_n}$, which we denote by the symbol A_g, will be called a differential operator of infinite order in the space $S_{b_k}^{b_n}$.

Theorem 1. *If the entire function g is a multiplier in the space $S_{b_k}^{b_n}(\mathbb{C})$, then in this space the continuous operator $g(D)$ is defined, and*

$$A_g\varphi(x) = F^{-1}[g(\sigma)F[\varphi](\sigma)](x), \quad \{x,\sigma\} \subset \mathbb{R}, \quad \varphi \in S_{b_k}^{b_n}. \tag{4}$$

Proof. Let us write down (so far formally) the relation

$$F[\psi](\sigma) = \sum_{n=0}^{\infty} c_n F[(iD)^n \varphi](\sigma) = \sum_{n=0}^{\infty} c_n \sigma^n F[\varphi](\sigma) = g(\sigma)F[\varphi](\sigma), \quad \varphi \in S_{b_k}^{b_n}, \quad \sigma \in \mathbb{R}. \tag{5}$$

Since $F[\varphi] \in S_{b_k}^{b_n}$, and g is a multiplier in this space, we have $gF[\varphi] \in S_{b_k}^{b_n}$. Then the function $gF[\varphi]$ can be analytically extended onto the whole complex plane, and $(gF[\varphi])(z) \in S_{b_k}^{b_n}(\mathbb{C})$, $z = x + iy \in \mathbb{C}$. Therefore, it suffices to prove the correctness of the transformations and justify the correctness of the formulas (5); hence the statements (4) will already follow. Therefore it suffices to establish that

$$r_n(z) := \sum_{k=n+1}^{\infty} c_k z^k F[\varphi](z) \to 0, \quad n \to \infty,$$

in the space $S_{b_k}^{b_n}(\mathbb{C})$. In other words, we need to show that: 1) $\{r_n : n \geq 1\} \subset S_{b_k}^{b_n}(\mathbb{C})$; 2) the sequence converges uniformly to zero in each bounded domain of the complex plane and the following inequalities are true

$$|r_n(z)| \leq c\gamma(a\sigma)\rho(by), \quad \gamma = 1/\rho, \quad z = \sigma + iy \in \mathbb{C}, \quad n \in \mathbb{N},$$

with some constants $a, b, c > 0$ independent of n.

Taylor coefficients c_n, $n \in \mathbb{Z}_+$, of function g are calculated by Cauchy's formula

$$c_n = \frac{1}{2\pi i} \int\limits_{\Gamma_R} \frac{g(z)}{z^{n+1}} dz, \quad n \in \mathbb{Z}_+,$$

where Γ_R is a circle of radius R with center at the point $z_0 = 0$. Hence, from the condition of the theorem (g is the multiplier in $S_{b_k}^{b_n}(\mathbb{C})$) it follows that

$$|c_n| \leq c_\varepsilon \inf_R \frac{(\gamma(\varepsilon R))^{-1}}{R^{n/2}} \inf_R \frac{\rho(\varepsilon R)}{R^{n/2}} = c_\varepsilon \inf_R \frac{\rho(\varepsilon R)}{R^{n/2}} \inf_R \frac{\rho(\varepsilon R)}{R^{n/2}}, \quad \varepsilon > 0.$$

Let us estimate separately the coefficients c_{2k} and c_{2k+1}, $k \in \mathbb{Z}_+$. So,

$$|c_{2k}| \leq c_\varepsilon \Big(\inf_R \frac{\rho(\varepsilon R)}{R^k} \Big)^2 = c_\varepsilon \varepsilon^{2k} \Big(\inf_R \frac{\rho(\varepsilon R)}{(\varepsilon R)^k} \Big)^2 = c_\varepsilon \varepsilon^{2k} \rho_k^2. \tag{6}$$

Similarly,

$$|c_{2k+1}| \leq c_\varepsilon \inf_R \frac{\rho(\varepsilon R)}{R^k} \inf_R \frac{\rho(\varepsilon R)}{R^{k+1}} \leq c_\varepsilon \varepsilon^{2k+1} \rho_k \rho_{k+1} \leq c_\varepsilon \varepsilon^{2k+1} \rho_k^2 \tag{7}$$

(it is taken into account that the sequence $\{\rho_k : k \in \mathbb{Z}_+\}$ is monotonically decreasing). Next, we estimate the function $\alpha_n(z) := |c_n z^n F[\varphi](z)|$, $z \in \mathbb{C}$, with a fixed $n \in \mathbb{N}$, if $n = 2k$ and $n = 2k+1$, $k \in \mathbb{Z}_+$, taking into account the inequalities (6) and (7), respectively.

Let $n = 2k$. Since $F[\varphi] \in S_{b_k}^{b_n}(\mathbb{C})$, we have

$$\exists\, c, a, b > 0 \quad \forall\, z = \sigma + iy \in \mathbb{C}: \quad |F[\varphi](z)| \leq c\gamma(a\sigma)\rho(by), \quad \gamma = 1/\rho.$$

In addition,

$$|z|^{2k} = (\sigma^2 + y^2)^k \leq (2\max\{\sigma^2, y^2\})^k \leq 2^k(|\sigma|^{2k} + |y|^{2k}).$$

So,

$$\begin{aligned} \alpha_{2k}(z) \leq\ & c c_\varepsilon z^k \varepsilon^{2k} \rho_k^2 (|\sigma|^{2k} + |y|^{2k}) \gamma(a\sigma)\rho(by) \\ &= c c_\varepsilon z^k \varepsilon^{2k} (\rho_k^2 (|\sigma|^{2k} \gamma(a\sigma)\rho(by) + \rho_k^2 |y|^{2k} \gamma(a\sigma)\rho(by)) = c c_\varepsilon \varepsilon^{2k} (\Delta_k'(z) + \Delta_k''(z)). \end{aligned}$$

Since

$$\rho_k = \inf_{\sigma \neq 0} \frac{\rho(\sigma)}{|\sigma|^k} = \Big(\frac{a}{4}\Big)^k \inf \frac{\rho(a\sigma/4)}{|a\sigma/4|^k},$$

we have

$$\rho_k^2 |\sigma|^{2k} \leq \Big(\frac{a}{4}\Big)^{2k} \frac{\rho^2(a\sigma/4)}{|a\sigma/4|^{2k}} |\sigma|^{2k} = \rho^2\Big(\frac{a}{4}\sigma\Big).$$

From the inequality (3) it follows

$$\gamma\Big(\frac{a}{2}\sigma\Big) = \gamma\Big(\frac{a}{4}\sigma\Big) + \gamma\Big(\frac{a}{4}\sigma\Big) \leq \gamma^2\Big(\frac{a}{4}\sigma\Big). \tag{8}$$

Since $\rho = 1/\gamma$, taking into account (8), we find that

$$\begin{aligned} \Delta_k'(z) = \rho_k^2 |\sigma|^{2k} \gamma(a\sigma)\rho(b\tau) &\leq \rho^2\Big(\frac{a}{4}\sigma\Big)\gamma\Big(\frac{a}{2}\sigma\Big)\gamma\Big(\frac{a}{2}\sigma\Big)\rho(by) \\ &\leq \frac{\gamma^2(a\sigma/4)}{\gamma^2(a\sigma/4)}\gamma\Big(\frac{a}{2}\sigma\Big)\rho(by) = \gamma\Big(\frac{a}{2}\sigma\Big)\rho(by). \end{aligned}$$

Let us estimate $\Delta_k''(z)$. Taking into account the convexity of the function $\ln\rho$, we obtain the following relations

$$\begin{aligned} \Delta_k''(z) =\ \rho_k^2 y^{2k} \gamma(a\sigma) e^{\ln\rho(by)} &= \rho_k^2 y^{2k} e^{-\ln\rho(\varepsilon_0 y)} e^{\ln\rho(by) + \ln\rho(\varepsilon_0 y)} \gamma(a\sigma) \\ &\leq \rho_k^2 y^{2k} e^{-\ln\rho(\varepsilon_0 y)} e^{\ln\rho((b+\varepsilon_0)y)} \gamma(a\sigma), \end{aligned}$$

($\varepsilon_0 > 0$ is arbitrarily fixed number). Next, we use the fact that $\rho_k = \nu_k^{-k}\rho(\nu_k) = \nu_k^{-k}e^k$, $k \geq 1$, where ν_k is the solution of the equation $x\mu(x) = k$, $x \geq 0$, $k \in \mathbb{N}$, $\mu = \rho'/\rho$, $\mu(1) > 1$. In fact, since

$$\ln \rho(y) = \int_0^y \mu(\xi)d\xi,$$

by the mean value theorem for a certain integral we have

$$\ln \rho(\nu_k) = \int_0^{\nu_k} \mu(\xi)d\xi = \nu_k\mu(\tilde{\nu}_k) \leq \nu_k\mu(\nu_k) < k, \quad 0 < \tilde{\nu}_k < \nu_k,$$

(here it is taken into account that μ is a monotonically increasing and continuous on $[0, +\infty)$ function [7]). Then $\rho(\nu_k) \leq e^k$, $k \in \mathbb{N}$. Next, we directly find that

$$\sup_{y \geq 0}(y^{2k}e^{-\ln \rho(\varepsilon_0 y)}) = \tilde{\tilde{\nu}}_k^{2k}e^{-\ln \rho(\varepsilon_0 \tilde{\tilde{\nu}}_k)} \leq \tilde{\tilde{\nu}}_k^{2k},$$

where $\tilde{\tilde{\nu}}_k$ is the solution of the equation $x\mu(x) = 2k$, $k \in \mathbb{N}$. Note that

$$\frac{\tilde{\tilde{\nu}}_k}{\nu_k} = \frac{\tilde{\tilde{\nu}}_k\mu(\nu_k)}{\nu_k \cdot \mu(\nu_k)} = \frac{\tilde{\tilde{\nu}}_k\mu(\nu_k)}{k}, \quad k \in \mathbb{N}.$$

Since $\nu_k \leq \tilde{\tilde{\nu}}$, a μ is increasing and continuous on $[0, +\infty)$ function, then $\mu(\nu_k) \leq \mu(\tilde{\tilde{\nu}})$. Therefore

$$\frac{\tilde{\tilde{\nu}}_k}{\nu_k} \leq \frac{\tilde{\tilde{\nu}}_k\mu(\tilde{\tilde{\nu}}_k)}{k} = \frac{2k}{k} = 2, \quad k \in \mathbb{N}, \qquad \text{and} \qquad \Delta_k''(z) \leq (2e)^{2k}\gamma(a\sigma)\rho((b+\varepsilon_0)y).$$

So, $\alpha_{2k}(z) \leq \beta A^k\gamma(a_1\sigma)\rho(b_1 y)$, $z \in \mathbb{C}$, $k \in \mathbb{Z}_+$, $b_1 = b + \varepsilon_0$. Similarly, we estimate $\alpha_{2k+1}(z)$, $k \in \mathbb{Z}_+$, $z \in \mathbb{C}$. As a result, we get $\alpha_n(z) \leq \tilde{\beta}\tilde{A}^n\varepsilon^n\gamma(a_2\sigma)\rho(b_2 y)$, $n \in \mathbb{Z}_+$, $z \in \mathbb{C}$, and all constants do not depend on n. So

$$|r_n(z)| \leq \tilde{\beta} \sum_{k=n+1}^{\infty} \tilde{A}^k\varepsilon^k\gamma(a_2\sigma)\rho(b_2 y), \quad z \in \mathbb{C}.$$

Let $\varepsilon = (2\tilde{A})^{-1}$. Then $\sum_{k=n+1}^{\infty} \tilde{A}^k\varepsilon^k = 2^{-n}$, i.e.

$$|r_n(z)| \leq \frac{\tilde{\beta}}{2^n}\gamma(a_2\sigma)\rho(b_2 y), \quad z \in \mathbb{C}. \tag{9}$$

From (9) it follows that: a) $r_n \in S_{b_k}^{b_n}(\mathbb{C})$ for each $n \in \mathbb{N}$ (i.e. condition 1) is satisfied); b) the sequence $\{r_n : n \geq 1\}$ converges uniformly to zero as $n \to \infty$ in any bounded domain $Q \subset \mathbb{C}$, while $|r_n(z)| \leq \tilde{\beta}\gamma(a_2\sigma)\rho(b_2 y)$, $n \in \mathbb{N}$, $z \in \mathbb{C}$, where the constants $\tilde{\beta}$, a_2, $b_2 > 0$ do not depend on n. So, the sequence $\{r_n : n \geq 1\}$ converges to zero in the space $S_{b_k}^{b_n}(\mathbb{C})$. This proves that the operator $g(D)$ is defined in the space $S_{b_k}^{b_n}(\mathbb{C})$, and it maps each bounded set of this space into a bounded set of the same space. Thus, the operator $g(D)$ is continuous in the space $S_{b_k}^{b_n}(\mathbb{C})$, and the operator Ag is defined and continuous in the space $S_{b_k}^{b_n}$, and from the relation (5) it follows that the equality (4) is correct. $\square$

3 Nonlocal multipoint by time problem

Let us consider the evolution equation

$$\partial u(t,x)/\partial t = A_g u(t,x), \quad (t,x) \in (0,T] \times \mathbb{R} \equiv \Omega, \tag{10}$$

where $A_g = F^{-1}_{\sigma\to x}[g(\sigma)F_{x\to\sigma}]$ is a pseudodifferential operator in space $S^{b_n}_{b_k}$ which is constructed by the function $g(\sigma)$, $\sigma \in \mathbb{R}$. This function is a multiplier in this space and $e^g \in S^{b_n}_{b_k}$. Recall (see Section 2) that the operator A_g can be understood as a differentiation operator of infinite order in space $S^{b_n}_{b_k}$. The symbol $P^{b_n}_{b_k}$ will denote the class of functions (symbols) g that satisfy these conditions. For example, let $g(z) = -\sigma^2$, $\sigma \in \mathbb{R}$. In this case, $A_g = F[-\sigma^2 F] = -(iD_x)^2 = D_x^2$, and the equation (10) is the heat equation $\partial u/\partial t = \partial^2 u/\partial x^2$.

Since $|e^{-z^2}| = |e^{-(\sigma+iy)^2}| = e^{-\sigma^2+y^2}$, $z = \sigma + iy \in \mathbb{C}$, from this and from the characteristics of the spaces S^β_α (see [4]) it follows that $e^{-x^2} \in S^{1/2}_{1/2} \equiv S^{n^{n/2}}_{k^{k/2}}$. In addition, the function $-\sigma^2$ is a multiplier in the space $S^{1/2}_{1/2}$. Therefore, the function $g(\sigma) = -\sigma^2$, $\sigma \in \mathbb{R}$, is an element of the space $P^{n^{n/2}}_{k^{k/2}}$.

For the equation (10) we define a nonlocal multipoint by time problem: find the solution of the equation (10), which satisfies the condition

$$\mu u(t,\cdot)|_{t=0} - \sum_{k=1}^{m} \mu_k B_k u(t,\cdot)|_{t=t_k} = f. \tag{11}$$

Here $m \in \mathbb{N}$, $\{\mu, \mu_1, \ldots, \mu_m\} \subset (0,+\infty)$, $\{t_1,\ldots,t_m\} \subset (0,T]$ are fixed parameters, and $\mu > \sum_{k=1}^m \mu_k$, $0 < t_1 < t_2 < \ldots < t_m \le T$, $f \in S^{b_n}_{b_k}$, $B_1,\ldots,B_m$ are pseudodifferential operators in space $S^{b_n}_{b_k}$, which are constructed by functions (symbols) $g_1,\ldots,g_m$, respectively, and they satisfy conditions:

$$\exists\, a > 0 \quad \forall\, \varepsilon > 0: \quad 0 \le g_k(\sigma) \le \exp\{\varepsilon \ln \rho(a\sigma)\},$$
$$\exists\, L_k > 0 \quad \exists\, a > 0 \quad \forall\, \varepsilon > 0: \quad |D^s_\sigma g_k(\sigma)| \le L_k^s s! \rho_s \exp\{\varepsilon \ln \rho(a\sigma)\},$$

where $s \in \mathbb{N}$, $\sigma \in \mathbb{R}$, $k \in \{1,\ldots,m\}$.

We find the solution of the problem (10), (11) using the Fourier transform in the form $u(t,x) = F^{-1}_{\sigma\to x}[v(t,\sigma)](x)$. For the function $v : \Omega \to \mathbb{R}$ we get the following problem with the parameter σ

$$\frac{dv(t,\sigma)}{dt} = g(\sigma)v(t,\sigma), \quad (t,\sigma) \in \Omega, \tag{12}$$

$$\mu v(t,\sigma)|_{t=0} - \sum_{k=1}^{m} \mu_k g_k(\sigma) v(t,\sigma)|_{t=t_k} = \tilde{f}(\sigma), \quad \sigma \in \mathbb{R}, \tag{13}$$

where $\tilde{f}(\sigma) = F^{-1}[f](\sigma)$. The general solution of the equation (12) has the form

$$v(t,\sigma) = c\exp\{tg(\sigma)\}, \quad (t,\sigma) \in \Omega, \tag{14}$$

where $c = c(\sigma)$ is defined by the condition (13). Substituting (14) into (13), we find

$$c = \widetilde{f}(\sigma)\Big(\mu - \sum_{k=1}^{m} \mu_k g_k(\sigma)\exp\{t_k g(\sigma)\}\Big)^{-1}, \quad \sigma \in \mathbb{R}.$$

Therefore, the formal solution of the problem (10), (11) is the function

$$u(t,x) = (2\pi)^{-1} \int_{\mathbb{R}} v(t,\sigma) e^{-ix\sigma} d\sigma.$$

Let us introduce the notation $G(t,x) = F^{-1}_{\sigma \to x}[Q(t,\sigma)](x)$, where $Q(t,\sigma) = Q_1(t,\sigma)Q_2(\sigma)$, $Q_1(t,\sigma) = \exp\{tg(\sigma)\}$,

$$Q_2(\sigma) = \left(\mu - \sum_{k=1}^{m} \mu_k g_k(\sigma) \exp\{t_k g(\sigma)\}\right)^{-1} = \left(\mu - \sum_{k=1}^{m} \mu_k g_k(\sigma) Q_1(t_k,\sigma)\right)^{-1}.$$

Then, considering formally, we find that

$$u(t,x) = \int_{\mathbb{R}} G(t,x-\xi) f(\xi) d\xi = G(t,x) * f(x), \quad (t,x) \in \Omega.$$

Indeed,

$$\begin{aligned} u(t,x) =& (2\pi)^{-1} \int_{\mathbb{R}} Q(t,\sigma) \left(\int_{\mathbb{R}} f(\xi) e^{-i\sigma\xi} d\xi \right) e^{i\sigma x} d\sigma \\ =& \int_{\mathbb{R}} \left((2\pi)^{-1} \int_{\mathbb{R}} Q(t,\sigma) e^{i\sigma(x-\xi)} d\sigma \right) f(\xi) d\xi \\ =& \int_{\mathbb{R}} G(t,x-\xi) f(\xi) d\xi = G(t,x) * f(x), \qquad (t,x) \in \Omega. \end{aligned} \tag{15}$$

The correctness of the transformations here and the convergence of the corresponding integrals, and hence the correctness of the formula (15) follows from the properties of the function G, which we present below. The properties of the function G are related to the properties of the function Q, since $G = F^{-1}[Q]$.

Since $g \in P^{b_n}_{b_k}$, we have $e^g \in S^{b_n}_{b_k}$. Then (see Section 1), there are numbers $c_0, a, b > 0$ such that

$$|e^{g(z)}| \le c_0 e^{-\ln \tilde{\gamma}(a\sigma) + \ln \rho(b\tau)}, \quad \tilde{\gamma} = 1/\gamma = \rho, \quad z = \sigma + i\tau \in \mathbb{C}. \tag{16}$$

Further, we assume that the constant c_0 in the inequality (16) satisfies $c_0 \le 1$. Then

$$|e^{tg(z)}| = |e^{g(z)}|^t \le [c_0 \exp\{-\ln \tilde{\gamma}(a\sigma) + \ln \rho(b\tau)\}]^t \le \exp\{-t \ln \tilde{\gamma}(a\sigma) + t \ln \rho(b\tau)\}. \tag{17}$$

Lemma 1. *For the function $Q_1(t,\sigma) = \exp\{tg(\sigma)\}$, $\sigma \in \mathbb{R}$, $t \in (0,T]$ and its derivatives (for the variable σ) the estimates*

$$|D^s_\sigma Q_1(t,\sigma)| \le \tilde{b}^s s! \rho_s \exp\{-t \ln \tilde{\gamma}(a\sigma)\}, \quad s \in \mathbb{Z}_+, \tag{18}$$

are correct, here $a > 0$ is constant from inequality (17), $\tilde{b} > 0$ is independent of t, $\rho_s = \inf_\tau(\rho(\tau)/|\tau|^s)$.

Proof. For $t > 1$ the inequality $t \ln \rho(b\tau) \le \ln \rho(bt\tau)$, $\tau \in [0,\infty)$ is correct. This property follows from the relations

$$\ln \rho(bt\tau) = \int_0^{tb\tau} \mu(\xi) d\xi = t \int_0^{b\tau} \mu(ty) dy \ge t \int_0^{b\tau} \mu(y) dy = t \ln \rho(b\tau), \tag{19}$$

where $\mu(\xi) = \rho'(\xi)/\rho(\xi)$, while μ is a non-negative, continuous function on $\mathbb{R}$, monotonically increasing on the interval $[0, \infty)$ [7]. Then

$$|Q_1(t,z)| \leq \exp\{-t \ln \tilde{\gamma}(a\sigma) + \ln \rho(tb\tau)\}, \quad z \in \mathbb{C}, \quad t > 1. \tag{20}$$

Due to the Cauchy integral formula we have

$$D_\sigma^s Q_1(t,\sigma) = \frac{s!}{2\pi i} \int_{\Gamma_R} \frac{Q_1(t,z)}{(z-\sigma)^{s+1}} dz, \quad s \in \mathbb{Z}_+,$$

where Γ_R is a circle of radius R centered at the point σ. Using (20), we come to the inequalities

$$|D_\sigma^s Q_1(t,\sigma)| \leq \frac{s!}{R^s} \max_{z \in \Gamma_R} |Q_1(t,z)| \leq \frac{s!}{R^s} \exp\{-t \ln \tilde{\gamma}(a\sigma_0) + \ln \rho(tbR)\},$$

where σ_0 is the maximum value of the function $\exp\{-t \ln \tilde{\gamma}(a\xi)\}$, $\xi \in [\sigma - R,\ \sigma + R]$. Since $\ln \tilde{\gamma}(a\xi)$ is an even function on $\mathbb{R}$ increasing on the interval $[0, +\infty)$,

$$\sigma_0 = \begin{cases} 0, & |\xi| \leq R, \\ \sigma + R, & \xi < -R, \\ \sigma - R, & \xi > R. \end{cases}$$

Using the inequality $-\ln \tilde{\gamma}(\sigma_1 + \sigma_2) + \ln \tilde{\gamma}(\sigma_1) \leq -\ln \tilde{\gamma}(\sigma_2)$, $\sigma_1, \sigma_2 > 0$, we prove the existence of constant $a_1 > 0$, such that

$$\forall \sigma \geq 0,\ \forall R > 0 : \exp\{-t \ln \tilde{\gamma}(a\sigma_0)\} \leq \exp\{-t \ln \tilde{\gamma}(a\sigma)\} \exp\{t \ln \tilde{\gamma}(a_1 R)\}.$$

Then

$$\begin{aligned} |D_\sigma^s Q_1(t,\sigma)| &\leq \frac{s!}{R^s} \exp\{-t \ln \tilde{\gamma}(a\sigma)\} \exp\{t \ln \tilde{\gamma}(a_1 R)\} \exp\{\ln \rho(tbR)\} \\ &\leq \frac{s!}{R^s} \exp\{-t \ln \tilde{\gamma}(a\sigma)\} \exp\{\ln \rho(tb_1 R)\}, \quad b_1 = b + a_1. \end{aligned}$$

Here we used that $\tilde{\gamma} = \rho$ as well as the inequality $\ln \rho(tbR) + \ln \rho(ta_1 R) \leq \ln \rho(t(b + a_1)R)$. For each $s \in \mathbb{Z}_+$ the function $g_{s,t}(R) = R^{-s} \exp\{\ln \rho(b_1 tR)\} = R^{-s} \rho(b_1 tR)$ is differentiable on $(0, +\infty)$, and the properties of the function ρ implies the relations

$$\lim_{R \to +\infty} g_{s,t}(R) = +\infty, \quad s \in \mathbb{Z}_+; \quad \lim_{R \to +0} g_{s,t}(R) = \begin{cases} +\infty, & s \in \mathbb{N}, \\ 1, & s = 0. \end{cases}$$

Since $g_{s,t}(R) > 0$, $R \in (0, +\infty)$, this function reaches its infimum. So,

$$\begin{aligned} |D_\sigma^s Q_1(t,\sigma)| &\leq s! \inf_R g_{s,t}(R) \exp\{-t \ln \tilde{\gamma}(a\sigma)\} = s! {b_1}^s t^s \inf \frac{\rho(tb_1 R)}{(tb_1 R)^s} \exp\{-t \ln \tilde{\gamma}(a\sigma)\} \\ &= {b_1}^s t^s s! \rho_s \exp\{-t \ln \tilde{\gamma}(a\sigma)\} \leq \tilde{b}^s s! \rho_s \exp\{-t \ln \tilde{\gamma}(a\sigma)\}, \end{aligned}$$

where $\tilde{b} = b_1 T^*$, $T^* = \max\{1, T\}$. The case of $0 < t \leq 1$ is considered similarly. As a result, we arrive at estimates (18). □

Remark. *From the inequality* (17) *it follows that* $Q_1(t,\cdot) \in S_{b_k}^{b_n}$, $b_n = n!\rho_n$, *for each* $t \in (0,T]$. *In fact, if* $t \in (0,1]$, *then from* (19) *the inequality*

$$\exp\{-t \ln \tilde{\gamma}(a\sigma)\} \leq \exp\{-\ln \tilde{\gamma}(a\sigma)\}, \quad a_1 = at, \quad \tilde{\gamma} = 1/\gamma = \rho,$$

follows. Then $|Q_1(t,z)| \leq \exp\{-\ln \tilde{\gamma}(a_1\sigma) + \ln \rho(b\tau)\} = \gamma(a_1\sigma)\rho(b\tau)$.

If $t > 1$ *and* t *is non-integer, then* $t = [t] + \{t\}$. *Then (see* (19)*)*

$$e^{-t \ln \tilde{\gamma}(a\sigma)} \leq e^{-[t]\ln \tilde{\gamma}(a\sigma)} e^{-\{t\}\ln \tilde{\gamma}(a\sigma)} \leq e^{-\{t\}\ln \tilde{\gamma}(a\sigma)} \leq e^{-\ln \tilde{\gamma}(a_2\sigma)},$$
$$a_2 = a\{t\}, \quad e^{t \ln \rho(b\tau)} \leq e^{\ln \rho(bt\tau)} = e^{\ln \rho(b_1\tau)}, \quad b_1 = bt.$$

So, $|Q_1(t,z)| \leq \exp\{-\ln \tilde{\gamma}(a_2\sigma)\} \exp\{\ln \rho(b_1\tau)\} = \gamma(a_2\sigma)\rho(b_1\tau)$ *for every* $t > 1$.

If $t = n$, $n \in \{2,3,\ldots\}$, *then we may write* $t = 1 + n - 1$. *In this case the estimation* $|Q_1(t,z)| \leq \exp\{-\ln \tilde{\gamma}(a_2\sigma)\} \exp\{\ln \rho(b_2\tau)\} = \gamma(a_2\sigma)\rho(b_2\tau)$, $b_2 = n$, *is correct.*

Lemma 2. *The function*

$$Q_2 = \Big(\mu - \sum_{k=1}^{m} \mu_k g_k(\sigma) \exp\{t_k g(\sigma)\}\Big)^{-1}, \quad \sigma \in \mathbb{R},$$

is a multiplier in the space $S_{b_k}^{a_n}$, *where* $a_n = n^{2n}$, $b_k = k!\rho_k$, $\{k,n\} \in \mathbb{Z}_+$.

Proof. To prove the assertion we estimate the derivatives of Q_2. For this purpose we use Faà di Bruno's formula for differentiating a complex function

$$D_\sigma^s F(\varphi(\sigma)) = \sum_{p=1}^{s} \frac{d^p}{d\varphi^p} F(\varphi) \sum \frac{s!}{m_1! \ldots m_l!} \Big(\frac{d}{d\sigma}\varphi(\sigma)\Big)^{m_1} \ldots \Big(\frac{1}{l!}\frac{d^l}{d\sigma^l}\varphi(\sigma)\Big)^{m_l},$$

where the above sum is taken over all solutions in non-negative integers of the equation $s = m_1 + 2m_2 + \ldots + lm_l$, $p = m_1 + \ldots + m_l$. In this formula we put $F = \varphi^{-1}$, $\varphi = R$, where $R(\sigma) = \mu - \sum_{k=1}^{m} \mu_k g_k(\sigma) Q_1(t_k,\sigma)$. Then $Q_2(\sigma) = F(R(\sigma))$ and

$$\frac{d^p}{d\varphi^p} F(R) = \frac{d^p}{dR^p} R^{-1} = (-1)^p p! R^{-(p+1)}.$$

Taking into account the inequalities (18) and the properties of functions $g_1, \ldots, g_m$, we find

$$\begin{aligned}\left|\frac{1}{l!}\frac{d^l}{d\sigma^l} R(\sigma)\right| &\leq \frac{1}{l!} \sum_{k=1}^{m} \mu_k \sum_{i=0}^{l} C_l^i |D_\sigma^i g_k(\sigma)| \cdot |D_\sigma^{l-i} e^{t_k g(\sigma)}| \\ &\leq \frac{1}{l!} \sum_{k=1}^{m} \mu_k \sum_{i=0}^{l} C_l^i L_k^i i! \rho_i \tilde{b}^{l-i} (l-i)! \rho_i e^{-t_k \ln \tilde{\gamma}(a\sigma)} e^{\varepsilon \ln \rho(a\sigma)}.\end{aligned}$$

Note that $\exp\{-t_k \ln \tilde{\gamma}(a\sigma)\} = \exp\{-t_k \ln \rho(a\sigma)\} \leq \exp\{-t_1 \ln \rho(a\sigma)\}$, $k \in \{2,3,\ldots,m\}$, $\tilde{\gamma} = \rho$, $i!(l-i)! \leq l!$, $\rho_i \leq \rho_0 = 1$, $\rho_{l-i} \leq \rho_0 = 1$, $i \in \{0,1,\ldots,l\}$ (here we used the fact that the sequence $\{\rho_n : n \in \mathbb{Z}_+\}$ is monotonically decreasing, $\rho_0 = 1$). Therefore the inequality

$$\left|\frac{1}{l!}\frac{d^l}{d\sigma^l} R(\sigma)\right| \leq \frac{1}{l!} \sum_{k=1}^{m} \mu_k l! \tilde{L}_1^l \exp\{-t_1 \ln \rho(a\sigma) + \varepsilon \ln \rho(a\sigma)\}$$

is correct, where $\tilde{L}_1 = 2\max\{\tilde{L}, \tilde{B}\}$, $\tilde{L} = \max\{1, L_1, \dots, L_m\}$. If we put $\varepsilon = t_1$, we get the estimate

$$\left|\frac{1}{l!}\frac{d^l}{d\sigma^l}R(\sigma)\right| \le cL_1^l, \quad c = \sum_{k=1}^{m}\mu_k.$$

So,

$$\begin{aligned}\left|\left(\frac{d}{d\sigma}R(\sigma)\right)^{m_1}\right| \dots \left|\left(\frac{1}{l!}\frac{d^l}{d\sigma^l}R(\sigma)\right)^{m_l}\right| &\le c^{m_1}L_1^{m_1}c^{m_2}L_1^{2m_2}\dots c^{m_l}L_1^{lm_l}\\ &= c^{m_1+m_2+\dots+m_l}L_1^{m_1+2m_2+\dots+lm_l} = c^pL_1^s \le \tilde{c}^sL_1^s,\end{aligned}$$

where $\tilde{c} = \max\{1, c\}$. Taking into account the properties of the functions $g_1, \dots, g_m$ and the inequality (17), we find

$$\mu_k g_k(\sigma)e^{t_k g_k(\sigma)} \le \mu_k e^{\varepsilon \ln \rho(a\sigma)}e^{-t_k \ln \tilde{\gamma}(a\sigma)} \le \mu_k e^{\varepsilon \ln \rho(a\sigma) - t_1 \ln \rho(a\sigma)} = \mu_k,$$

where $\varepsilon = t_1$. Then

$$R_2(\sigma) = \mu - \sum_{k=1}^{m}\mu_k g_k(\sigma)e^{t_k g(\sigma)} \ge \mu - \sum_{k=1}^{m}\mu_k, \quad Q_2(\sigma) = R^{-1}(\sigma) \le \left(\mu - \sum_{k=1}^{m}\mu_k\right)^{-1} \equiv \beta_0 > 0.$$

Therefore, $R^{-(p+1)} \le \beta_0^{p+1}$. Summing up, we find $|D_\sigma^s Q_2(\sigma)| \le b_0B_0^s(s!)^2 \le bB^s s^{2s}$, $s \in \mathbb{Z}_+$.

From the last inequality and boundedness of the function Q_2 on $\mathbb{R}$ it follows that Q_2 is a multiplier in the space $S_{b_k}^{a_n}$, where $a_n = n^{2n}$, $b_k = k!\rho_k$.

From Lemma 1 and Lemma 2 it follows that for every fixed $t \in (0, T]$ the function $Q(t,\sigma) = Q_1(t,\sigma)Q_2(\sigma)$ as function of variable σ is an element of space $S_{b_k}^{a_n}$, $a_n = n^{2n}$ and the inequality

$$|D_\sigma^s Q(t,\sigma)| \le cB^s s^{2s}\exp\{-t\ln\tilde{\gamma}(a\sigma)\} \tag{21}$$

is correct with constants $c, B, a > 0$ independent of t.

Since $G(t,\cdot) = F^{-1}[Q(t,\cdot)]$, taking into account the properties of direct and inverse Fourier transforms in spaces of type S, we obtain $G(t,\cdot) \in S_{a_k}^{b_n}$, $b_n = n!\rho_n$, $a_k = k^{2k}$ for each $t \in (0, T]$.

Note that representation

$$\begin{aligned}Q_2(\sigma) &= \left(\mu - \sum_{k=1}^{m}\mu_k g_k(\sigma)e^{t_k g(\sigma)}\right)^{-1} = \frac{1}{\mu}\left(1 - \frac{1}{\mu}\sum_{k=1}^{m}\mu_k g_k(\sigma)e^{t_k g(\sigma)}\right)^{-1}\\ &= \frac{1}{\mu}\sum_{r=0}^{\infty}\mu^{-r}\left(\sum_{k=1}^{m}\mu_k g_k(\sigma)e^{t_k g(\sigma)}\right)^r\\ &= \sum_{r=0}^{\infty}\mu^{-(r+1)}\sum_{r_1+\dots+r_m=r}\frac{r!}{r_1!\dots r_m!}(\mu_1 g_1(\sigma)e^{t_1 g(\sigma)})^{r_1}\dots(\mu_m g_m(\sigma)e^{t_m g(\sigma)})^{r_m}\\ &= \sum_{r=0}^{\infty}\mu^{-(r+1)}\sum_{r_1+\dots+r_m=r}\frac{r!}{r_1!\dots r_m!}\mu_1^{r_1}\dots\mu_m^{r_m}g_1^{r_1}(\sigma)\dots g_m^{r_m}(\sigma)e^{(t_1r_1+\dots+t_mr_m)g(\sigma)}\end{aligned}$$

is also correct for the function Q_2. Here we used polynomial formula and inequality

$$\frac{1}{\mu}\sum_{k=1}^{m}\mu_k g_k(\sigma)e^{t_k g(\sigma)} < 1.$$

From this fact we get

$$G(t,x) = (2\pi)^{-1}\int_{\mathbb{R}} e^{tg(\sigma)}Q_2(\sigma)e^{-i\sigma x}d\sigma = \sum_{r=0}^{\infty}\frac{1}{\mu^{r+1}}\sum_{r_1+\ldots+r_m=r}\frac{r!\mu_1^{r_1}\ldots\mu_m^{r_m}}{r_1!\ldots r_m!}\tilde{G}(\lambda+t,x),$$

where

$$\tilde{G}(\lambda+t,x) = (2\pi)^{-1}\int_{\mathbb{R}} e^{(\lambda+t)g(\sigma)}g_1^{r_1}(\sigma)\ldots g_m^{r_m}(\sigma)e^{-i\sigma x}d\sigma,$$

$\lambda := t_1r_1+\ldots+t_mr_m+t$, and $G(t,x)$ is the fundamental solution of Cauchy problem for equation (10). □

Lemma 3. *The function* $G(t,\cdot)$, $t\in(0,T]$, *as an abstract function of the parameter* t *with values in the space* $S_{a_k}^{b_n}\equiv S_{k^{2k}}^{b_n}$ *differentiable by* t.

Proof. Continuity of direct and inverse Fourier transforms in spaces of type S implies that we only need to establish that the function $F[G(t,\cdot)] = Q(t,\cdot)$ is differentiable by t as an abstract function of the parameter t with values in the space $F[S_{a_k}^{b_n}] = S_{b_k}^{a_n}$, i.e. we need to prove that the limit relation

$$\Phi_{\Delta t}(\sigma) := \frac{1}{\Delta t}[Q(t+\Delta t,\sigma)-Q(t,\sigma)] \to \frac{\partial}{\partial t}Q(t,\sigma), \quad \Delta t\to 0,$$

is performed in the following sense: 1) $D_\sigma^s\Phi_{\Delta t}(\sigma) \underset{\Delta t\to 0}{\longrightarrow} D_\sigma^s g(\sigma)Q(t,\sigma)$, $s\in\mathbb{Z}_+$, uniformly on each segment $[a,b]\subset\mathbb{R}$; 2) $|D_\sigma^s\Phi_{\Delta t}(\sigma)| \leq \bar{c}\bar{B}^s s^{2s}e^{-\ln\tilde{\gamma}(\bar{a}\sigma)}$, $s\in\mathbb{Z}_+$, where the constants $\bar{c},\bar{a},\bar{B}>0$ do not depend on Δt for rather small values of Δt.

The function $Q(t,\sigma)$, $(t,\sigma)\in(0,T]\times\mathbb{R}$, is differentiable by t in the usual sense, therefore, due to Lagrange's mean value theorem, we have $\Phi_{\Delta t}(\sigma) = g(\sigma)Q(t+\theta\Delta t,\sigma)$, $0<\theta<1$, $t+\theta\Delta t\leq T$. So,

$$D_\sigma^s\Phi_{\Delta t}(\sigma) = \sum_{l=0}^{s} C_s^l D_\sigma^l g(\sigma) D_\sigma^{s-l}Q(t+\theta\Delta t,\sigma)$$

and

$$D_\sigma^s\left(\Phi_{\Delta t}(\sigma) - \frac{\partial}{\partial t}Q(t,\sigma)\right) = \sum_{l=0}^{s} C_s^l D_\sigma^l g(\sigma)[D_\sigma^{s-l}Q(t+\theta\Delta t,\sigma) - D_\sigma^{s-l}Q(t,\sigma)].$$

From the equality $D_\sigma^{s-l}Q(t+\theta\Delta t,\sigma) - D_\sigma^{s-l}Q(t,\sigma) = D_\sigma^{s-l+1}Q(t+\theta_1\Delta t,\sigma)\theta\Delta t$, $0<\theta_1<1$, and from (21) it follows that $D_\sigma^{s-l+1}Q(t+\theta_1\Delta t,\sigma)\theta\Delta t\to 0$, $\Delta t\to 0$, uniformly on each segment $[a,b]\subset\mathbb{R}$. Then

$$D_\sigma^s\Phi_{\Delta t}(\sigma) \to D_\sigma^s\left(\frac{\partial}{\partial t}Q(t,\sigma)\right), \quad \Delta t\to 0,$$

uniformly on the segment $[a,b]\subset\mathbb{R}$. So, condition 1) holds.

Since g is a multiplier in the space $S_{b_k}^{n!\rho_n}$, we obtain

$$\forall\,\varepsilon>0 \quad \exists\, c_\varepsilon>0 \quad \forall\, z=\sigma+i\tau\in\mathbb{C}: \quad |g(z)|\leq c_\varepsilon e^{\ln\tilde{\gamma}(\varepsilon\sigma)+\ln\rho(\varepsilon\tau)}. \tag{22}$$

Due to the Cauchy integral formula we have that

$$g^{(n)}(\sigma) = \frac{n!}{2\pi i}\int_{\Gamma_R}\frac{g(z)}{(z-\sigma)^{n+1}}dz, \quad n\in\mathbb{Z}_+,$$

where Γ_R is a circle of radius R centered at the point $\sigma \in \mathbb{R}$. Then, taking into account (22), we get the inequalities

$$|g^{(n)}(\sigma)| \leq \frac{n!}{R^n} \max_{z\in\Gamma_R} |g(z)| \leq c_\varepsilon \frac{n!}{R^n} e^{\ln\tilde{\gamma}(\varepsilon(\sigma+R))+\ln\rho(\varepsilon R)}$$
$$\leq c_\varepsilon n! \inf_R \frac{\rho(\varepsilon R)}{R^n} e^{\ln\tilde{\gamma}(\varepsilon(\sigma+R))} = c_\varepsilon \varepsilon^n n! \rho_n e^{\ln\tilde{\gamma}(\varepsilon(\sigma+R))}, \quad \sigma \geq 0.$$

For sufficiently large values $\sigma \geq 0$, the inequality $\varepsilon(\sigma+R) \leq (\varepsilon+R)\sigma$ holds. Since the function $\ln\tilde{\gamma}$ increases monotonically for $\sigma \geq 0$, we have for the same values of σ that $\ln\tilde{\gamma}(\varepsilon(\sigma+R)) \leq \ln\tilde{\gamma}(\varepsilon+R)\sigma)$. For all $\sigma \geq 0$ the inequality $\ln\tilde{\gamma}(\varepsilon(\sigma+R)) \leq \ln\tilde{\gamma}((\varepsilon+R)\sigma)+c_R$ holds. So, for $\sigma \geq 0$ we have

$$\exp\{\ln\tilde{\gamma}(\varepsilon(\sigma+R))\} \leq \tilde{c}_R \exp\{\ln\tilde{\gamma}((\varepsilon+R)\sigma)\}.$$

In what follows, for a given $\varepsilon > 0$, we assume that $R = \varepsilon$. Then

$$|g^{(n)}(\sigma)| \leq \tilde{c}_\varepsilon \varepsilon^n n! \rho_n e^{\ln\tilde{\gamma}(2\varepsilon\sigma)} \leq \tilde{c}'_\varepsilon \varepsilon^n n! e^{\ln\tilde{\gamma}(2\varepsilon\sigma)}, \quad n \in \mathbb{Z}_+,$$

(here we take into account that the sequence $\{\rho_n\}$ is monotonically increasing). Due to above inequality and estimates for the derivatives of $Q(t,\sigma)$, we find that

$$|D^s_\sigma \Phi_{\Delta t}(\sigma)| \leq \tilde{\tilde{c}}_\varepsilon \sum_{l=0}^{s} C^l_s \varepsilon^l l! B^{s-l}(s-l)^{2(s-l)} e^{\ln\tilde{\gamma}(2\varepsilon\sigma)-(t+\theta\Delta t)\ln\tilde{\gamma}(a\sigma)}$$
$$\leq \bar{c}\bar{B}^s s^{2s} e^{\ln\tilde{\gamma}(2\varepsilon\sigma)-t\ln\tilde{\gamma}(a\sigma)} \leq \bar{c}\bar{B}^s s^{2s} e^{\ln\tilde{\gamma}(2\varepsilon\sigma)-t\ln\tilde{\gamma}(a\sigma)}$$

(we assume $t+\theta\Delta t > 0$, $t \in (0,T]$ is fixed number). Take $\varepsilon = ta/4$. From the convexity of the function $\ln\tilde{\gamma}$ it follows

$$\ln\tilde{\gamma}(2\varepsilon\sigma) - \ln\tilde{\gamma}(at\sigma) \leq -\ln\tilde{\gamma}((at-2\varepsilon)\sigma) \equiv -\ln\tilde{\gamma}(\bar{a}\sigma), \quad \bar{a} = at - 2\varepsilon = at/2 > 0.$$

Then

$$|D^s_\sigma \Phi_{\Delta t}(\sigma)| \leq \bar{c}\bar{B}^s s^{2s} e^{-\ln\tilde{\gamma}(\bar{a}\sigma)}, \quad \sigma \geq 0,$$

moreover, the constants $\bar{c}, \bar{a}, \bar{B} > 0$ do not depend on Δt (for enough small values of Δt). The case of $\sigma < 0$ may be considered similarly. So, the condition 2) also holds. □

Lemma 4. *The formula*

$$\frac{\partial}{\partial t}(f * G(t,x)) = f * \frac{\partial G(t,x)}{\partial t}, \quad \forall\, f \in \left(S^{b_n}_{a_k}\right)', \quad t \in (0,T], \quad a_k = k^{2k}, \quad k \in \mathbb{Z}_+,$$

is correct.

Proof. By the definition of the convolution of a generalized function with a test function, we have

$$f * G(t,x) = \langle f_\xi, T_{-x}\check{G}(t,\xi)\rangle, \quad \check{G}(t,\xi) = G(t,-\xi).$$

Then

$$\frac{\partial}{\partial t}(f * G(t,x)) = \lim_{\Delta t\to 0} \frac{1}{\Delta t}[f * G(t+\Delta t,\cdot) - f * G(t,\cdot)]$$
$$= \lim_{\Delta t\to 0} \left\langle f_\xi, \frac{1}{\Delta t}[T_{-x}\check{G}(t+\Delta t,\xi) - T_{-x}\check{G}(t,\xi)]\right\rangle.$$

Due to Lemma 3 the limit relation

$$\frac{1}{\Delta t}[T_{-x}\check{G}(t+\Delta t,\cdot) - T_{-x}\check{G}(t,\cdot)] \xrightarrow[\Delta t\to 0]{} \frac{\partial}{\partial t}T_{-x}\check{G}(t,\cdot)$$

is performed in the sense of convergence in the topology of space $S_{a_k}^{b_n} \equiv S_{k^{2k}}^{n!\rho_n}$, therefore, taking into account the continuity of the functional f, we have that

$$\begin{aligned}\frac{\partial}{\partial t}(f * G(t,\cdot)) &= \Big\langle f_\xi, \lim_{\Delta t\to 0}\frac{1}{\Delta t}[T_{-x}\check{G}(t+\Delta t,\xi) - T_{-x}\check{G}(t,\xi)]\Big\rangle \\ &= \Big\langle f_\xi, \frac{\partial}{\partial t}T_{-x}\check{G}(t,\xi)\Big\rangle = \Big\langle f_\xi, T_{-x}\frac{\partial}{\partial t}\check{G}(t,\xi)\Big\rangle = f * \frac{\partial G(t,x)}{\partial t}.\end{aligned}$$

□

Lemma 5. *In the space $(S_{a_k}^{b_n})' \equiv (S_{k^{2k}}^{n!\rho_n})'$ the boundary relation*

$$\mu \lim_{t\to+0} G(t,\cdot) - \sum_{l=1}^{m}\mu_l \lim_{t\to t_l} B_l G(t,\cdot) = \delta \tag{23}$$

holds (here δ is the Dirac delta function).

Proof. Using the continuity of the Fourier transform and the function $G(t,\cdot)$ as an abstract function of the parameter t with values in the space $S_{k^{2k}}^{b_n}$, from the relation (23) we obtain

$$\mu \lim_{t\to+0} F[G(t,\cdot)] - \sum_{l=1}^{m}\mu_l \lim_{t\to t_l} F[B_l G(t,\cdot)] = F[\delta] \tag{24}$$

in the space $(S_{b_k}^{n^{2n}})'$. Considering the representation of the function G, (24) is given as

$$\mu \lim_{t\to+0} Q(t,\sigma) - \sum_{l=1}^{m}\mu_l \lim_{t\to t_l} g_l(\sigma)Q(t,\sigma) = 1. \tag{25}$$

To prove (25) we take an arbitrary function $\varphi \in S_{b_k}^{n^{2n}}$ and using the limit transition theorem under the Lebesgue integral and treating $Q(t,\cdot)$, $g_l Q(t,\cdot)$, $l \in \{1,\dots,m\}$, as regular distributions from the space $(S_{b_k}^{n^{2n}})'$, we find that

$$\begin{aligned}\mu \lim_{t\to+0}\langle Q(t,\cdot),\varphi\rangle - &\sum_{l=1}^{m}\mu_l \lim_{t\to t_l}\langle g_l(\cdot)Q(t,\cdot),\varphi\rangle \\ &= \mu \lim_{t\to+0}\int_{\mathbb{R}} Q(t,\sigma)\varphi(\sigma)d\sigma - \sum_{l=1}^{m}\mu_l \lim_{t\to t_l}\int_{\mathbb{R}} g_l(\sigma)Q(t_l,\sigma)\varphi(\sigma)d\sigma \\ &= \int_{\mathbb{R}}\left[\mu Q(0,\sigma) - \sum_{l=1}^{m}\mu_l g_l(\sigma)Q(t_l,\sigma)\right]\varphi(\sigma)d\sigma \\ &= \int_{\mathbb{R}}\left[\frac{\mu}{\mu - \sum_{k=1}^m \mu_k g_k(\sigma)Q_1(t_k,\sigma)} - \sum_{l=1}^{m}\mu_l\frac{g_l(\sigma)Q_1(t_l,\sigma)}{\mu - \sum_{k=1}^m \mu_k g_k(\sigma)Q_1(t_k,\sigma)}\right]\varphi(\sigma)d\sigma \\ &= \int_{\mathbb{R}}\frac{\mu - \sum_{l=1}^m \mu_l g_l(\sigma)Q_1(t_l,\sigma)}{\mu - \sum_{k=1}^m \mu_k g_k(\sigma)Q_1(t_k,\sigma)}\varphi(\sigma)d\sigma = \int_{\mathbb{R}}\varphi(\sigma)d\sigma = \langle 1,\varphi\rangle.\end{aligned}$$

It follows that the relation (25) holds in the space $(S_{b_k}^{n^{2n}})'$, and therefore the relation (23) is correct. □

The symbol $(S^{b_n}_{k^{2k}}, *)'$ denote the class of generalized functions from $(S^{b_n}_{k^{2k}})'$, which are convolutors in the space $S^{b_n}_{k^{2k}}$.

Lemma 6. *Let*

$$\omega(t,x) = f * G(t,x), \quad f \in (S^{b_n}_{k^{2k}}, *)', \quad (t,x) \in \Omega.$$

Then in the space $(S^{b_n}_{k^{2k}})'$ the following limit relation holds

$$\mu \lim_{t\to+0} \omega(t,\cdot) - \sum_{k=1}^{m} \mu_k \lim_{t\to t_k} B_k\omega(t,\cdot) = f. \tag{26}$$

Proof. Since

$$\omega(t,x) = f * G(t,x) = \langle f_\xi, T_{-x}\check{G}(t,\xi)\rangle, \quad f \in (S^{b_n}_{k^{2k}}, *)',$$

from the continuity of $G(t,\cdot)$ as an abstract function of the parameter t with values in the space $S^{b_n}_{k^{2k}}$ it follows the continuity of $\omega(t,\cdot)$ as an abstract function of the parameter t with values in the same space. Taking into account the continuity of the Fourier transform and the formula $F[f*G] = F[f]F[G] = F[f]Q$, which is correct for an arbitrary generalized function f from the class $(S^{b_n}_{k^{2k}}, *)'$, from (26) we get the relation

$$\mu \lim_{t\to+0} F[\omega(t,\cdot)] - \sum_{k=1}^{m} \mu_k \lim_{t\to t_k} F[B_k\omega(t,\cdot)] = F[f]$$

in the space $(S^{n^{2n}}_{b_k})'$ or

$$\mu \lim_{t\to+0} Q(t,\cdot) - \sum_{k=1}^{m} \mu_k \lim_{t\to t_k} g_k(\sigma)Q(t,\cdot) = 1,$$

which, as proved earlier (see (25)), is correct in this space. This proves that the relation (26) holds in the space $(S^{b_n}_{k^{2k}})'$. □

The function G is the solution of the equation (10). Indeed,

$$\frac{\partial}{\partial t}G(t,x) = \frac{\partial}{\partial t}F^{-1}[Q(t,\sigma)](x) = F^{-1}\Big[\frac{\partial}{\partial t}Q(t,\sigma)\Big](x).$$

On the other hand,

$$A_g G(t,x) = F^{-1}_{\sigma\to x}[g(\sigma)F_{x\to\sigma}[G(t,x)]] = F^{-1}[g(\sigma)Q(t,\sigma)](x) = F^{-1}\Big[\frac{\partial}{\partial t}Q(t,\sigma)\Big](x).$$

It follows that the function G satisfies the equation (10).

Further, let us call the function G the fundamental solution of a nonlocal multipoint by time problem for the equation (10).

From the Lemma 6 it follows that for the equation (10) nonlocal multipoint by time problem can be formulated as follows: find the solution of the equation (10), which satisfies the condition

$$\mu \lim_{t\to+0} u(t,\cdot) - \sum_{k=1}^{m} \mu_k \lim_{t\to t_k} B_k u(t,\cdot) = f, \quad f \in (S^{b_n}_{k^{2k}}, *)', \tag{27}$$

where the limit relation (27) is considered in the space $(S^{b_n}_{k^{2k}})'$ (restrictions on parameters $\mu,\ \mu_1,\ \ldots,\ \mu_m,\ t_1,\ \ldots,\ t_m$ are the same as in the case of problem (10) , (11)).

Theorem 2. *The nonlocal multipoint by time problem* (10), (27) *is solvable; the solution is given by the formula* $u(t,x) = f * G(t,x)$, $(t,x) \in \Omega$, *where* G *is the fundamental solution of the multipoint problem for the equation* (10).

Proof. The function $u(t,x)$ is the solution of the equation (10). In fact (see Lemma 4),

$$\frac{\partial u(t,x)}{\partial t} = \frac{\partial}{\partial t}(f * G(t,x)) = f * \frac{\partial G(t,x)}{\partial t},$$

$$A_g u(t,x) = F^{-1}[g(\sigma)F[f * G]](x).$$

Since f is a convolutor in the space $S^{b_n}_{k^{2k}}$, we obtain

$$F[f * G(t,x)](\sigma) = F[f](\sigma)F[G(t,x)](\sigma) = F[f](\sigma)Q(t,\sigma).$$

So,

$$\begin{aligned} A_g u(t,x) &= F^{-1}[g(\sigma)Q(t,\sigma)F[f](\sigma)](x) = F^{-1}\Big[\frac{\partial}{\partial t}Q(t,\sigma)F[f](\sigma)\Big](x) \\ &= F^{-1}\Big[F\Big[\frac{\partial}{\partial t}G\Big](t,\sigma) \cdot F[f](\sigma)\Big](x) = F^{-1}\Big[F\Big[f * \frac{\partial G}{\partial t}\Big]\Big](x) = f * \frac{\partial G(t,x)}{\partial t}. \end{aligned}$$

Hence, the function $u(t,x)$, $(t,x) \in \Omega$, satisfies the equation (10). From the Lemma 6 it follows that u satisfies the condition (27) in the specified sense. □

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Received 10.10.2020
Revised 02.02.2021

Городецький В.В., Мартинюк О.В. *Еволюційні псевдодиференціальні рівняння з аналітичними символами в просторах типу S* // Карпатські матем. публ. — 2021. — Т.13, №1. — С. 160–179.

Досліджується нелокальна багатоточкова за часом задача для еволюційного рівняння з псевдодиференціальним оператором, який трактується як оператор диференціювання нескінченного порядку в узагальнених просторах типу S у випадку, коли початкова умова є елементом простору узагальнених функцій типу ультрарозподілів, а нелокальна умова містить псевдодиференціальні оператори. Встановлено розв'язність такої задачі, досліджено властивості фундаментального розв'язку, знайдено аналітичне зображення розв'язку.

Ключові слова і фрази: нелокальна багатоточкова задача, псевдодиференціальний оператор, узагальнена функція.