



MATHEMATICAL MODELING OF FINITE TOPOLOGIES

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Integer programming is a tool for solving some combinatorial optimization problems. In this paper, we deal with combinatorial optimization problems on finite topologies. We use the binary representation of the sets to characterize finite topologies as the solutions of a Boolean quadratic system. This system is used as a basic model for formulating other types of topologies (e.g. door topology and T_0 -topology) and some combinatorial optimization problems on finite topologies. As an example of the proposed model, we found that the smallest number $m(k)$ for which the topology exists on an $m(k)$ -elements set containing exactly k open sets, for $k = 8$ and $k = 15$ is 3 and 5, respectively.

Key words and phrases: finite topology, mathematical modeling.

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1 INTRODUCTION

Integer programming is a useful tool for solving combinatorial optimization problems. The first step in this solving process is building a mathematical model from the combinatorial problem. Some modeling techniques have been discussed in [5,6] and the references therein. There are some combinatorial optimization problems on finite topological spaces. The most well-known of them concerns the number of finite topologies on n elements, $T(n)$. K. Ragnarsson and B.E. Tenner in [4] state that this problem can be refined by counting $T(n, k)$, the number of topologies on n elements having k open sets. They also point out the number $T(n)$ is asymptotically equal to $T_0(n)$, the number of T_0 -topologies on n elements. Proving the equivalence between T_0 -topologies and cellular complexes, V.A. Kovalevsky [1] solves some contradictions or paradoxes in image analysis. The other combinatorial problem arises in finite topological spaces introduced by K. Ragnarsson and B.E. Tenner [4] is to find the smallest positive integer $m(k)$ such that there exists a topology on $m(k)$ elements having k open sets.

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In this paper, we consider the mathematical modeling of combinatorial problems on finite topological spaces. As well as modeling problems related to obtaining $m(k)$, $T_0(n)$, $T(n, k)$, the mathematical models of other topological concepts such as connected components, door topologies, connected topologies, etc. are also presented.

In Section 2 we recall some essential definitions and lemmas. In Section 3, first, we propose a quadratic zero-one system for general finite topological spaces and, then, we use it as the feasible region of models related to the other types of finite topologies. In Section 4, using the notion of topological basis we offer some model improvements which reduce the size of the model in constraints. Then we present an algorithm for the problem of counting the number of finite topologies on an n -set. In Section 5 we exhibit an example of combinatorial problems, which are solved by this mathematical programming approach.

2 DEFINITIONS

In this section, some basic definitions are presented. Many of them are well-known in the literature [3] and presented here for the sake of completeness. We begin by the definition of a topology.

Definition 1. Let X be a nonempty set. Denote by $P(X)$ the set of all subsets of X . A subset τ of $P(X)$ is called a topology on X if it satisfies the following

- (a) $\emptyset, X \in \tau$,
- (b) the intersection of finitely many elements of τ is in τ ,
- (c) the arbitrary union of elements of τ is in τ .

The pair (X, τ) is called a topological space. Obviously, $P(X)$ is itself a topology on X which is called the discrete topology. A set $A \subseteq X$ is said to be open (closed) in X with respect to τ if $A \in \tau$ ($A^c \in \tau$). A set which is open and also closed is called a clopen set. A topological space (X, τ) is called finite if X is a finite set. In this case (c) may be replaced by

- (c') finite union of elements of τ is in τ .

We now introduce some topological concepts which are closely related to our work.

Definition 2. Consider a topological space (X, τ) .

- (a) A subset β of $P(X)$ is called a topology basis if for any $x \in X$ and $U, V \in \beta$ with $x \in U \cap V$ there is a set W belonging to β such that $x \in W$ and $W \subseteq U \cap V$.
- (b) τ is called a door topology if each element of $P(X)$ is either open or close.
- (c) τ has T_0 -property if for each $x, y \in X$ there is an open set in X which contains exactly one of x or y . The topology which has the T_0 -property is called T_0 -topology.
- (d) Let A be a subset of X . The collection $\tau_A = \{A \cap G : G \in \tau\}$ is a topology on A , which is called the induced topology. With this topology, A is called a subspace of X .

- (e) X is called *connected* if it is not a union of two nonempty disjoint elements of τ . A subset A of X is called a *connected set* if it is connected as a subspace of X . It is easy to see that X is connected if and only if the only its clopen subsets are X and \emptyset .
- (f) A connected subset A of X is called a *connected component* if it is not a proper subset of any connected set.
- (g) A function $f : X \rightarrow X$ is called *continuous with respect to τ* if for any open set G , the set $f^{-1}(G)$ is also open.
- (h) Let T be a subset of $P(X)$. The intersection of all topologies containing T , denoted by $\langle T \rangle$, is called the *topology generated by T* .

We conclude this section by a simple lemma.

Lemma 1. Let x be an integer in $[0, n]$ and y be a binary number which satisfy $\frac{x}{n} \leq y \leq x$. Then $x > 0$ if and only if $y = 1$.

3 MODELING

In this section, we propose mathematical models of various types of topologies introduced in Section 2.

Let $X = \{1, \dots, n\}$ be an n -element set and $P(X) = \{\emptyset = G_0, G_1, \dots, G_{2^n-1} = X\}$. Suppose $b_i \in \{0, 1\}^n$ is the binary representation of G_i , i. e. $b_{ij} = 1$ if $j \in G_i$ and $b_{ij} = 0$ otherwise. There are some bitwise operations on b_i 's: "bitwise or", "bitwise and" and "negation". Let b and c are two binary n -vectors, then bitwise or, bitwise and of b, c and negation of b are denoted by $b \vee c$, $b \wedge c$ and b' , respectively.

Consider the decision variables as follows: x_{b_i} is a binary variable which is equal to 1 if $G_i \in \tau$ and equal to zero otherwise, $i = 0, \dots, 2^n - 1$.

Thereinafter, for convenience, we use i instead of b_i if there is no ambiguity. Now, any solution $x = (x_0, x_1, \dots, x_{2^n-1})$ of the system

$$x_{i \wedge j} \geq x_i x_j, \quad \begin{array}{l} i = 1, \dots, 2^n - 1, \\ j = i + 1, \dots, 2^n - 1, \end{array} \quad (1)$$

$$x_{i \vee j} \geq x_i x_j, \quad \begin{array}{l} i = 1, \dots, 2^n - 1, \\ j = i + 1, \dots, 2^n - 1, \end{array} \quad (2)$$

$$\begin{array}{l} x_0 = 1, \\ x_{2^n-1} = 1, \\ x_i \in \{0, 1\}, \quad i = 0, \dots, 2^n - 1, \end{array} \quad (3)$$

corresponds to a general topology on the finite set. We denote this system and its solution set by $GFT(n)$ and \mathcal{T} , respectively. It is easy to see that (1), (2) could be replaced by the following inequalities

$$x_{i \vee j} + x_{i \wedge j} \geq 2x_i x_j, \quad \begin{array}{l} i = 1, \dots, 2^n - 1, \\ j = i + 1, \dots, 2^n - 1. \end{array}$$

3.1 Modeling special types of topologies

In this section, we formulate some other types of topologies as a zero-one quadratic system or as a zero-one quadratic optimization problem by adding appropriate equalities/inequalities to $GFT(n)$.

3.1.1 Door topology

By definition τ is a door topology when exactly one of G_i or $G_i^c = G_{i'}$ is in τ . Thus, by adding the constraints $x_i + x_{i'} = 1$, for $i = 1, \dots, 2^n - 1$ to $GFT(n)$, the door topology's model is at hand.

3.1.2 T_0 -Topology

Let $i, j \in X$. The value of $(b_{ki} - b_{kj})^2$ is equal to 1 if and only if $b_{ki} \neq b_{kj}$. Equivalently $(b_{ki} - b_{kj})^2 = 1$ if and only if either $i \in G_k$ or $j \in G_k$. Thus the sum $\sum_{k=0}^{2^n-1} x_k (b_{ki} - b_{kj})^2$ is the number of elements of the topology which contains exactly one of i or j . Hence, the inequalities

$$\sum_{k=0}^{2^n-1} x_k (b_{ki} - b_{kj})^2 \geq 1, \quad i = 1, \dots, n-1, j = i+1, \dots, n, \quad (4)$$

guarantee T_0 -property. We therefore add the inequality (4) to $GFT(n)$ to achieve a T_0 -topology.

3.1.3 Topologies for which the induced topology on a fixed subset A (the subspace topology on A) have the T_0 -property

Clearly, A has the T_0 -property if and only if for any two elements of A , there is an open set in X , contains exactly one of them. Similar to (4) it suffices to add the following constraints

$$\sum_{k=0}^{2^n-1} x_k (b_{ki} - b_{kj})^2 \geq 1, \quad i, j \in A.$$

3.1.4 Generated topology by a subset T of $P(X)$

It is easy to see that $\langle T \rangle$ is the smallest topology containing T . Thus the following nonlinear integer program gives the generated topology by T

$$\min \sum_{i=0}^{2^n-1} x_i \quad x_i = 1, \quad i : G_i \in T \quad x \in \mathcal{T}.$$

3.1.5 Topology τ for which a given function $f : X \rightarrow X$ is continuous with respect to τ

Let $f : X \rightarrow X$ be an arbitrary function. We desire to find topology τ for which f is continuous. Let $F = (f_{ij})$ where $f_{ij} = 1$ if $f(j) = i$ and $f_{ij} = 0$ otherwise. We have

$$f^{-1}(G) = (\max_{i \in G} f_{i1} \quad \max_{i \in G} f_{i2} \quad \dots \quad \max_{i \in G} f_{in}),$$

since if b is the binary representation of $f^{-1}(G)$, then $k \in f^{-1}(G)$ if and only if $b_k = 1$. On the other hand $\max_{i \in G} f_{ik} = 1$ if and only if at least one of f_{ik} s equal to one, equivalently if and only if there is an $i \in X$ such that $f(i) = k$.

We wish $f^{-1}(G_i)$ in τ , for each i . This is attained by the following equations

$$x_{f_i^{-1}} = 1, \quad i = 1, \dots, 2^n - 1,$$

where f_i^{-1} is the binary representation of $f^{-1}(G_i)$.

3.1.6 Connected topology

Adding the inequalities

$$x_i + x_{i'} \leq 1, \quad i = 1, \dots, 2^{n-1} - 1,$$

implies that for any nonempty proper subset A of X , at most one of A and A^c is in τ , and by definition, we have a connected topology.

3.1.7 Topology with the smallest number of connected components

Suppose some types of topologies are modeled as a mathematical program say (MP). By theorem below one can easily show that including objective function $\sum_i x_i x_{i'}$ to (MP), will result in topology which has the smallest number of connected components.

Theorem 1. *Let (X, τ) be a finite topological space. Then $A \subseteq X$ is clopen if and only if A is a union of some connected components of X . Thus, if F is the number of connected components of X then X has exactly 2^F clopen subsets.*

Proof. Let G_1, G_2, \dots, G_p are the connected components of X . Since X is the union of its connected components, then $G_j^c = \bigcup_{i \neq j} G_i$. Therefore G_j^c is closed, since G_i 's are closed by Theorem 1 [2, p. 139]. Thus G_j is clopen and it proves that an arbitrary union of connected components is a clopen set.

Conversely, let A be a clopen subset of X . We have $X = A \cup A^c$. So by Theorem 1 in [2, p. 131] G_i entirely belongs to A or A^c . On the other hand $X = \bigcup_i G_i$ thus

$$A = \bigcup (A \cap G_i) = \bigcup_{G_i \cap A \neq \emptyset} (A \cap G_i) = \bigcup_{G_i \cap A \neq \emptyset} G_i.$$

□

3.1.8 The topology in which nonempty open sets have at least k_1 elements and nonempty closed sets have at least k_2 elements, $0 \leq k_1, k_2 \leq n$

Let G be any nonempty proper open set. Then G and G^c have at least k_1, k_2 elements, respectively. On the other hand $|G^c| = n - |G|$, where $|A|$ denotes number of elements in A . Therefore $k_1 \leq |G| \leq n - k_2$. The last inequality is achieved by adding the following equations to GFT(n)

$$x_i k_1 \leq x_i \sum_{j=1}^n b_{ij} \leq x_i (n - k_2), \quad i = 1, \dots, 2^n - 1.$$

3.2 Some combinatorial optimization problems on finite topologies

This section concerns two combinatorial optimization problems on finite topologies. First, we propose a formulation for computing $m(k)$, the smallest positive integer such that there exists a topology on $m(k)$ elements having k open sets. Then, we propose an algorithm for computing the number $T(n)$ of topologies on an n -set.

3.2.1 Computing $m(k)$

Topologies with exactly k open sets could be handled by including the equation

$$\sum_{i=0}^{2^n-1} x_i = k$$

to GFT(n).

K. Ragnarsson and B.E. Tenner in [4] prove that $m(k) \leq \frac{4}{3} \log_2 k + 2$. In addition it's clear that $m(k) \geq \log_2 k$. Let $n = \lceil \frac{4}{3} \log_2 k \rceil + 2$ and $l = \lceil \log_2 k \rceil$. We use the following theorem to formulate the problem as an integer program.

Theorem 2. *Suppose τ, τ' are topologies on*

$$X = \{1, 2, \dots, n\} \quad \text{and} \quad X' = \{1, 2, \dots, n, n+1, \dots, m\},$$

respectively and let x, x' are corresponding binary vectors. Then

$$\sum_{i=0}^{2^n-1} 2^i x_i < \sum_{i=0}^{2^m-1} 2^i x'_i.$$

Proof. We have

$$\sum_{i=0}^{2^n-1} 2^i x_i \leq \sum_{i=0}^{2^n-1} 2^i = 2^{2^n} < 2^{2^m-1}.$$

Since $\{1, 2, \dots, m\} \in \tau'$ then $\sum_{i=0}^{2^m-1} 2^i x'_i \geq 2^{2^m-1}$ and this completes the proof. □

As a consequence of Theorem 2, $m(k)$ is the optimal solution of the following nonlinear integer program:

$$\begin{aligned} \text{MM}(k) \quad & \min \sum_{i=0}^{2^n-1} 2^i x_i, \\ \text{s.t.} \quad & x_{i \wedge j} \geq x_i x_j, & i = 1, \dots, 2^n - 1, \\ & & j = i + 1, \dots, 2^n - 1, \\ & x_{i \vee j} \geq x_i x_j, & i = 1, \dots, 2^n - 1, \\ & & j = i + 1, \dots, 2^n - 1, \\ & \sum_{i=0}^{2^n-1} x_i = k, \\ & x_0 = 1, \\ & x_i \in \{0, 1\}, & i = 0, \dots, 2^n - 1. \end{aligned}$$

Note that we exclude the equality $x_{2^n-1} = 1$, since if β is the topology attained by the system then β is the topology on $\cup_{A \in \beta} A$.

The $MM(k)$ may not be useful in practice, because in the objective function when i takes the upper bound of the summation, the value of 2^{2^n-1} is very large even for small values of n . We therefore add new variables y_j for $j = 1, \dots, n$ and the following constraints

$$\frac{1}{n} \sum_{i=0}^{2^n-1} x_i b_{ij} \leq y_j \leq \sum_{i=0}^{2^n-1} x_i b_{ij}, \quad j = 1, \dots, n.$$

If $\sum_{i=0}^{2^n-1} x_i b_{ij} > 0$ then $x_i b_{ik} = 1$ for some $0 \leq i \leq 2^n - 1$ and $k \in X$. Thus by Lemma 1, $y_k = 1$ if and only if k is in some open set. Hence, the sum $\sum_{j=1}^n y_j$ gives the number of topology elements. Consequently we can use this sum for an objective function of $MM(k)$ as follows

$$\begin{aligned} MM(k) \quad & \min \sum_{j=1}^n y_j, \\ \text{s.t.} \quad & x_{i \wedge j} \geq x_i x_j, & i = 1, \dots, 2^n - 1, \\ & & j = i + 1, \dots, 2^n - 1, \\ & x_{i \vee j} \geq x_i x_j, & i = 1, \dots, 2^n - 1, \\ & & j = i + 1, \dots, 2^n - 1, \\ & \frac{1}{n} \sum_{i=0}^{2^n-1} x_i b_{ij} \leq y_j \leq \sum_{i=0}^{2^n-1} x_i b_{ij}, & j = 1, \dots, n, \\ & \sum_{j=1}^n y_j \geq l, \\ & \sum_{i=0}^{2^n-1} x_i = k, \\ & x_0 = 1, \\ & x_i \in \{0, 1\}, & i = 0, \dots, 2^n - 1. \end{aligned}$$

We conclude this section with an example.

Example 1. For $k = 8$ we have $n = 6$ and $l = 3$. We solve $MM(k)$ using LINGO software and obtain $m(8) = 3$ with topology $\tau_8 = P(\{1, 2, 3\})$. Similarly for $k = 15$ again $n = 6$ but $l = 4$ and $m(15) = 5$ and the corresponding topology is

$$\begin{aligned} \tau_{15} = \{ & \emptyset, \{2\}, \{1, 2\}, \{3\}, \{2, 3\}, \{1, 2, 3\}, \{4\}, \{2, 4\}, \{1, 2, 4\}, \\ & \{3, 4\}, \{2, 3, 4\}, \{1, 2, 3, 4\}, \{3, 4, 5\}, \{2, 3, 4, 5\}, \{1, 2, 3, 4, 5\} \}. \end{aligned}$$

3.2.2 Number of finite topologies

Suppose that any indiscrete topology has at most $2^n - t(n)$ elements. While adding the inequality

$$\sum_{i=0}^{2^n-1} x_i \leq 2^n - t(n) - 1$$

to $GFT(n)$ drops discrete topology from solution space. It may improve solving process (e.g. it decreases the number of nodes in branch and bound method). Using this fact one may provides an algorithm for counting number of topologies on the finite set, $T(n)$.

$$\text{Let } \mathcal{T}_k^{\leq} = \{x \in \mathcal{T} : \sum x_i \leq k\}.$$

Algorithm 1: Computing $T(n)$

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1  $T(n) \leftarrow 1;$ 
2  $s \leftarrow 2^n - t(n);$ 
3 while Problem  $\max \{ \sum x_i : x \in \mathcal{T}_s^{\leq} \}$  is not infeasible do
4    $k = \max \{ \sum x_i : x \in \mathcal{T}_s^{\leq} \};$ 
5    $T(n) \leftarrow T(n) + T(n, k);$ 
6    $s \leftarrow s - 1;$ 
7 end
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Note that $T(n) = \sum_{k=1}^{2^n-1} T(n, k)$. In line 4 of the algorithm we omit to compute zero terms of this sum.

4 MODEL IMPROVEMENT

In this section, we mention some improvements in models. It is well-known that some properties on topologies could be restricted to a topology basis. Let us consider the problem of the existence of such topologies. For example, is there any T_0 -topology on n -set for which a given function f is continuous? To answer such problems we can consider only the topology basis. For this goal it suffices to keep the constraints related to closeness under intersection (constraints (1)) and remove constraints (2) and (3) from $GFT(n)$. By this approach if there is a T_0 -topology for which a given function f is continuous, then we find its basis, else, the model is infeasible.

5 CONCLUSION

In this paper, we exhibited mathematical models of some combinatorial problems on finite topologies. As well as modeling problems related to obtaining $m(k)$, $T_0(n)$, $T(n, k)$, the mathematical models of other topological concepts such as connected components, door topologies, connected topologies, etc. are also presented. First, we proposed a quadratic zero-one system for general finite topological spaces and then use it as the feasible region of the other types of finite topologies models. We also provided an algorithm for the problem of counting the number of finite topologies on n elements set. It is possible to incorporate models introduced in section 3, to handle another type of problems. For example to answer the question: "is there any door topology τ , for which given function f continuous with respect to τ ?", it suffices to combine 3.1.1 and 3.1.5.

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Цілочисельне програмування — це інструмент для вирішення деяких комбінаторних задач оптимізації. У цій роботі ми розглядаємо комбінаторні задачі оптимізації на скінченних топологіях. Ми використовуємо двійкове представлення множин для характеристики скінченних топологій як розв'язків булевої квадратичної системи. Ця система використовується як базова модель для формулювання інших типів топологій (наприклад, топологія дверей та T_0 -топологія) та деяких комбінаторних задач оптимізації на скінченних топологіях. Як приклад запропонованої моделі ми виявили, що найменше число $m(k)$, для якого існує топологія на наборі $m(k)$ елементів, що містить рівно k відкритих наборів, при $k = 8$ і $k = 15$ — це 3 і 5 відповідно.

Ключові слова і фрази: скінченна топологія, математичне моделювання.