



SOME PROPERTIES OF THE POLYNOMIALLY BOUNDED O-MINIMAL EXPANSIONS OF THE REAL FIELD AND OF SOME QUASIANALYTIC LOCAL RINGS

BERRAHO M.

In this paper, we study the Weierstrass division theorem over the rings of smooth germs that are definable in an arbitrary polynomially bounded o-minimal expansion of the real field by giving some criteria for satisfying this theorem. Afterwards, we study some topological properties of some quasianalytic subrings of the ring of smooth germs for the (x_1) -adic topology by showing that these rings are separable metric spaces. Also, we cite a criterion for their completeness with respect to the (x_1) -adic topology.

Key words and phrases: Weierstrass division theorem, polynomially bounded o-minimal structure, quasianalytic ring, (x_1) -adic topology.

Faculty of Sciences, Ibn Tofail University, PO Box 242, Kenitra, Morocco
E-mail: b.mourad87@hotmail.com

1 INTRODUCTION

In this paper, we will study the Weierstrass division theorem over some polynomially bounded o-minimal structures in order to tackle a question asked by L. Van den Dries in [18]: does the Weierstrass division theorem hold over the ring of real analytic germs that are definable in an o-minimal structure (not necessarily polynomially bounded), extending the structure of real numbers? In [9], there is a positive answer to this question for the semialgebraic setting and also for the structure of globally subanalytic sets and functions, but, a negative answer for the structure $(\mathbb{R}, +, -, \cdot, 0, 1, <, \exp)$. The Weierstrass division theorem is the key tool for local complex analytic geometry (see, for example, [8, Chapter II and III]). It is also used e.g. in the proof of the important Oka's coherence theorem (see [8, Chapter IV]). So, V. Theilliez have shown in [17] that over the ring of germs of smooth functions in some fixed quasianalytic Carleman class that is closed under derivation, this theorem holds only for hyperbolic polynomials (see also [6]). In [7], it has been shown that for the ring of smooth germs that are definable in a polynomially bounded o-minimal expansion of the real field, this theorem holds just for the hyperbolic polynomials. Also in [14], it has been shown in particular that the Weierstrass division theorem does not hold for the ring of the smooth germs that are definable in a polynomially bounded o-minimal expansion of the real field that contains strictly the ring of real analytic germs. So, in the third section, we try to generalize the result given in [7] by studying the Weierstrass division theorem over a polynomially bounded o-minimal expansion of the real field for arbitrary functions that are regular of an order p with respect to the last

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variable, so we give a negative answer to this problem and thanks to the result proved in [13], we will form some criterions for satisfying this theorem.

In commutative algebra, the filtration on a commutative ring R by the powers of a proper ideal I determines the Krull topology (after Wolfgang Krull) or the I -adic topology on R . The case of a maximal ideal $I = m$ is especially important, for example the distinguished maximal ideal of a valuation ring. The basis of open neighbourhoods of 0 in R is given by the powers $(I^n)_{n \in \mathbb{N}}$. In the last section, we are going to study some (x_1) -adic topological properties over some quasianalytic subrings of the ring of smooth germs, we endow these rings with a metric which turns them into a separable metric spaces, also we study their completeness for the (x_1) -adic topology.

2 NOTATIONS AND DEFINITIONS

Throughout this paper, \mathcal{R} denotes a fixed (but arbitrary) expansion of the structure $\overline{\mathbb{R}} = (\mathbb{R}, <, 0, 1, +, -, \cdot)$. Definable means first order definable in \mathcal{R} with parameters from \mathbb{R} . A function $f : X \rightarrow \mathbb{R}$ is said to be definable if its graph is definable. We say that \mathcal{R} is o-minimal if every definable subset of \mathbb{R} is just a finite union of intervals and points. We say that \mathcal{R} is polynomially bounded if for every definable function $f : \mathbb{R} \rightarrow \mathbb{R}$ there exists $N \in \mathbb{N}$ such that $|f(t)| \leq t^N$ for all sufficiently large positive t .

Example 1. *The structure $\overline{\mathbb{R}} := (\mathbb{R}, +, -, \cdot, 0, 1, <)$ is polynomially bounded and o-minimal (by Tarski-Seidenberg); the sets definable in this structure are exactly the semialgebraic sets.*

The structure of the ordered real field with restricted analytic functions \mathbb{R}_{an} , whose definable sets are the finitely subanalytic sets, is a polynomially bounded o-minimal structure by [10].

For each $n \in \mathbb{N}$, we denote by \mathcal{E}_n the ring of smooth germs at the origin of \mathbb{R}^n and by $\mathbb{R}[[x_1, \dots, x_n]]$ the ring of formal power series with coefficients in \mathbb{R} .

For every $f \in \mathcal{E}_n$, we denote by $\hat{f} \in \mathbb{R}[[x_1, \dots, x_n]]$ its (infinite) Taylor expansion at the origin. The mapping $\mathcal{E}_n \ni f \mapsto \hat{f} \in \mathbb{R}[[x_1, \dots, x_n]]$ is called the Borel mapping.

Let $\hat{f} = \sum_{n \in \mathbb{N}} a_n x_1^n \in \mathbb{R}[[x_1]]$ be a formal power series. We denote by $Supp \hat{f} = \{n \in \mathbb{N} : a_n \neq 0\}$.

Definition 1. *For each $n \in \mathbb{N}$, let $\mathcal{C}_n \subset \mathcal{E}_n$ be a subring of the ring of germs of smooth functions at the origin of \mathbb{R}^n . We say that \mathcal{C}_n is a quasianalytic ring if the Borel mapping $\hat{\cdot} : \mathcal{C}_n \rightarrow \mathbb{R}[[x_1, \dots, x_n]]$ is injective. In other words, if the ring \mathcal{C}_n does not contain any nonzero smooth flat germ at 0.*

Example 2. a) *The rings \mathcal{A}_n , $n \in \mathbb{N}$, of germs of the real analytic functions at $0 \in \mathbb{R}^n$ are quasianalytic rings.*

b) *The ring of smooth germs \mathcal{E}_1 is not a quasianalytic ring as it contains some nonzero smooth germs that vanish at 0 together with all their derivatives.*

By quasianaliticity, we have that $\hat{\mathcal{C}}_n \subset \mathbb{R}[[x_1, \dots, x_n]]$ for each $n \in \mathbb{N}$.

A germ $f \in \mathcal{C}_n$ is called regular in x_n of order p with respect to the variable x_n if there exists a formal power series $h \in \mathbb{R}[[x_1, \dots, x_n]]$ such that $\hat{f} = x_n^p h(0, x_n)$ with $h(0) \neq 0$.

Let $\{\mathcal{C}_n : n \in \mathbb{N}\}$ be a system of quasianalytic rings, we say that this system satisfies Weierstrass division theorem if $f, g \in \mathcal{C}_n$ such that the germ g is regular of order p with respect to the variable x_n , there exist $q \in \mathcal{C}_n$ and $r_1, \dots, r_p \in \mathcal{C}_{n-1}$ such that

$$f = gq + \sum_{j=1}^p r_j(x_1, \dots, x_{n-1})x_n^{p-j}.$$

Example 3. *The system of the rings of real analytic germs at the origin $\{\mathcal{A}_n : n \in \mathbb{N}\}$ satisfies the Weierstrass division theorem.*

3 WEIERSTRASS DIVISION THEOREM OVER SOME POLYNOMIALLY BOUNDED O-MINIMAL STRUCTURES

Fix a polynomially bounded o-minimal structure \mathcal{R} that is an expansion of the structure $\overline{\mathbb{R}} = (\mathbb{R}, <, 0, 1, +, -, \cdot)$ and denote by $(\mathcal{D}_n)_{n \in \mathbb{N}}$ the rings of those smooth functions germs at 0 which are definable in \mathcal{R} . We recall that if $\mathcal{R} = (\mathbb{R}, <, 0, 1, +, -, \cdot)$, then the rings $(\mathcal{D}_n)_{n \in \mathbb{N}}$ are exactly the rings of algebraic smooth functions germs called Nash germs (see [5] for a thorough treatment of Nash germs).

It is well known thanks to [11], that the rings $(\mathcal{D}_n)_{n \in \mathbb{N}}$ are also a famous example of quasi-analytic local rings.

We recall that in every polynomially bounded o-minimal structure \mathcal{R} , the family $(\mathcal{D}_n)_{n \in \mathbb{N}}$ satisfies the following assertions:

- (C1) the rings $(\mathcal{D}_n)_{n \in \mathbb{N}}$ are closed under monomial division, so each \mathcal{D}_1 is a principal ideal domain;
- (C2) the closedness under inverse implies that each \mathcal{D}_n is a local ring. The unique maximal ideal of each ring \mathcal{D}_n is generated by the coordinates x_1, \dots, x_n ;
- (C3) the rings $(\mathcal{D}_n)_{n \in \mathbb{N}}$ are closed under partial differentiation.

We know thanks to [5, theorem 8.2.2] that the ring of smooth germs that are definable in the structure $\overline{\mathbb{R}}$ (called Nash germs) satisfies the Weierstrass division theorem, but unfortunately, there are some rings \mathcal{D}_n which don't satisfy the Weierstrass division theorem even if we take them to be the rings of real analytic functions that are definable in a polynomially bounded o-minimal structure \mathcal{R} , which is the aim of the following proposition.

We recall that the o-minimal structure $\mathcal{R} = (\mathbb{R}, \sin \upharpoonright [0, 1])$ (where $\sin \upharpoonright [0, 1]$ denotes the restriction of the sin function to the interval $[0, 1]$) is polynomially bounded as a reduct (in the sense of definability) of the polynomially bounded o-minimal structure \mathbb{R}_{an} (the definable sets in \mathbb{R}_{an} are exactly the finitely subanalytic sets).

Remark 1. *By [3, Proposition 3.1], if the system $(\mathcal{D}_n)_{n \in \mathbb{N}}$ satisfies the Weierstrass division theorem, then so does the system of the rings of real analytic definable germs in such structures. But the reciprocal does not hold true by [3, Remark 3.3].*

Proposition 1. *Let $\mathcal{R} = (\overline{\mathbb{R}}, \sin \upharpoonright [0, 1])$, then the Weierstrass division theorem does not hold in the ring \mathcal{D}_2 .*

Proof. Suppose that the Weierstrass division theorem holds in \mathcal{D}_2 , the function \cos is also definable in this structure \mathcal{R} as it is closed under partial differentiation. So, we deduce the existence of Q_1, Q_2 in \mathcal{D}_2 , and R_i, T_i in $\mathcal{D}_1, i = 0, 1$, such that:

$$\sin(y) = (x^2 + y^2)Q_1(x, y) + R_1(x)y + R_0(x), \tag{1}$$

$$\cos(y) = (x^2 + y^2)Q_2(x, y) + T_1(x)y + T_0(x). \tag{2}$$

It is well known that $\exp(x) = \cos(ix) - i \sin(ix)$ for all $x \in \mathbb{R}$. By taking $y = ix$ in (1) and (2), we have that

$$\exp(x) = T_0 + ixT_1 - i(R_0 + ixR_1) = (T_0 + xR_1) + i(xT_1 - R_0).$$

So, $\exp = T_0 + xR_1$.

Therefore, the restriction of the function \exp to the interval $[0, 1]$ is definable in the structure $\mathcal{R} = (\overline{\mathbb{R}}, \sin \upharpoonright [0, 1])$, which contradicts Bianconi's theorem in [4]. Hence, the Weierstrass division theorem does not hold in the ring \mathcal{D}_2 . \square

So, under what conditions this system of the rings $(\mathcal{D}_n)_{n \in \mathbb{N}}$ satisfies the Weierstrass division theorem? That is why, we will give some criteria for satisfying this theorem.

For each $n \in \mathbb{N}$, let $f, g \in \mathcal{D}_n$ be such that g is regular of order p with respect to x_n , by the Weierstrass division theorem in the ring of formal power series $\mathbb{R}[[x_1, \dots, x_n]]$, there exist $\tilde{q} \in \mathbb{R}[[x_1, \dots, x_n]]$, $\tilde{r}_1, \dots, \tilde{r}_p \in \mathbb{R}[[x_1, \dots, x_{n-1}]]$ such that

$$\hat{f} = \hat{g}\tilde{q} + \sum_{j=1}^p \tilde{r}_j(x_1, \dots, x_{n-1})x_n^{p-j}. \tag{3}$$

Put $\tilde{R} = \sum_{j=1}^p \tilde{r}_j(x_1, \dots, x_{n-1})x_n^{p-j}$.

Theorem 1. For each $n \in \mathbb{N}$, let f and g be elements of the ring \mathcal{D}_n and the division as in the equation (3), so, if there exists $R \in \mathcal{D}_n$ such that $\hat{R} = \tilde{R}$, then this division holds in the ring \mathcal{D}_n .

Proof. We have that $\tilde{R} = \sum_{j=1}^p \tilde{r}_j(x_1, \dots, x_{n-1})x_n^{p-j}$, hence we have

$$\frac{\partial^{p-1}\tilde{R}}{\partial x_n^{p-1}} = (p-1)!\tilde{r}_1(x_1, \dots, x_{n-1}).$$

But we have

$$\left(\widehat{\frac{\partial^{p-1}R}{\partial x_n^{p-1}}}\right) = \frac{\partial^{p-1}\tilde{R}}{\partial x_n^{p-1}} = (p-1)!\tilde{r}_1(x_1, \dots, x_{n-1}).$$

As the rings $(\mathcal{D}_n)_{n \in \mathbb{N}}$ are closed under partial differentiation, we deduce that there exists r_1 such that $\hat{r}_1 = \tilde{r}_1$.

We have $\frac{\partial^{p-2}\tilde{R}}{\partial x_n^{p-2}} = (p-1)!\tilde{r}_1(x_1, \dots, x_{n-1})x_n + (p-2)!\tilde{r}_2(x_1, \dots, x_{n-1})$, for the same reason we deduce that there exists $r_2 \in \mathcal{D}_{n-1}$ such that $\hat{r}_2 = \tilde{r}_2$. By continuing this process, we prove that there exist

$$r_1, \dots, r_p \in \mathcal{D}_{n-1} \quad \text{such that} \quad \hat{r}_j = \tilde{r}_j, \forall j = 1, \dots, p.$$

By (3), we have

$$\left(f - \widehat{\sum_{j=1}^p r_j x_n^{p-j}} \right) = \tilde{q} \hat{g}, \text{ where } \tilde{q} \in \mathbb{R}[[x_1, \dots, x_{n-1}]].$$

By a result of Krzysztof Jan Nowak [13], Criterion of Divisibility, there exists $q \in \mathcal{D}_n$ such that

$$f - \sum_{j=1}^p r_j x_n^{p-j} = qg,$$

which proves the theorem. □

We know that the quasianalytic rings $(\mathcal{D}_n)_{n \in \mathbb{N}}$ contain the polynomial ring and are closed under composition and partial differentiation. So, they satisfy the assertions of Definition 1.1 in [16], by [16, Corollary 2.4], the Borel mapping $\wedge : \mathcal{D}_n \rightarrow \mathbb{R}[[x_1, \dots, x_n]]$ is not surjective if $n \geq 2$, that is why we must restrict the following two propositions just to the case when $n = 1$.

Recall that a polynomial $a_k x_2^k + a_{k-1} x_2^{k-1} + \dots + a_0 \in \mathbb{R}[[x_1]][x_2]$ is a distinguished polynomial if $a_k = 1$ and $a_0(0) = \dots = a_{k-1}(0) = 0$.

Proposition 2. *If the Borel mapping $\wedge : \mathcal{D}_1 \rightarrow \mathbb{R}[[x_1]]$ is surjective, then the Weierstrass preparation theorem holds in the ring \mathcal{D}_2 .*

Proof. Let $f \in \mathcal{D}_2$ such that \hat{f} is regular of order p with respect to x_2 , so, by the Weierstrass preparation theorem in the ring of formal power series $\mathbb{R}[[x_1, x_2]]$, there exists a distinguished polynomial $P \in \mathbb{R}[[x_1]][x_2]$ of degree p with respect to x_2 , such that $\hat{f} = gP$, where $g \in \mathbb{R}[[x_1, x_2]]$, and $g(0) \neq 0$, and $P = x_2^p + \tilde{a}_{p-1}(x_1)x_2^{p-1} + \dots + \tilde{a}_0(x_1)$.

As the Borel mapping is surjective, there exist $a_0, \dots, a_{p-1} \in \mathcal{D}_1$ such that

$$\hat{a}_j = \tilde{a}_j \text{ with } a_j(0) = 0, \quad \forall j = 0, \dots, p-1.$$

Put $P_f = x_2^p + a_{p-1}(x_1)x_2^{p-1} + \dots + a_0(x_1)$, so, $\hat{f} = g\hat{P}_f$.

By the result of Krzysztof Jan Nowak in [13], Criterion of Divisibility, there exists $q \in \mathcal{D}_2$ such that $f = qP_f$, as $\hat{q} = g$ we deduce that $q(0) \neq 0$, which proves the proposition. □

Proposition 3. *If the Borel mapping $\wedge : \mathcal{D}_1 \rightarrow \mathbb{R}[[x_1]]$ is surjective, then the Weierstrass division theorem holds in the ring \mathcal{D}_2 .*

Proof. Let $f, g \in \mathcal{D}_2$ be such that \hat{g} is regular of order p with respect to x_2 , by the Weierstrass division theorem in the ring of formal power series $\mathbb{R}[[x_1, x_2]]$, there exist $\tilde{q} \in \mathbb{R}[[x_1, x_2]]$, $\tilde{r}_1, \dots, \tilde{r}_p \in \mathbb{R}[[x_1]]$ such that

$$\hat{f} = \hat{g}\tilde{q} + \sum_{j=1}^p \tilde{r}_j(x_1)x_2^{p-j}. \tag{4}$$

By assumption, the Borel mapping is surjective, so there exist $r_1, \dots, r_p \in \mathcal{D}_1$ such that

$$\hat{r}_j = \tilde{r}_j, \quad \forall j = 1, \dots, p,$$

so (4) implies that

$$\left(f - \widehat{\sum_{j=1}^p r_j(x_1)x_2^{p-j}} \right) = \hat{g}\tilde{q}, \text{ where } \tilde{q} \in \mathbb{R}[[x_1, x_2]].$$

By the result of Krzysztof Jan Nowak in [13], Criterion of Divisibility, there exists $q \in \mathcal{D}_2$ such that

$$f - \sum_{j=1}^p r_j(x_1)x_2^{p-j} = gq,$$

which proves the proposition. \square

Remark 2. Proposition 2 is an immediate consequence of Proposition 3, therefore, it suffices to prove Proposition 3 to obtain the Proposition 2.

Remark 3. By imitating the classical proof of the noetherianity of the ring of convergent power series as a consequence of the satisfaction of the Weierstrass division theorem, we deduce that if the Borel mapping $\wedge : \mathcal{D}_1 \rightarrow \mathbb{R}[[x_1]]$ is surjective, then the ring \mathcal{D}_2 is noetherian.

We end this section by another application of the result of Krzysztof Jan Nowak. For this aim, let us recall the following theorem proved in [15, Theorem 10.1].

Put $f = \sum_{n=0}^{+\infty} f_n x_1^n \in \mathbb{R}[[x_1]]$, let g_1, \dots, g_s be elements of $\mathbb{R}[[x_1]]$. Set

$$e(f) := \min\{n \in \mathbb{N} : f_n \neq 0\} = \inf \text{Supp}(f),$$

$$\Delta_1 = e(g_1) + \mathbb{N},$$

and

$$\Delta_i = e(g_i) + \mathbb{N} \setminus \bigcup_{1 \leq j < i} \Delta_j \text{ for all } 2 \leq i \leq s.$$

Finally, set $\Delta_0 := \mathbb{N} \setminus \bigcup_{1 \leq i \leq s} \Delta_i$.

Theorem 2. Let $f \in \mathbb{R}[[x_1]]$. Then there exist some unique power series $q_1, \dots, q_s, r \in \mathbb{R}[[x_1]]$ such that

$$f = g_1 q_1 + \dots + g_s q_s + r,$$

and

$$e(g_i) + \text{Supp}(q_i) \subset \Delta_i, \text{ and } \text{Supp}(r) \subset \Delta_0.$$

The power series r is called the remainder of the division of f by g_1, \dots, g_s .

Proposition 4. Let I be an ideal of the ring \mathcal{D}_1 , then every element of this ring is equivalent modulo I to a polynomial.

Proof. Let I be a nonzero ideal of the ring \mathcal{D}_1 , where \mathcal{D}_1 is a principal ideal domain and this ideal I is generated by a single element g . Let f be an arbitrary element of the ring \mathcal{D}_1 and $e(g) = \alpha$. By applying Theorem 2 above, there exist $h, r \in \mathbb{R}[[x_1]]$ such that $\hat{f} = \hat{g}h + r$, as $\text{Supp}(r)$ is finite. We deduce that r is a polynomial and therefore $r \in \mathcal{D}_1$, so

$$\widehat{f - r} = \hat{g}h.$$

By applying of the result of Krzysztof Jan Nowak [13], Criterion of Divisibility, there exists $q \in \mathcal{D}_2$ such that

$$f - r = gq.$$

Therefore, f is equivalent to r modulo the ideal I . \square

4 THE (x_1) -ADIC TOPOLOGY OVER SOME QUASIANALYTIC LOCAL RINGS

In the sequel, our work will be restricted just to the case of one variable. Firstly, let us recall some basic facts about the (x_1) -adic topology.

Let m be an ideal of a ring A , the m -adic topology is a linear topology of the ring A in which the fundamental system of neighbourhoods of zero consists of the powers m^n , where $n \in \mathbb{N}$.

If the m -adic topology is separated (i. e. $\bigcap_{i \geq 0} m^i = \{0\}$), then this topology is metrizable. Indeed, for all $x, y \in A$, let n be the largest integer such that $x - y \in m^n$, where $m^0 = A$. Then we define the metric $d(x, y) = \frac{1}{2^n}$ when n exists and 0 otherwise.

A local ring (A, m) is called complete when it is complete with respect to the above metric, that is, when all its Cauchy sequences converge.

The m -adic completion of a ring A is equal to the projective limit of the factor rings $(A/m^n)_{n \in \mathbb{N}}$. For example, the (x_1) -adic completion of the polynomial ring $\mathbb{R}[x_1]$ is the ring of formal power series $\mathbb{R}[[x_1]]$ up to isomorphism.

The interested reader will find more information about this in a very readable form in [1, Chapter 10]. Let $\mathcal{C}_1 \subset \mathcal{E}_1$ be a quasianalytic subring of the ring \mathcal{E}_1 which contains the polynomial ring $\mathbb{R}[x_1]$. Assume that the ring \mathcal{C}_1 satisfies the following property, called the stability under monomial division.

Let $\hat{f} \in \hat{\mathcal{C}}_1$ and $\hat{f} = x_1 \hat{\varphi}$, where $\hat{\varphi} \in \mathbb{R}[[x_1]]$, then $\varphi \in \mathcal{C}_1$.

Remark 4. *By the property of the stability under monomial division, the ring \mathcal{C}_1 is a principal ideal domain (see [2, Remark 3.1]).*

Remark 5. *As the polynomial ring $\mathbb{R}[x_1]$ is included in the ring \mathcal{C}_1 and by quasianalyticity, we may also assume that $\mathcal{C}_1 \subset \mathbb{R}[[x_1]]$, and as the (x_1) -adic completion of the ring $\mathbb{R}[x_1]$ is isomorphic to $\mathbb{R}[[x_1]]$, we deduce that the (x_1) -adic completion of the ring \mathcal{C}_1 is also the ring of formal power series $\mathbb{R}[[x_1]]$ up to isomorphism.*

Proposition 5. *The polynomial ring $\mathbb{R}[x_1]$ is dense in the ring \mathcal{E}_1 for the (x_1) -adic topology.*

Proof. Let f be a smooth germ and $n \in \mathbb{N}$. We need to find a polynomial g_n such that the germ of $f - g_n$ at 0 is in $(x_1^n)\mathcal{E}_1$. Thanks to Hadamard’s Lemma 2.8 in [12], we deduce that there exists a smooth function h_1 such that $f - f(0) = x_1 h_1$, and by the same way, there exists $h_2 \in \mathcal{E}_1$ such that $h_1 - h_1(0) = x_1 h_2$.

By iterated application of the same Lemma, we can write

$$f(x_1) = g_n(x_1) + x_1^n h_n(x_1),$$

where $g_n(x_1) = f(0) + f'(0)x_1 + \dots + \frac{f^{(n-1)}(0)}{(n-1)!}x_1^{n-1}$, and h_n is a smooth function germ. Then the germ of $f - g_n$ is in $(x_1^n)\mathcal{E}_1$. □

Remark 6. *Proposition 5 still holds true if we replace the ring of smooth germs \mathcal{E}_1 by the subring $\mathcal{C}_1 \subset \mathcal{E}_1$ as it is closed under monomial division.*

Remark 7. From the above properties and thanks to the quasianalyticity, we deduce that the ring \mathcal{C}_1 is a metrizable separable metric space for the (x_1) -adic topology.

We end this paper by giving a criterion that allows us to test the non surjectivity of the Borel mapping over the ring \mathcal{C}_1 .

Proposition 6. If the Borel mapping $\hat{\cdot} : \mathcal{C}_1 \rightarrow \mathbb{R}[[x_1]]$ is surjective, then the ring \mathcal{C}_1 is complete for the (x_1) -adic topology.

Proof. Suppose that the mapping $\hat{\cdot}$ is surjective, and let us take $f \in \mathcal{C}_1$ such that $\hat{f} = x_1^n H$, where $n \in \mathbb{N}$ and $H \in \mathbb{R}[[x_1]]$, so by the property of the stability under monomial division, there exists $h \in \mathcal{C}_1$ such that $\hat{h} = H$ and by quasianalyticity, we have $f = x_1^n h$. It is clear that if $f \in (x_1^n)\mathcal{C}_1$ then $\hat{f} \in (x_1^n)\mathbb{R}[[x_1]]$, consequently, we have the equivalence $f \in (x_1^n)\mathcal{C}_1 \Leftrightarrow \hat{f} \in (x_1^n)\mathbb{R}[[x_1]]$ for all $n \in \mathbb{N}$, therefore, the mapping $\hat{\cdot}$ is a homeomorphism for the (x_1) -adic topology. As the mapping $\hat{\cdot}$ is a linear homeomorphism and the ring $\mathbb{R}[[x_1]]$ is complete, we deduce that the ring \mathcal{C}_1 is also complete for the (x_1) -adic topology. \square

Remark 8. Proposition 6 still holds if we skip the assumption of stability under monomial division and by assuming just that \mathcal{C}_1 is a quasianalytic local ring which is closed under derivation.

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Беррахо М. Деякі властивості поліноміально обмежених o -мінімальних продовжень дійсного поля і деяких квазіаналітичних локальних кілець // Карпатські матем. публ. — 2020. — Т.12, №2. — С. 483–491.

У цій роботі ми досліджуємо теорему Вейерштрасса про подільність над кільцями гладких ростків, які можна визначити у довільному поліноміально обмеженому o -мініальному розширенні дійсного поля, давши деякі критерії, що задовольняють цю теорему. Потім досліджуємо деякі топологічні властивості деяких квазіаналітичних підкілець кільця гладких ростків для (x_1) -адичної топології показуючи, що ці кільця є сепарабельними метричними просторами. Також наводимо критерій їх повноти щодо (x_1) -адичної топології.

Ключові слова і фрази: теорема Вейерштрасса про подільність, поліноміально обмежена o -мінімальна структура, квазіаналітичне кільце, (x_1) -адична топологія.