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ON REPRESENTATION OF SEMIGROUPS OF INCLUSION HYPERSPACES

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Given a group X we study the algebraic structure of the compact right-topological semigroup $G(X)$ consisting of inclusion hyperspaces on X. This semigroup contains the semigroup $\lambda(X)$ of maximal linked systems as a closed subsemigroup. We construct a faithful representation of the semigroups $G(X)$ and $\lambda(X)$ in the semigroup $\mathsf{P}(X)^{\mathsf{P}(X)}$ of all self-maps of the power-set $\mathsf{P}(X)$. Using this representation we prove that each minimal left ideal of $\lambda(X)$ is topologically isomorphic to a minimal left ideal of the semigroup $\mathsf{pT}^{\mathsf{pT}}$, where by pT we denote the family of pretwin subsets of X.

Introduction

After discovering a topological proof of Hindman theorem [8] (see [10, p.102], [9]), topological methods become a standard tool in the modern combinatorics of numbers, see [10], [11]. The crucial point is that any semigroup operation $*$ defined on a discrete space X can be extended to a right-topological semigroup operation on $\beta(X)$, the Stone-Čech compactification of X. The extension of the operation from X to $\beta(X)$ can be defined by the simple formula

$$\mathcal{A}\circ\mathcal{B}=\big\{A\subset X:\{x\in X:x^{-1}A\in\mathcal{B}\}\in\mathcal{A}\big\}, \tag{1}$$

where $\mathcal{A},\mathcal{B}$ are ultrafilters on X. Endowed with the so-extended operation, the Stone-Čech compactification $\beta(X)$ becomes a compact right-topological semigroup. The algebraic properties of this semigroup (for example, the existence of idempotents or minimal left ideals) have important consequences in combinatorics of numbers, see [10], [11].

The Stone-Čech compactification $\beta(X)$ of X is the subspace of the double power-set $\mathsf{P}(\mathsf{P}(X))$, which is a complete lattice with respect to the operations of union and intersection. In [7] it was observed that the semigroup operation extends not only to $\beta(X)$ but also to the complete sublattice $G(X)$ of $\mathsf{P}(\mathsf{P}(X))$ generated by $\beta(X)$. This complete sublattice consists of all inclusion hyperspaces over X.

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By definition, a family $\mathcal{F}$ of non-empty subsets of a discrete space X is called an *inclusion hyperspace* if $\mathcal{F}$ is monotone in the sense that a subset $A \subset X$ belongs to $\mathcal{F}$ provided A contains some set $B \in \mathcal{F}$. Besides the operations of union and intersection, the set $G(X)$ possesses an important transversality operation assigning to each inclusion hyperspace $\mathcal{F} \in G(X)$ the inclusion hyperspace

$$\mathcal{F}^{\perp} = \{A \subset X : \forall F \in \mathcal{F}\ (A \cap F \neq \varnothing)\}.$$

This operation is involutive in the sense that $(\mathcal{F}^{\perp})^{\perp} = \mathcal{F}$.

It is known that the family $G(X)$ of inclusion hyperspaces on X is closed in the double power-set $\mathsf{P}(\mathsf{P}(X)) = \{0,1\}^{\mathsf{P}(X)}$ endowed with the natural product topology. The induced topology on $G(X)$ can be described directly: it is generated by the sub-base consisting of the sets

$$U^{+} = \{\mathcal{F} \in G(X) : U \in \mathcal{F}\} \text{ and } U^{-} = \{\mathcal{F} \in G(X) : U \in \mathcal{F}^{\perp}\}$$

where U runs over subsets of X. Endowed with this topology, $G(X)$ becomes a Hausdorff supercompact space. The latter means that each cover of $G(X)$ by the sub-basic sets has a 2-element subcover. Let also $N_2(X) = \{\mathcal{A} \in G(X) : \mathcal{A} \subset \mathcal{A}^{\perp}\}$ denote the family of all linked inclusion hyperspaces on X and $\lambda(X) = \{\mathcal{F} \in G(X) : \mathcal{F} = \mathcal{F}^{\perp}\}$ the family of all maximal linked systems on X.

By [6], both the subspaces $\lambda(X)$ and $N_2(X)$ are closed in the space $G(X)$. Observe that $U^{+} \cap \lambda(X) = U^{-} \cap \lambda(X)$ and hence the topology on $\lambda(X)$ is generated by the sub-basis consisting of the sets

$$U^{\pm} = \{\mathcal{A} \in \lambda(X) : U \in \mathcal{A}\},\ \ U \subset X.$$

The extension of a binary operation $*$ from X to $G(X)$ can be defined in the same manner as for ultrafilters, i.e., by the formula (1) applied to any two inclusion hyperspaces $\mathcal{A}, \mathcal{B} \in G(X)$. In [7] it was shown that for an associative binary operation $*$ on X the space $G(X)$ endowed with the extended operation becomes a compact right-topological semigroup. The structure of this semigroup was studied in details in [7]. In particular, it was shown that for each group X the minimal left ideals of $G(X)$ are singletons containing *invariant* inclusion hyperspaces. Besides the Stone-Čech extension, the semigroup $G(X)$ contains many important spaces as closed subsemigroups. In particular, the space $\lambda(X)$ of maximal linked systems on X is a closed subsemigroup of $G(X)$. The space $\lambda(X)$ is well-known in General and Categorial Topology as the *superextension* of X, see [12].

We call an inclusion hyperspace $\mathcal{A} \in G(X)$ *invariant* if $x\mathcal{A} = \mathcal{A}$ for all $x \in X$. It follows from the definition of the topology on $G(X)$ that the set $\overleftrightarrow{G}(X)$ of all invariant inclusion hyperspaces is closed and non-empty in $G(X)$. Moreover, the set $\overleftrightarrow{G}(X)$ coincides with the minimal ideal of $G(X)$, which is a closed semigroup of right zeros. The latter means that $\mathcal{A} \circ \mathcal{B} = \mathcal{B}$ for all $\mathcal{A}, \mathcal{B} \in \overleftrightarrow{G}(X)$.

The minimal ideal $\overleftrightarrow{G}(X)$ contains the closed subset $\overleftrightarrow{N}_2(X) = N_2(X) \cap \overleftrightarrow{G}(X)$ of invariant linked systems on X. The subset $\max \overleftrightarrow{N}_2(X)$ of *maximal invariant linked systems* on X is denoted by $\overleftrightarrow{\lambda}(X)$. It can be shown that $\overleftrightarrow{\lambda}(X)$ is a closed subsemigroup of $\overleftrightarrow{N}_2(X)$. By [2, 2.2], this semigroup has cardinality $|\overleftrightarrow{\lambda}(X)| = 2^{2^{|X|}}$ for every infinite group X.

The thorough study of algebraic properties of semigroups of inclusion hyperspaces and the superextensions of groups was started in [7] and continued in [1], [2] and [3]. In this paper we construct a faithful representation of the semigroups $G(X)$ and $\lambda(X)$ in the semigroup $\mathsf{P}(X)^{\mathsf{P}(X)}$ of all self-maps of the power-set $\mathsf{P}(X)$ and show that the image of $\lambda(X)$ in $\mathsf{P}(X)^{\mathsf{P}(X)}$ coincides with the semigroup $\lambda(X, \mathsf{P}(X))$ of all functions $f : \mathsf{P}(X) \to \mathsf{P}(X)$ that are equivariant, monotone and symmetric in the sense that $f(X \setminus A) = X \setminus f(A)$ for all $A \subset X$. Using this representation we prove that each minimal left ideal of $\lambda(X)$ is topologically isomorphic to a minimal left ideal of the semigroup $\mathsf{pT}^{\mathsf{pT}}$, where by pT we denote the family of pretwin subsets of X. A subset A of a group X is called a *pretwin subset* if $xA \subset X \setminus A \subset yA$ for some $x, y \in X$.

1 Right-topological semigroups

In this section we recall some information from [10] related to right-topological semigroups. By definition, a right-topological semigroup is a topological space S endowed with a semigroup operation $* : S \times S \to S$ such that for every $a \in S$ the right shift $r_a : S \to S$, $r_a : x \mapsto x * a$, is continuous. If the semigroup operation $* : S \times S \to S$ is (separately) continuous, then $(S, *)$ is a *(semi-)topological semigroup*.

From now on, S is a compact Hausdorff right-topological semigroup. We shall recall some known information concerning ideals in S, see [10].

A non-empty subset I of S is called a *left* (resp. *right*) *ideal* if $SI \subset I$ (resp. $IS \subset I$). If I is both a left and right ideal in S, then I is called an *ideal* in S. Observe that for every $x \in S$ the set $SxS = \{sxt : s, t \in S\}$ (resp. $Sx = \{sx : s \in S\}$, $xS = \{xs : s \in S\}$) is an ideal (resp. left ideal, right ideal) ideal in S. Such an ideal is called *principal*. An ideal $I \subset S$ is called *minimal* if any ideal of S that lies in I coincides with I. By analogy we define minimal left and right ideals of S. It is easy to see that each minimal left (resp. right) ideal I is principal. Moreover, $I = Sx$ (resp. $I = xS$) for each $x \in I$. This simple observation implies that each minimal left ideal in S, being principal, is closed in S. By [10, 2.6], each left ideal in S contains a minimal left ideal.

We shall use the following known fact, see [3, Lemma 1.1].

Proposition 1.1. *If a homomorphism $h : S \to S'$ between two semigroups is injective on some minimal left ideal of S, then h is injective on each minimal left ideal of S.*

2 The function representation of the semigroup $G(X)$

In this section given a group X we introduce the function representation $\Phi : G(X) \to \mathsf{P}(X)^{\mathsf{P}(X)}$ of the semigroup $G(X)$ in the semigroup $\mathsf{P}(X)^{\mathsf{P}(X)}$ of all self-maps of the power-set $\mathsf{P}(X)$ of X. The semigroup $\mathsf{P}(X)^{\mathsf{P}(X)}$ endowed with the Tychonov product topology is a compact right-topological semigroup naturally homeomorphic to the Cantor cube $(\{0,1\}^X)^{\mathsf{P}(X)} = \{0,1\}^{X \times \mathsf{P}(X)}$. The sub-base of the topology of $\mathsf{P}(X)^{\mathsf{P}(X)}$ consists of the sets

$$\langle x, A\rangle^+ = \{f \in \mathsf{P}(X)^{\mathsf{P}(X)} : x \in f(A)\},$$
$$\langle x, A\rangle^- = \{f \in \mathsf{P}(X)^{\mathsf{P}(X)} : x \notin f(A)\}.$$

Given an inclusion hyperspace $\mathcal{A} \in G(X)$ consider the function

$$\Phi_{\mathcal{A}} : \mathsf{P}(X) \to \mathsf{P}(X), \;\; \Phi_{\mathcal{A}}(A) = \{x \in G : x^{-1}A \in \mathcal{A}\}$$

called the *function representation* of $\mathcal{A}$.

Proposition 2.1. *A function $\varphi : \mathsf{P}(X) \to \mathsf{P}(X)$ coincides with the function representation $\Phi_{\mathcal{A}}$ of some (invariant) inclusion hyperspace $\mathcal{A} \in G(X)$ if and only if φ is*

1) equivariant *in the sense that $\varphi(xA) = x\varphi(A)$ for any $A \subset X$ and $x \in X$;*

2) monotone *in the sense that $\varphi(A) \subset \varphi(B)$ for any subsets $A \subset B$ of X;*

3) $\varphi(\varnothing) = \varnothing$, $\varphi(X) = X$ (and $\varphi(\mathsf{P}(X)) \subset \{\varnothing, X\}$).

Proof. To prove the "only if" part, take any inclusion hyperspace $\mathcal{A} \in G(X)$ and consider its function representation $\Phi_{\mathcal{A}}$.

It is equivariant because

$$\Phi_{\mathcal{A}}(xA) = \{y \in X : y^{-1}xA \in \mathcal{A}\} = \{xy : y^{-1}A \in \mathcal{A}\} = x\,\Phi_{\mathcal{A}}(A)$$

for any $x \in X$ and $A \subset X$.

Also it is monotone because

$$\Phi_{\mathcal{A}}(A) = \{x \in G : x^{-1}A \in \mathcal{A}\} \subset \{x \in G : x^{-1}B \in \mathcal{A}\} = \Phi_{\mathcal{A}}(B)$$

for any subsets $A \subset B$ of X.

It is clear that $\Phi_{\mathcal{A}}(\varnothing) = \varnothing$ and $\Phi_{\mathcal{A}}(X) = X$.

If $\mathcal{A}$ is invariant, then for every $A \in \mathcal{A}$ we get $\Phi_{\mathcal{A}}(A) = X$ and for each $A \in \mathsf{P}(X) \setminus \mathcal{A}$ we get $\Phi_{\mathcal{A}}(A) = \varnothing$.

To prove the "if" part, fix any equivariant monotone map $\varphi : \mathsf{P}(X) \to \mathsf{P}(X)$ with $\varphi(\varnothing) = \varnothing$ and $\varphi(X) = X$ and observe that the family

$$\mathcal{A}_\varphi = \{x^{-1}A : A \subset X, \; x \in \varphi(A)\}$$

is an inclusion hyperspace with $\Phi_{\mathcal{A}_\varphi} = \varphi$. If $\varphi(\mathsf{P}(X)) \subset \{\varnothing, X\}$, then the inclusion hyperspace $\mathcal{A}_\varphi$ is invariant. □

Remark 2.1. *If X is a left-topological group and $\mathcal{A}$ is the filter of neighborhoods of the identity element e of X, then the functional representations $\Phi_{\mathcal{A}}$ and $\Phi_{\mathcal{A}^\perp}$ have transparent topological interpretations: for any subset $A \subset X$ the set $\Phi_{\mathcal{A}}(A)$ coincides with the interior of a set $A \subset X$ while $\Phi_{\mathcal{A}^\perp}(A)$ with the closure of A in X!*

The correspondence $\Phi : \mathcal{A} \mapsto \Phi_{\mathcal{A}}$ determines a map $\Phi : G(X) \to \mathsf{P}(X)^{\mathsf{P}(X)}$ called the *function representation* of the semigroup $G(X)$.

Theorem 1. *The function representation $\Phi : G(X) \to \mathsf{P}(X)^{\mathsf{P}(X)}$ is a continuous injective semigroup homomorphism.*

Proof. To check that Φ is a semigroup homomorphism, take any two inclusion hyperspaces $\mathcal{X}, \mathcal{Y} \in G(X)$ and let $\mathcal{Z} = \mathcal{X} \circ \mathcal{Y}$. We need to check that $\Phi_{\mathcal{Z}}(A) = \Phi_{\mathcal{X}} \circ \Phi_{\mathcal{Y}}(A)$ for every $A \subset X$. Observe that

$$\begin{aligned}\Phi_{\mathcal{Z}}(A) &= \{z \in G : z^{-1}A \in \mathcal{Z}\} = \{z \in G : \{x \in G : x^{-1}z^{-1}A \in \mathcal{Y}\} \in \mathcal{X}\} = \\ &= \{z \in G : \Phi_{\mathcal{Y}}(z^{-1}A) \in \mathcal{X}\} = \{z \in G : z^{-1}\Phi_{\mathcal{Y}}(A) \in \mathcal{X}\} = \Phi_{\mathcal{X}}(\Phi_{\mathcal{Y}}(A)).\end{aligned}$$

To see that Φ is injective, take any two distinct inclusion hyperspaces $\mathcal{X}, \mathcal{Y} \in G(X)$. Without loss of generality, $\mathcal{X} \setminus \mathcal{Y}$ contains some set $A \subset X$. It follows that $e \in \Phi_{\mathcal{X}}(A)$ but $e \notin \Phi_{\mathcal{Y}}(A)$ and hence $\Phi_{\mathcal{X}} \neq \Phi_{\mathcal{Y}}$.

To prove that $\Phi : G(X) \to \mathsf{P}(X)^{\mathsf{P}(X)}$ is continuous we first define a convenient sub-base of the topology on the spaces $\mathsf{P}(X)$ and $\mathsf{P}(X)^{\mathsf{P}(X)}$. The product topology of $\mathsf{P}(X)$ is generated by the sub-base consisting of the sets

$$x^+ = \{A \subset X : x \in A\} \text{ and } x^- = \{A \subset X : x \notin A\}$$

where $x \in X$. On the other hand, the product topology on $\mathsf{P}(X)^{\mathsf{P}(X)}$ is generated by the sub-base consisting of the sets

$$\langle x, A\rangle^+ = \{f \in \mathsf{P}(X)^{\mathsf{P}(X)} : x \in f(A)\} \text{ and } \langle x, A\rangle^- = \{f \in \mathsf{P}(X)^{\mathsf{P}(X)} : x \notin f(A)\}$$

where $A \in \mathsf{P}(X)$ and $x \in X$.

Now observe that the preimage

$$\Phi^{-1}(\langle x, A\rangle^+) = \{\mathcal{A} \in G(X) : x \in \Phi_{\mathcal{A}}(A)\} = \{\mathcal{A} \in G(X) : x^{-1}A \in \mathcal{A}\} = (x^{-1}A)^+$$

is open in $G(X)$. The same is true for the preimage

$$\Phi^{-1}(\langle x, A\rangle^-) = \{\mathcal{A} \in G(X) : x \notin \Phi_{\mathcal{A}}(A)\} = \{\mathcal{A} \in G(X) : x^{-1}A \notin \mathcal{A}\} = (X \setminus x^{-1}A)^-$$

which also is open in $G(X)$. □

3 THE SEMIGROUP $\lambda(X, \mathsf{P}(X))$ AND ITS PROJECTIONS $\lambda(X, \mathsf{F})$

Since for a group X the function representation $\Phi : G(X) \to \mathsf{P}(X)^{\mathsf{P}(X)}$ is an isomorphic embedding, instead of the semigroup $\lambda(X)$ we can study its isomorphic copy $\lambda(X, \mathsf{P}(X)) = \Phi(\lambda(X)) \subset \mathsf{P}(X)^{\mathsf{P}(X)}$. Our strategy is to study $\lambda(X, \mathsf{P}(X))$ via its projections $\lambda(X, \mathsf{F})$ onto the faces $\mathsf{P}(X)^{\mathsf{F}}$ of the cube $\mathsf{P}(X)^{\mathsf{P}(X)}$, where F is a suitable subfamily of $\mathsf{P}(X)$.

Given a subfamily $\mathsf{F} \subset \mathsf{P}(X)$ by

$$\mathrm{pr}_{\mathsf{F}} : \mathsf{P}(X)^{\mathsf{P}(X)} \to \mathsf{P}(X)^{\mathsf{F}}, \;\; \mathrm{pr}_{\mathsf{F}} : f \mapsto f|\mathsf{F},$$

we denote the projection of $\mathsf{P}(X)^{\mathsf{P}(X)}$ onto its F-face $\mathsf{P}(X)^{\mathsf{F}}$. Let

$$\Phi_{\mathsf{F}} = \mathrm{pr}_{\mathsf{F}} \circ \Phi : \lambda(X) \to \mathsf{P}(X)^{\mathsf{F}}$$

and

$$\lambda(X, \mathsf{F}) = \Phi_{\mathsf{F}}(\lambda(X)) = \mathrm{pr}_{\mathsf{F}}(\lambda(X, \mathsf{P}(X)) = (\mathrm{pr}_{\mathsf{F}} \circ \Phi)(\lambda(X)).$$

Now we detect functions $f : \mathsf{F} \to \mathsf{P}(X)$ belonging to the image $\lambda(X, \mathsf{F})$. Let us call a family $\mathsf{F} \subset \mathsf{P}(X)$

- *X-invariant* if $xF \in \mathsf{F}$ for every $F \in \mathsf{F}$ and every $x \in X$;
- *symmetric* if for each $A \in \mathsf{F}$ we get $X \setminus A \in \mathsf{F}$.

Theorem 2. *A function $f : \mathsf{F} \to \mathsf{P}(X)$ defined on a symmetric X-invariant subfamily $\mathsf{F} \subset \mathsf{P}(X)$ belongs to the image $\lambda(X, \mathsf{F}) = \Phi_{\mathsf{F}}(\lambda(X))$ if and only if*

1) f is equivariant;

2) f is monotone;

3) f is symmetric in the sense that $f(X \setminus A) = X \setminus f(A)$ for each $A \in \mathsf{F}$.

Proof. To prove the "only if" part, take any maximal linked system $\mathcal{L} \in \lambda(X)$ and consider its function representation $f = \Phi_{\mathcal{L}} : \mathsf{P}(X) \to \mathsf{P}(X)$.

By Proposition 2.1, the function f is equivariant and monotone. Consequently, the restriction $f|\mathsf{F}$ satisfies the items (1), (2). To prove the third item, take any set $A \in \mathsf{F}$ and observe that

$$f(X \setminus A) = \{x \in X : x^{-1}(X \setminus A) \in \mathcal{L}\} = \{x \in X : X \setminus x^{-1}A \in \mathcal{L}\} = \\ = \{x \in X : x^{-1}A \notin \mathcal{L}\} = X \setminus \{x \in X : x^{-1}A \in \mathcal{L}\} = X \setminus f(A).$$

This completes the proof of the "only if" part.

To prove the "if" part, take any function $f : \mathsf{F} \to \mathsf{P}(X)$ satisfying the conditions 1)–3) and consider the family

$$\mathcal{L}_f = \{x^{-1}A : A \in \mathsf{F},\ x \in f(A)\}.$$

We claim that this family is linked. Assuming the converse, find two sets $A, B \in \mathsf{F}$ and two points $x \in f(A)$ and $y \in f(B)$ with $x^{-1}A \cap y^{-1}B = \varnothing$. Then $yx^{-1}A \subset X \setminus B$ and hence $yx^{-1}f(A) \subset f(X \setminus B) = X \setminus f(B)$ by the properties 1)–3) of the map f. Then $x^{-1}f(A) \subset X \setminus y^{-1}f(B)$, which is not possible because the neutral element e of the group X belongs to $x^{-1}f(A) \cap y^{-1}f(B)$.

Enlarge the linked family $\mathcal{L}_f$ to a maximal linked family $\mathcal{L} \in \lambda(X)$. We claim that $\Phi_{\mathcal{L}}|\mathsf{F} = f$. Indeed, take any set $A \in \mathsf{F}$ and observe that

$$f(A) \subset \{x \in X : x^{-1}A \in \mathcal{L}_f\} \subset \{x \in X : x^{-1}A \in \mathcal{L}\} = \Phi_{\mathcal{L}}(A).$$

To prove the reverse inclusion, observe that for any $x \in X \setminus f(A) = f(X \setminus A)$ we get $x^{-1}(X \setminus A) = X \setminus x^{-1}A \in \mathcal{L}_f \subset \mathcal{L}$. Since $\mathcal{L}$ is linked, $x^{-1}A \notin \mathcal{L}$ and hence $x \notin \Phi_{\mathcal{L}}(A)$. □

A subfamily $\mathsf{F} \subset \mathsf{P}(X)$ is called *$\subset$-incomparable* if for any subset $A, B \in \mathsf{F}$ the inclusion $A \subset B$ implies the equality $A = B$. In this case each function $f : \mathsf{F} \to \mathsf{P}(X)$ is monotone, so the characterization Theorem 2 simplifies as follows.

Corollary 3.1. *A function $f : \mathsf{F} \to \mathsf{P}(X)$ defined on a $\subset$-incomparable symmetric X-invariant subfamily $\mathsf{F} \subset \mathsf{P}(X)$ belongs to the image $\lambda(X, \mathsf{F}) = \Phi_{\mathsf{F}}(\lambda(X))$ if and only if f is equivariant and symmetric.*

A subfamily $\mathsf{F} \subset \mathsf{P}(X)$ is called λ-*invariant* if $\Phi_{\mathcal{L}}(\mathsf{F}) \subset \mathsf{F}$ for every maximal linked system $\mathcal{L} \in \lambda(X)$. In this case $\lambda(X, \mathsf{F}) \subset \mathsf{F}^{\mathsf{F}}$ is a subsemigroup of the right-topological group F^{F} of all self-maps of F.

Now we see that Theorem 1 implies

Proposition 3.1. *For any λ-invariant subfamily $\mathsf{F} \subset \mathsf{P}(X)$ the map*

$$\Phi_{\mathsf{F}} = \mathrm{pr}_{\mathsf{F}} \circ \Phi : \lambda(X) \to \lambda(X, \mathsf{F}) \subset \mathsf{F}^{\mathsf{F}}$$

is a continuous semigroup homomorphism and $\lambda(X, \mathsf{F})$ is a compact right-topological semigroup.

4 Self-linked sets in groups

Our strategy in studying minimal left ideals of the semigroup $\lambda(X)$ consists in finding a relatively small λ-invariant subfamily $\mathsf{F} \subset \mathsf{P}(X)$ such that the function representation $\Phi_{\mathsf{F}} : \lambda(X) \to \lambda(X, \mathsf{F})$ is injective on some (equivalently all) minimal left ideals of $\lambda(X)$.

The first step in finding such a family F is to consider the family of self-linked sets in X.

Definition 4.1. *A subset A of a group X is* self-linked *if $xA \cap yA \neq \varnothing$ for all $x, y \in X$.*

Self-linked sets in (finite) groups were studied in details in [1]. The following simple characterization can be easily derived from the definitions.

Proposition 4.1. *For a subset $A \subset X$ the following conditions are equivalent:*

1) *A is self-linked;*

2) *the family of shifts $\{xA : x \in X\}$ is linked;*

3) *$AA^{-1} = X$;*

4) *A belongs to an invariant linked system $\mathcal{A} \in \overset{\leftrightarrow}{N}_2(X)$;*

5) *A belongs to a maximal invariant linked system $\mathcal{A} \in \overset{\leftrightarrow}{\lambda}(X) = \max \overset{\leftrightarrow}{N}_2(X)$.*

The following proposition was first proved in [3, 4.1]. Here we present a short proof for completeness.

Proposition 4.2. *For any invariant linked system $\mathcal{L}_0 \in \overset{\leftrightarrow}{N}_2(X)$ the upper set*

$$\uparrow\mathcal{L}_0 = \{\mathcal{L} \in \lambda(X) : \mathcal{L} \supset \mathcal{L}_0\}$$

is a closed left ideal in $\lambda(X)$.

Proof. Let $\mathcal{A}, \mathcal{B} \in \lambda(X)$ be maximal linked systems with $\mathcal{L}_0 \subset \mathcal{B}$. Then for every subset $L \in \mathcal{L}_0$ we get

$$L = \bigcup_{x \in X} x(x^{-1}L) \in \mathcal{A} * \mathcal{L}_0 \subset \mathcal{A} * \mathcal{B}$$

which means that $\mathcal{L}_0 \subset \mathcal{A} * \mathcal{B}$.

To show that $\uparrow\mathcal{L}_0$ is closed in $\lambda(X)$, take any maximal linked system $\mathcal{L} \in \lambda(X) \setminus \uparrow\mathcal{L}_0$ and find a set $A \in \mathcal{L}_0$ with $A \notin \mathcal{L}$. Since $\mathcal{L}$ is maximal linked, $X \setminus A \in \mathcal{L}$. Consequently, $(X \setminus A)^{\pm}$ is an open neighborhood of $\mathcal{L}$ that does not intersect $\uparrow\mathcal{L}_0$. □

Observe that any linked system $\mathcal{L} \in N_2(X)$ extending an invariant linked system $\mathcal{L}_0 \in \overset{\leftrightarrow}{N}_2(X)$ lies in the inclusion hyperspace $\mathcal{L}_0^{\perp}$. It turns out that sets from $\mathcal{L}_0^{\perp} \setminus \mathcal{L}_0$ have a specific structure described in the following theorem.

Theorem 3. *For any maximal invariant linked system $\mathcal{L}_0 \in \overset{\leftrightarrow}{\lambda}(X)$ and any $A \in \mathcal{L}_0^{\perp} \setminus \mathcal{L}_0$ there are points $a, b \in X$ such that $aA \subset X \setminus A \subset bA$.*

Proof. Fix a subset $A \in \mathcal{L}_0^{\perp} \setminus \mathcal{L}_0$. We claim that

$$aA \cap A = \varnothing \tag{2}$$

for some $a \in X$. Assuming the converse, we would conclude that the family $\{xA : x \in X\}$ is linked and then the invariant linked system $\mathcal{L}_0 \cup \{xA : x \in X\}$ is strictly larger than $\mathcal{L}_0$, which impossible because of the maximality of $\mathcal{L}_0$.

Next, we find $b \in X$ with

$$A \cup bA = X. \tag{3}$$

Assuming that no such a point b exist, we conclude that for any $x, y \in X$ the union $xA \cup yA \neq X$. Then $(X \setminus xA) \cap (X \setminus yA) = X \setminus (xA \cup yA) \neq \varnothing$, which means that the family $\{X \setminus xA : x \in X\}$ is linked and invariant. We claim that $X \setminus A \in \mathcal{L}_0^{\perp}$. Assuming the converse, we would conclude that $X \setminus A$ misses some set $L \in \mathcal{L}_0$. Then $L \subset A$ and hence $A \in \mathcal{L}_0$ which is not the case. Thus $X \setminus A \in \mathcal{L}_0^{\perp}$ and hence $\{X \setminus xA : x \in X\} \subset \mathcal{L}_0^{\perp}$ because $\mathcal{L}_0^{\perp}$ is invariant. Since $\mathcal{L}_0 \cup \{X \setminus xA : x \in X\}$ is an invariant linked system containing $\mathcal{L}_0$, the maximality of $\mathcal{L}_0$ guarantees that $G \setminus A \in \mathcal{L}_0$ which contradicts $A \in \mathcal{L}_0^{\perp}$.

Unifying the equalities (2) and (3) we get the required inclusions

$$aA \subset X \setminus A \subset bA.$$

□

5 Twin and pretwin sets in groups

Having in mind the sets appearing in Theorem 3 we introduce the following two notions.

Definition 5.1. *A subset A of a group X is called*

- a twin subset *if $X \setminus A = xA$ for some $x \in X$;*
- a pretwin subset *if $xA \subset X \setminus A \subset yA$ for some $x, y \in X$.*

By T and pT we denote the families of twin and pretwin subsets of X, respectively.

Proposition 5.1. *The families* pT *and* T *are* λ*-invariant.*

Proof. Take any maximal linked system $\mathcal{L} \in \lambda(X)$ and consider its function representation $f = \Phi_{\mathcal{L}} : \mathsf{P}(X) \to \mathsf{P}(X)$, which is equivariant, monotone, and symmetric according to Theorem 2.

To show that the family pT is λ-invariant, take any pretwin set $A \in \mathsf{pT}$ and find two points $x, y \in X$ with $xA \subset X \setminus A \subset yA$. Applying to those inequalities the monotone equivariant symmetric function f we get

$$xf(A) = f(xA) \subset f(X \setminus A) = X \setminus f(A) \subset f(yA) = yf(A),$$

which means that $f(A)$ is pretwin.

If a set A is twin, then $X \setminus A = xA$ for some $x \in X$ and then $X \setminus f(A) = f(X \setminus A) = f(xA) = xf(A)$, which means that $f(A)$ is a twin set. $\square$

Propositions 5.1 and 3.1 imply that $\lambda(X, \mathsf{T})$ and $\lambda(X, \mathsf{pT})$ both are compact right-topological semigroups. The importance of the family pT is explained by the following

Theorem 4. *For every maximal invariant linked system* $\mathcal{L}_0 \in \overset{\leftrightarrow}{\lambda}(X)$ *the restriction* $\Phi_{\mathsf{pT}}|{\uparrow}\mathcal{L}_0 : {\uparrow}\mathcal{L}_0 \to \lambda(X, \mathsf{pT})$ *is a topological isomorphism of the compact right-topological semigroups.*

Proof. Since Φ_{pT} is continuous and the semigroups $\lambda(X)$ and $\lambda(X, \mathsf{pT})$ are compact. It suffices to check that the restriction $\Phi_{\mathsf{pT}}|{\uparrow}\mathcal{L}_0$ is bijective.

To show that it is surjective, take any function $f \in \lambda(X, \mathsf{pT})$, which is equivariant, monotone, and symmetric according to Theorem 2.

By the proof of Theorem 2, the family

$$\mathcal{L}_f = \{x^{-1}A : A \in \mathsf{pT},\ x \in f(A)\}$$

is linked. We claim that so is the family $\mathcal{L}_0 \cup \mathcal{L}_f$. Assuming the opposite we could find disjoint sets $A \in \mathcal{L}_f$ and $B \in \mathcal{L}_0$. Since A is pretwin, $xA \subset X \setminus A \subset yA$ for some $x, y \in X$. Now we see that

$$B \subset X \setminus A \subset yA \subset X \setminus yB,$$

which is not possible as B is self-linked and hence meets its shift yB.

Now extend the linked family $\mathcal{L}_0 \cup \mathcal{L}_f$ to a maximal linked family $\mathcal{L} \in \lambda(X)$ and show that $\Phi_{\mathcal{L}}|\mathsf{pT} = f$ (repeating the argument of the proof of Theorem 2).

Next, we show that the restriction $\Phi_{\mathsf{pT}}|{\uparrow}\mathcal{L}_0$ is injective. Take any two distinct maximal linked systems $\mathcal{X}, \mathcal{Y} \in {\uparrow}\mathcal{L}_0$. It follows that there is a set $A \in \mathcal{X} \setminus \mathcal{Y}$. This set belongs to $\mathcal{L}_0^{\perp} \setminus \mathcal{L}_0$ and hence is pretwin by Theorem 3. Now the definition of the function representation yields that $e \in \Phi_{\mathcal{X}}(A) \setminus \Phi_{\mathcal{Y}}(A)$, witnessing that $\Phi_{\mathsf{pT}}(\mathcal{X}) \neq \Phi_{\mathsf{pT}}(\mathcal{Y})$. $\square$

Since the function representation Φ_{pT} is injective on the left ideal ${\uparrow}\mathcal{L}_0$ of $\lambda(X)$, it is injective on some minimal left ideal of $\lambda(X)$ and hence is injective on each minimal left ideal of $\lambda(X)$, see Proposition 1.1. In such a way we prove

Corollary 5.1. *The function representation $\Phi_{\mathsf{pT}} : \lambda(X) \to \lambda(X, \mathsf{pT})$ is injective on each minimal left ideal of $\lambda(X)$. Consequently, each minimal left ideal of $\lambda(X)$ is topologically isomorphic to a minimal left ideal of the semigroup $\lambda(X, \mathsf{pT})$.*

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В роботі вивчається алгебраїчна структура компактної правотопологічної напівгрупи $G(X)$, яка складається зі всіх гіперпросторів включення на групі X. Дана напівгрупа містить напівгрупу $\lambda(X)$ всіх максимальних зчеплених систем як замкнену піднапівгрупу. Побудовано точне зображення напівгруп $G(X)$ та $\lambda(X)$ в напівгрупі $\mathsf{P}(X)^{\mathsf{P}(X)}$ всіх відображень степінь-множини $\mathsf{P}(X)$ в себе. Використовуючи це зображення доведено, що кожен мінімальний лівий ідеал напівгрупи $\lambda(X)$ топологічно ізоморфний мінімальному лівому ідеалу напівгрупи $\mathsf{pT}^{\mathsf{pT}}$.

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В работе изучается алгебраическая структура компактной правотопологической полугруппы $G(X)$, которая содержит все гиперпространства включения на группе X. Эта полугруппа содержит полугруппу $\lambda(X)$ всех максимальных сцепленных систем в качестве замкнутой подполугруппы. Построено точное представление полугрупп $G(X)$ и $\lambda(X)$ в полугруппе $\mathsf{P}(X)^{\mathsf{P}(X)}$ всех отображений степень-множества $\mathsf{P}(X)$ в себя. Используя это представление доказано, что каждый минимальный левый идеал полугруппы $\lambda(X)$ топологически изоморфен минимальному левому идеалу полугруппы $\mathsf{pT}^{\mathsf{pT}}$.