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APPLICATIONS ON OPERATIONS ON WEAKLY COMPACT GENERALIZED TOPOLOGICAL SPACES

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In this paper, we have introduced the notion of operations on a generalized topological space (X, μ) to investigate the notion of γ_μ-compact subsets of a generalized topological space and to study some of its properties. It is also shown that, under some conditions, γ_μ-compactness of a space is equivalent to some other weak forms of compactness. Characterizations of such sets are given. We have then introduced the concept of γ_μ-T_2 spaces to study some properties of γ_μ-compact spaces. This operation enables us to unify different results due to S. Kasahara.

Key words and phrases: operation, γ_μ-open set, γ_μ-compact space.

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Introduction

The notion of an operation on a topological space was introduced by S. Kasahara [6] in 1979. After then D.S. Janković [5] introduced the concept of α-closed sets and investigated some properties of functions with α-closed graphs. The notion of γ-open sets was studied by H. Ogata [8] to investigate some new separation axioms. Recently, the notion of operations on the family of all semi-open sets and pre-open sets is investigated in [7, 12].

In this paper, our aim is to study the concept of γ_μ-compact subsets that are defined via operations, where an operation is defined on a collection of generalized open sets instead of a topology. The notion of generalized open sets was introduced by Á. Császár. We recall some notions defined in [2]. Let X be a non-empty set and $\mathcal{P}(X)$ be the power set of X. We call a class $\mu \subseteq expX$ a generalized topology [2] (briefly, GT) if $\varnothing \in \mu$ and any union of elements of μ belongs to μ. A set X with a GT μ on it is said to be a generalized topological space (briefly, GTS) and is denoted by (X, μ). A GT μ on X is said to be strong if $X \in \mu$. A GTS (X, μ) is said to be quasi topological space [4] if it is closed under finite intersection.

For a GTS (X, μ), the elements of μ are called μ-open sets and the complement of μ-open sets are called μ-closed sets. For $A \subseteq X$, we denote by $c_\mu(A)$ the intersection of all μ-closed sets containing A, i.e. the smallest μ-closed set containing A; and by $i_\mu(A)$ the union of all μ-open sets contained in A, i.e., the largest μ-open set contained in A (see [1, 2]).

It is easy to observe that i_μ and c_μ are idempotent and monotonic, where $\gamma\colon \mathcal{P}(X) \to \mathcal{P}(X)$ is said to be idempotent iff for each $A \subseteq X$, $\gamma(\gamma(A)) = \gamma(A)$, and monotonic iff $\gamma(A) \subseteq \gamma(B)$ whenever $A \subseteq B \subseteq X$. It is also well known [2, 3] that let μ be a GT on X and $A \subseteq X$,

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$x \in X$, then $x \in c_\mu(A)$ if and only if $M \cap A \neq \varnothing$ for every $M \in \mu$ containing x and that $c_\mu(X \setminus A) = X \setminus i_\mu(A)$. We note that $x \in i_\mu(A)$ if and only if there exists some μ-open set U containing x such that $U \subseteqq A$. A subset A of X is μ-open (resp. μ-closed) if and only if $A = i_\mu(A)$ (resp. $A = c_\mu(A)$).

1 γ_μ-OPEN SETS

Definition 1 ([9]). *Let (X,μ) be a GTS. Let $\gamma_\mu : \mu \to \mathcal{P}(X)$ be a function from μ to $\mathcal{P}(X)$ such that $U \subseteqq \gamma_\mu(U)$ for each $U \in \mu$. The function γ_μ is called an operation on μ and the image $\gamma_\mu(U)$ will be denoted by U^{γ_μ}.*

Definition 2 ([9]). *Let (X,μ) be a GTS and γ_μ be an operation on μ. A subset A of X is called γ_μ-open if for each $x \in A$ there exists $U \in \mu$ such that $x \in U \subseteqq U^{\gamma_\mu} \subseteqq A$. The family of all γ_μ-open sets of (X,μ) is denoted by γ_μ-$O(X)$. We assume that $\varnothing$ is a γ_μ-open set.*

Theorem 1. *Let (X,μ) be a strong GTS and γ_μ be an operation on μ. For γ_μ-$O(X)$, the following properties hold:*

(i) $\varnothing, X \in \gamma_\mu$-$O(X)$;

(ii) γ_μ-$O(X)$ is closed under arbitrary union and hence γ_μ-$O(X)$ is a GT on X;

(iii) γ_μ-$O(X) \subseteqq \mu$.

Proof. (i) We assumed that $\varnothing \in \gamma_\mu$-$O(X)$. For each $x \in X$, there exists $X \in \mu$ such that $x \in X \subseteqq X^{\gamma_\mu} \subseteqq X$. Thus $X \in \gamma_\mu$-$O(X)$.

(ii) Let $\{A_\alpha : \alpha \in \Lambda\}$ be a family of γ_μ-open sets in (X,μ) and $x \in \cup\{A_\alpha : \alpha \in \Lambda\}$. Then there exists an $\alpha_0 \in \Lambda$ such that $x \in A_{\alpha_0}$. Since A_{α_0} is a γ_μ-open set, there exists a μ-open set U such that $x \in U \subseteqq U^{\gamma_\mu} \subseteqq A_{\alpha_0} \subseteqq \cup\{A_\alpha : \alpha \in \Lambda\}$. Thus $\cup\{A_\alpha : \alpha \in \Lambda\}$ is a γ_μ-open set.

(iii) Let $A \in \gamma_\mu$-$O(X)$. Then for each $x \in A$, there exists a μ-open set U such that $x \in U \subseteqq U^{\gamma_\mu} \subseteqq A$. Hence $A = \cup\{U : x \in A\}$. Hence A is μ-open. Thus γ_μ-$O(X) \subseteqq \mu$. □

Remark 1. *It follows that γ_μ-$O(X)$ is a GT on X. But it is not closed under finite intersection, i.e. the intersection of two γ_μ-open sets is not always γ_μ-open. It follows from the example below.*

Example 1. *Let $X = \{1,2,3\}$ and $\mu = \{\varnothing, \{1\}, \{1,2\}, \{2,3\}, X\}$. Then (X,μ) is a GTS. Now $\gamma_\mu : \mu \to \mathcal{P}(X)$ defined by*

$$\gamma_\mu(A) = \begin{cases} A, & \text{if } 1 \in A, \\ \{2,3\}, & \text{otherwise,} \end{cases}$$

is an operation. It can be easily checked that $\{1,2\}$ and $\{2,3\}$ are two γ_μ-open sets but their intersection $\{2\}$ is not so.

Definition 3. *A GTS (X,μ) is said to be γ_μ-regular if for each $x \in X$ and each $U \in \mu$ containing x, there exists $V \in \mu$ such that $x \in V \subseteqq V^{\gamma_\mu} \subseteqq U$.*

Theorem 2. *For a strong GTS (X,μ), the following propreties are equivalent:*

(i) $\mu = \gamma_\mu$-$O(X)$;

(ii) (X,μ) is γ_μ-regular;

(iii) for each $x \in X$ and each $U \in \mu$ containing x, there exists $W \in \gamma_\mu$-$O(X)$ such that $x \in W \subseteqq W^{\gamma_\mu} \subseteqq U$.

Proof. (i) $\Leftrightarrow$ (ii) This follows immediately from Definitions 2 and 3.

(ii) $\Rightarrow$ (iii) For each $x \in X$ and $U \in \mu$ containing x, by (ii) there exists $W \in \mu$ such that $x \in W \subseteqq W^{\gamma_\mu} \subseteqq U$. Now by (i), $\mu = \gamma_\mu$-$O(X)$ and hence W is a γ_μ-open set such that $x \in W \subseteqq W^{\gamma_\mu} \subseteqq U$.

(iii) $\Rightarrow$ (i) By Theorem 1, γ_μ-$O(X) \subseteqq \mu$. Let $U \in \mu$. Then for any $x \in U$, by (iii) there exists $W_x \in \gamma_\mu$-$O(X)$ such that $x \in W_x \subseteqq U$. Thus by Theorem 1, we have $U = \cup\{W_x : x \in U\} \in \gamma_\mu$-$O(X)$. □

Definition 4. *An operation γ_μ on a GTS (X, μ) is said to be regular if for each $x \in X$ and each $U, V \in \mu$ containing x, there exists $W \in \mu$, such that $x \in W \subseteqq W^{\gamma_\mu} \subseteqq U^{\gamma_\mu} \cap V^{\gamma_\mu}$.*

Theorem 3. *Let $\gamma_\mu : \mu \to \mathcal{P}(X)$ be a regular operation on μ. Then $A \cap B$ is a γ_μ-open set for any γ_μ-open sets A and B. If μ is in addition strong, then γ_μ is a topology on X.*

Proof. Let A and B be two γ_μ-open sets in a GTS (X, μ). We shall show that $A \cap B$ is also a γ_μ-open set. Let $x \in A \cap B$. Then there exist two μ-open sets U and V containing x such that $U^{\gamma_\mu} \subseteqq A$ and $V^{\gamma_\mu} \subseteqq B$. Since $\gamma_\mu : \mu \to \mathcal{P}(X)$ is a regular operation, there exists a μ-open set W containing x such that $W^{\gamma_\mu} \subseteqq U^{\gamma_\mu} \cap V^{\gamma_\mu} \subseteqq A \cap B$. Thus $A \cap B$ is γ_μ-open. The rest follows from Theorem 1. □

Definition 5. *Let (X, μ) be a GTS and $\gamma_\mu : \mu \to \mathcal{P}(X)$ be an operation.*

(a) It follows from Theorem 1 (ii) that γ_μ is a GT on X. The γ_μ-closure [9] of a set A in X is denoted by $c_{\gamma_\mu}(A)$ and is defined as $c_{\gamma_\mu}(A) = \cap\{F : F$ is a γ_μ-closed set and $A \subseteqq F\}$. It is easy to check that for each $x \in X$, $x \in c_{\gamma_\mu}(A)$ if and only if $V \cap A \neq \varnothing$ for any $V \in \gamma_\mu$ with $x \in V$. Also it is to be observed that $c_{\gamma_\mu}(A)$ is a γ_μ-closed set.

(b) The γ_μ^*-closure of A is denoted by γ_μ-$c(A)$ and defined by γ_μ-$c(A) = \{x : A \cap U^{\gamma_\mu} \neq \varnothing$ for every μ-open set U containing $x\}$. A subset A ($\subseteqq X$) is called γ_μ^*-closed if γ_μ-$c(A) = A$.

Remark 2. *Let $\gamma_\mu : \mu \to \mathcal{P}(X)$ be an operation on a GTS (X, μ). It then follows that for any subset A of X, $A \subseteqq c_\mu(A) \subseteqq \gamma_\mu$-$c(A) \subseteqq c_{\gamma_\mu}(A)$.*

2 γ_μ-COMPACT SPACES AND RELATED PROPERTIES

Definition 6. *A subset A of a strong GTS (X, μ) is said to be μ-compact [11] (weakly μ-compact [10]) if every cover $\{U_\alpha : \alpha \in \Lambda\}$ of A by μ-open subsets of X has a finite subset Λ_0 of Λ such that $A \subseteqq \cup\{U_\alpha : \alpha \in \Lambda_0\}$ (resp. $A \subseteqq \cup\{c_\mu(U_\alpha) : \alpha \in \Lambda_0\}$).*

If $A = X$, then the μ-compact (resp. μ-closed) subset A is known as a μ-compact space (resp. μ-closed) space.

Definition 7. *Let (X, μ) be a strong GTS and $\gamma_\mu : \mu \to \mathcal{P}(X)$ be an operation on μ. A subset A of (X, μ) is said be γ_μ-compact if every cover $\{U_\alpha : \alpha \in \Lambda\}$ of A by μ-open subsets of X, there exists a finite subset Λ_0 of Λ such that $A \subseteqq \cup\{U_\alpha^{\gamma_\mu} : \alpha \in \Lambda_0\}$. If $A = X$, then the γ_μ-compact subset A is called a γ_μ-compact space.*

Remark 3. *If (X, μ) be a strong GTS and $\gamma_\mu : \mu \to \mathcal{P}(X)$ be an operation on μ. If γ_μ is an identity (resp. μ-closure) operation, then the notion of a γ_μ-compact space coincides with that of a μ-compact (weakly μ-compact) space.*

Theorem 4. *Let (X, μ) be a strong GTS and $\gamma_\mu : \mu \to \mathcal{P}(X)$ be a regular operation on μ. Then the following are equivalent:*

(i) (X, μ) is μ-compact;

(ii) (X, μ) is γ_μ-compact;

(iii) $(X, \gamma_\mu\text{-}O(X))$ is μ-compact;

(iv) $(X, \gamma_\mu\text{-}O(X))$ is γ_μ-compact.

The implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) hold without the assumption of γ_μ-regularity on (X, μ).

Proof. (i) $\Rightarrow$ (ii) Let (X, μ) be a μ-compact space. For any cover $\{U_\alpha : \alpha \in \Lambda\}$ of μ-open subsets of X, there exists a finite subset Λ_0 of Λ such that $X = \cup\{U_\alpha : \alpha \in \Lambda_0\} \subseteqq \cup\{U_\alpha^{\gamma_\mu} : \alpha \in \Lambda_0\}$. Therefore (X, μ) is γ_μ-compact.

(ii) $\Rightarrow$ (iii) Let (X, μ) be a γ_μ-compact space and $\{U_\alpha : \alpha \in \Lambda\}$ be a cover of X by γ_μ-open subsets of X. For each $x \in X$, there exists $\alpha(x) \in \Lambda$ such that $x \in U_{\alpha(x)}$. Since $U_{\alpha(x)}$ is γ_μ-open, there exists $V_{\alpha(x)} \in \mu$ such that $x \in V_{\alpha(x)} \subseteqq V_{\alpha(x)}^{\gamma_\mu} \subseteqq U_{\alpha(x)}$. Then $\{V_{\alpha(x)} : x \in X\}$ is a cover of X by μ-open subsets of X. Since (X, μ) is γ_μ-compact, there exists a finite subset $\{x_1, x_2, \ldots, x_n\}$ of X such that $\cup\{V_{\alpha_{(x_i)}}^{\gamma_\mu} : i = 1, 2, \ldots, n\} = X$ and hence $X = \cup\{U_{\alpha(x_i)} : i = 1, 2, \ldots, n\}$. Thus, $(X, \gamma_\mu\text{-}O(X))$ is μ-compact.

(iii) $\Rightarrow$ (iv) This follows from the fact that $\gamma_\mu\text{-}O(X) \subseteqq \mu$.

(iv) $\Rightarrow$ (i) Let (X, μ) be γ_μ-regular and $(X, \gamma_\mu\text{-}O(X))$ be γ_μ-compact. Then by Theorem 2, $\mu = \gamma_\mu\text{-}O(X)$ and (X, μ) is γ_μ-compact. Let $\{U_\alpha : \alpha \in \Lambda\}$ be a μ-open cover of X. Then for each $x \in X$, there exists $\alpha(x) \in \Lambda$ such that $x \in U_{\alpha(x)}$. Since (X, μ) is γ_μ-regular, there exists $V_{\alpha(x)} \in \mu$ such that $x \in V_{\alpha(x)} \subseteqq V_{\alpha(x)}^{\gamma_\mu} \subseteqq U_{\alpha(x)}$. Since $\{V_{\alpha(x)} : x \in X\}$ is a μ-open cover of X and (X, μ) is γ_μ-compact, there exists a finite subset $\{x_1, x_2, \ldots, x_n\}$ of X such that $\cup\{V_{\alpha_{(x_i)}}^{\gamma_\mu} : i = 1, 2, \ldots, n\} = X$. Thus, $\cup\{U_{\alpha_{(x_i)}} : i = 1, 2, \ldots, n\} = X$. Hence, (X, μ) is μ-compact. $\square$

Remark 4. *Let (X, μ) be a μ-compact space. If $\gamma_\mu : \mu \to \mathcal{P}(X)$ is an operation on μ then (X, μ) is γ_μ-compact.*

Example 2. *Let $\mathbb{N}$ be the set of natural numbers. Let $\mu = \mathcal{P}(\mathbb{N})$ (= the power set of $\mathbb{N}$). Now $\gamma_\mu : \mathcal{P}(\mathbb{N}) \to \mathcal{P}(\mathbb{N})$ defined by*

$$\gamma_\mu(A) = \begin{cases} \mathbb{N}, & \text{if } A \neq \{\varnothing\}, \\ \{\varnothing\}, & \text{otherwise.} \end{cases}$$

Then $(\mathbb{N}, \mu)$ is a γ_μ-compact space which is not μ-compact.

Definition 8. *Let (X, μ) be a GTS and $\gamma_\mu : \mu \to (X)$ be an operation on μ. A filterbase $\mathcal{F}$ on X is said to be*

(a) γ_μ-converge to a point $x \in X$ if for each μ-open set U containing x, there exists $F \in \mathcal{F}$ such that $F \subseteqq U^{\gamma_\mu}$;

(b) γ_μ-accumulate at $x \in X$ if for each $F \in \mathcal{F}$ and each μ-open set U containing x, $F \cap U^{\gamma_\mu} \neq \varnothing$.

Theorem 5. *Let (X,μ) be a strong GTS and $\gamma_\mu : \mu \to \mathcal{P}(X)$ be a μ-regular operation on μ. If a filterbase $\mathcal{F}$ on X γ_μ-accumulates at $x \in X$, then there exists a filterbase $\mathcal{G}$ on X such that $\mathcal{F} \subseteqq \mathcal{G}$ and $\mathcal{G}$ γ_μ-converges to x.*

Proof. Let $\mathcal{F}$ be the filterbase which γ_μ-accumulates at x. Hence for each μ-open set U containing x and each $A \in \mathcal{F}$, $A \cap U^{\gamma_\mu} \neq \varnothing$. Hence $x \in \gamma_\mu\text{-}c(A)$ for each $A \in \mathcal{F}$. Let $\mathcal{H} = \{A \cap U^{\gamma_\mu} : U$ is a μ-open set containing x and $A \in \mathcal{F}\}$. Suppose that $H_1, H_2 \in \mathcal{H}$. Then $H_1 \cap H_2 = (A_1 \cap U_1^{\gamma_\mu}) \cap (A_2 \cap U_2^{\gamma_\mu}) = (A_1 \cap A_2) \cap (U_1^{\gamma_\mu} \cap U_2^{\gamma_\mu})$ for every $A_1, A_2 \in \mathcal{F}$ and every μ-open set U_1, U_2 in X. Since γ_μ is μ-regular, there exists a μ-open set U_3 of X containing x such that $U_3^{\gamma_\mu} \subseteqq U_1^{\gamma_\mu} \cap U_2^{\gamma_\mu}$. Since $\mathcal{F}$ is a filterbase, there exists $A_3 \in \mathcal{F}$ such that $A_3 \subseteqq A_1 \cap A_2$. Hence $A_3 \cap U_3^{\gamma_\mu} \subseteqq H_1 \cap H_2$. Thus $\mathcal{H}$ is a filterbase. Now set $\mathcal{G} = \{B : \exists\, C \in \mathcal{H}$ with $C \subseteqq B\}$. Then $\mathcal{G}$ is a filter generated by $\mathcal{H}$. Now for each μ-open set U containing x and each $A \in \mathcal{F}$, $U^{\gamma_\mu} \supseteqq A \cap U^{\gamma_\mu} \in \mathcal{G}$, where $A \cap U^{\gamma_\mu} \in \mathcal{H}$. So $\mathcal{G}$ γ_μ-converges to x. Also for each $A \in \mathcal{F}$, $A = X^{\gamma_\mu} \cap A \in \mathcal{H}$. So $A \in \mathcal{G}$. Hence, $\mathcal{F} \subseteqq \mathcal{G}$. □

Corollary 1. *Let (X,μ) be a strong GTS and $\gamma_\mu : \mu \to \mathcal{P}(X)$ be a μ-monotonic operation on μ. If a maximal filter in X that γ_μ-accumulates at a point $x \in X$, then it also γ_μ-converges to x.*

Proof. Let $\mathcal{F}$ be a maximal filterbase which γ_μ-accumulates at some point x_0 of X. If $\mathcal{F}$ does not γ_μ-converge to x_0, then there exists $U_0 \in \mu$ containing x_0 such that $F \cap U_{x_0}^{\gamma_\mu} \neq \varnothing$ and $F \cap (X \setminus U_{x_0}^{\gamma_\mu}) \neq \varnothing$ for every $F \in \mathcal{F}$. Then $\mathcal{F} \cup \{F \cap \mathcal{U}^{\gamma_\mu} : F \in \mathcal{F}\}$ is a filterbase on X which strictly contains $\mathcal{F}$. This is a contradiction to the maximality of $\mathcal{F}$. □

Theorem 6. *If a GTS (X,μ) is γ_μ-compact, for some operation μ such that(X,μ) is γ_μ-regular, then (X,μ) is μ-compact.*

Proof. Let $\mathcal{U} = \{U_\alpha : \alpha \in \Lambda\}$ be a μ-open cover of X. For any $x \in X$, there exists $\alpha \in \Lambda$ such that $x \in U_\alpha$. Since X is γ_μ-regular, there exists $V_\alpha \in \mu$ such that $x \in V_\alpha \subseteqq V_\alpha^{\gamma_\mu} \subseteqq U_\alpha$. Since (X,μ) is γ_μ-compact and $\{V_\alpha : \alpha \in \Lambda\}$ is a μ-open cover of X, there is a finite subset Λ_0 of Λ such that $X = \cup\{V_\alpha^{\gamma_\mu} : \alpha \in \Lambda_0\} \subseteqq \cup\{U_\alpha : \alpha \in \Lambda_0\}$. □

Theorem 7. *Let (X,μ) be a strong GTS and $\gamma_\mu : \mu \to \mathcal{P}(X)$ be an operation on μ. Then the following are equivallent:*

(i) (X,μ) is γ_μ-compact;
(ii) each maximal filter $\mathcal{F}$ on X γ_μ-converges to some point of X;
(iii) each filterbase in X γ_μ-accumulates at some point of X.

Proof. (i) $\Rightarrow$ (ii) Let (X,μ) be γ_μ-compact and $\mathcal{F}_0$ be a maximal filter on X. Suppose that $\mathcal{F}_0$ does not γ_μ-converge to any point of X. Then by Corollary 1, $\mathcal{F}_0$ does not γ_μ-accumulate at any point of X. Then for each $x \in X$, there exist $F_x \in \mathcal{F}_0$ and $U_x \in \mu$ containing x such that $F_x \cap U_x^{\gamma_\mu} = \varnothing$. Then the family $\{U_x : x \in X\}$ is a cover of X by μ-open subsets of X. Thus by (i), there exists a finite subset $\{x_1, x_2, \ldots, x_n\}$ of X such that $X = \cup\{U_{x_i}^{\gamma_\mu} : i = 1,2,\ldots,n\}$. Since $\mathcal{F}_0$ is a filterbase on X, there exists $F_0 \in \mathcal{F}_0$, such that $F_0 \subseteqq \cap\{F_{x_i} : i = 1,2,\ldots,n\}$. Then $F_0 = F_0 \cap \cup\{U_{x_i}^{\gamma_\mu} : i = 1,2,\ldots,n\} = \cup\{F_0 \cap U_{x_i}^{\gamma_\mu} : i = 1,2,\ldots,n\} \subseteqq \cup\{F_{x_i} \cap U_{x_i}^{\gamma_\mu} : i = 1,2,\ldots,n\} = \varnothing$. This is a contradiction to the fact that $F_0 \in \mathcal{F}_0$. Thus $\mathcal{F}_0$ γ_μ-converges to some point of X.

(ii) $\Rightarrow$ (iii) Let $\mathcal{F}$ be a filterbase on X. Then there exists a maximal filterbase $\mathcal{F}_0$ such that $\mathcal{F} \subseteqq \mathcal{F}_0$. By (ii), $\mathcal{F}_0$ γ_μ-converges to some point $x_0 \in X$. For any μ-open set U containing x_0, there exists $F_0 \in \mathcal{F}_0$ such that $F_0 \subseteqq U^{\gamma_\mu}$. For any $F \in \mathcal{F}$, $F \in \mathcal{F}_0$ and $\varnothing \neq F \cap F_0 \subseteqq F \cap U^{\gamma_\mu}$. Thus each filterbase γ_μ-accumulates at some point of X.

(iii) $\Rightarrow$ (i) Suppose that (X, μ) is not γ_μ-compact and (iii) holds. Then there exists a cover $\{U_\alpha : \alpha \in \Lambda\}$ of X by μ-open sets of X such that $X \neq \cup\{U_\alpha^{\gamma_\mu} : \alpha \in \Lambda_0\}$ for every finite subset Λ_0 of Λ. Let $\Gamma(\Lambda)$ denotes the family of all finite subsets of Λ and $\mathcal{F} = \{X \setminus \cup_{\alpha \in \Lambda_\lambda} U_\alpha^{\gamma_\mu} : \Lambda_\lambda \in \Gamma(\Lambda)\}$. Then $\mathcal{F}$ is a filterbase on X and by (iii) $\mathcal{F}$ γ_μ-accumulates at some point x_0 of X. Since $\{U_\alpha : \alpha \in \Lambda\}$ is a cover of X by μ-open subsets of X, there exists $\alpha(x_0) \in \Lambda$ such that $x_0 \in U_{\alpha(x_0)}$. Then we have $(X \setminus U_{\alpha(x_0)}^{\gamma_\mu}) \cap U_{\alpha(x_0)}^{\gamma_\mu} = \varnothing$. This contradicts that $\mathcal{F}$ γ_μ-accumulates at x_0. $\square$

Theorem 8. *Let A be any subset of a strong GTS (X, μ) such that A and $X \setminus A$ are both γ_μ-compact subsets of X, then (X, μ) is γ_μ-compact.*

Proof. Let $\{V_\alpha : \alpha \in \Lambda\}$ be a μ-open cover of X. Then $\{V_\alpha : \alpha \in \Lambda\}$ is a μ-open cover of A and $X \setminus A$ also. Thus there exist finite subsets Λ_1 and Λ_2 of Λ such that $A \subseteqq \{V_\alpha^{\gamma_\mu} : \alpha \in \Lambda_1\}$ and $X \setminus A \subseteqq \{V_\alpha^{\gamma_\mu} : \alpha \in \Lambda_2\}$. Thus $X = A \cup (X \setminus A) \subseteqq \{V_\alpha^{\gamma_\mu} : \alpha \in \Lambda_1 \cup \Lambda_2\}$. This completes the proof. $\square$

Remark 5. *Finite union of γ_μ-compact subsets of X is also γ_μ-compact.*

Definition 9. *Let (X, μ) be a GTS and $\gamma_\mu : \mu \to \mathcal{P}(X)$ be an operation. Then (X, μ) is said to be γ_μ-T_2 if for any two distinct points x and $y \in X$, there exist μ-open sets U and V such that $x \in U, y \in V$ and $U^{\gamma_\mu} \cap V^{\gamma_\mu} = \varnothing$.*

We observe that every γ_μ-T_2 space is μ-T_2 space. But the coverse is false as shown by Example 2.

Theorem 9. *Let (X, μ) be QTS γ_μ-regular and $\gamma_\mu : \mu \to \mathcal{P}(X)$ be a μ-monotonic operation. If (X, μ) is γ_μ-T_2 and $K \subseteqq X$ is γ_μ-compact, then K is a γ_μ-closed set.*

Proof. It is sufficient to show that $X \setminus K$ is a γ_μ-open set. Let $x \in X \setminus K$. For each $y \in K$, there exist μ-open sets U_y and V_y such that $x \in U_y$, $y \in V_y$ and $U_y^{\gamma_\mu} \cap V_y^{\gamma_\mu} = \varnothing$. Thus we can construct a cover $\mathcal{U} = \{V_y : y \in K\}$ of K by μ-open sets of X. Since K is γ_μ-compact, there exists a finite collection $\{V_{y_1}, V_{y_2}, \ldots, V_{yn}\}$ of $\mathcal{U}$ such that $K \subseteqq \bigcup_{i=1}^{n} V_{y_i}^{\gamma_\mu}$. Let $U = \bigcap_{i=1}^{n} U_{y_i}$. Then U is a μ-open set containing x such that $U^{\gamma_\mu} \subseteqq X \setminus K$. Then by γ_μ-regularity of X, there exists a μ-open set W containing x such that $x \in W \subseteqq W^{\gamma_\mu} \subseteqq U \subseteqq U^{\gamma_\mu}$. Thus $W \subseteqq W^{\gamma_\mu} \subseteqq X \setminus K$. Hence, $X \setminus K$ is γ_μ-open. $\square$

We call an operation $\gamma_\mu : \mu \to \mathcal{P}(X)$ on a GTS (X, μ) to be additive if for any $A, B \in \mu$, $(A \cup B)^{\gamma_\mu} = A^{\gamma_\mu} \cup B^{\gamma_\mu}$.

Theorem 10. *Let (X, μ) be a QTS and $\gamma_\mu : \mu \to \mathcal{P}(X)$ be an additive, μ-monotonic operation on μ. If $Y \subseteqq X$ is γ_μ-compact, $x \in X \setminus Y$ and (X, μ) is γ_μ- T_2, then there exist μ-open sets U and V with $x \in U, Y \subseteqq V^{\gamma_\mu}$ and $U^{\gamma_\mu} \cap V^{\gamma_\mu} = \varnothing$.*

Proof. For each $y \in Y$, let V_y and U_y be μ-open sets such that $V_y^{\gamma_\mu} \cap U_y^{\gamma_\mu} = \varnothing$, with $y \in V_y$ and $x \in U_y$. The collection $\mathcal{V} = \{V_y : y \in Y\}$ is then a cover of Y by μ-open sets. Now since Y is γ_μ-compact, there exists a finite subcollection $\{V_{y_1}, V_{y_2}, \dots, V_{y_n}\}$ of $\mathcal{V}$ such that $Y \subseteqq \bigcup_{i=1}^{n} V_{y_i}^{\gamma_\mu}$. Let $U = \bigcap_{i=1}^{n} U_{y_i}$ and $V = \bigcup_{i=1}^{n} V_{y_i}$. Since $U \subseteqq U_{y_i}$ for every $i = 1, 2, \dots, n$ and γ_μ is monotonic, $U^{\gamma_\mu} \cap V_{y_i}^{\gamma_\mu} \subseteqq U_{y_i}^{\gamma_\mu} \cap V_{y_i}^{\gamma_\mu} = \varnothing$ for $i = 1, 2, \dots, n$. Thus, $U^{\gamma_\mu} \cap V^{\gamma_\mu} = \varnothing$ (as γ_μ is an additive operation on μ). Thus, $Y \subseteqq V^{\gamma_\mu}$ and $x \in U$. □

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У цій статті введено означення операцій на узагальненому топологічному просторі (X, μ) для того, щоб дослідити поняття γ_μ-компактних підмножин узагальненого топологічного простору та вивчити деякі їх властивості. Також показано, що при деяких умовах γ_μ-компактність простору еквівалентна деяким іншим слабшим формам компактності. Дано характеризацію таких множин. Також введено поняття γ_μ-T_2 просторів для вивчення деяких властивостей γ_μ-компактних просторів. Це дозволяє нам уніфікувати різні результати С. Касахари.

Ключові слова і фрази: операція, γ_μ-відкрита множина, γ_μ-компактний простір.