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ON CONVERGENCE CRITERIA FOR BRANCHED CONTINUED FRACTION

The starting point of the present paper is a result by E.A. Boltarovych (1989) on convergence regions, dealing with branched continued fraction

$$\sum_{i_1=1}^N \frac{a_{i(1)}}{1} + \sum_{i_2=1}^N \frac{a_{i(2)}}{1} + \cdots + \sum_{i_n=1}^N \frac{a_{i(n)}}{1} + \cdots,$$

where $|a_{i(2n-1)}| \leq \alpha/N$, $i_p = \overline{1, N}$, $p = \overline{1, 2n-1}$, $n \geq 1$, and for each multiindex $i(2n-1)$ there is a single index j_{2n} , $1 \leq j_{2n} \leq N$, such that $|a_{i(2n-1), j_{2n}}| \geq R$, $i_p = \overline{1, N}$, $p = \overline{1, 2n-1}$, $n \geq 1$, and $|a_{i(2n)}| \leq r/(N-1)$, $i_{2n} \neq j_{2n}$, $i_p = \overline{1, N}$, $p = \overline{1, 2n}$, $n \geq 1$, where $N > 1$ and α, r, R are real numbers that satisfying certain conditions. In the present paper, conditions for these regions are replaced by $\sum_{i_1=1}^N |a_{i(1)}| \leq \alpha(1-\varepsilon)$, $\sum_{i_{2n+1}=1}^N |a_{i(2n+1)}| \leq \alpha(1-\varepsilon)$, $i_p = \overline{1, N}$, $p = \overline{1, 2n}$, $n \geq 1$, and for each multiindex $i(2n-1)$ there is a single index j_{2n} , $1 \leq j_{2n} \leq N$, such that $|a_{i(2n-1), j_{2n}}| \geq R$ and $\sum_{i_{2n} \in \{1, 2, \dots, N\} \setminus \{j_{2n}\}} |a_{i(2n)}| \leq r$, $i_p = \overline{1, N}$, $p = \overline{1, 2n-1}$, $n \geq 1$, where ε, α, r and R are real numbers that satisfying certain conditions, and better convergence speed estimates are obtained.

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1 INTRODUCTION

All known general methods of proof of convergence criteria of continued fractions are based on value-region considerations. The interplay between element regions and value regions leads to convergence region criteria, that is, results of the form: if the elements of continued fraction lie in some regions then the continued fraction converges. In addition, the relationship between element regions and value regions provides one with knowledge of the location of approximants of continued fraction whose elements lie in some convergence regions. Both of these phenomena (i.e., the convergence regions and the information about the location of approximants) are not to be found for most common infinite processes, such as series and products [15, pp. 63–78].

It is well know (see, for example, [7]) that branched continued fractions (BCF) are multi-dimensional generalization of continued fractions. Let N be a fixed natural number. For BCF with the complex elements

$$\sum_{i_1=1}^N \frac{a_{i(1)}}{1} + \sum_{i_2=1}^N \frac{a_{i(2)}}{1} + \cdots + \sum_{i_n=1}^N \frac{a_{i(n)}}{1} + \cdots, \quad (1)$$

E.A. Boltarovych [9] proved the following theorem.

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Theorem 1. Let $N > 1$ and let there exist real numbers α, r and R such that $0 \leq \alpha \leq 1/4$, $0 \leq r < \infty$, $R(1 - \alpha) \geq (1 + \alpha)(r + 2 - 2\alpha)$,

$$Q = \frac{\alpha(R + r)(1 + \alpha)^2}{(R(1 - \alpha) - r(1 + \alpha) - 1 + \alpha^2)^2} < 1, \tag{2}$$

and such that BCF (1) with elements $a_{i(n)}$ satisfying

$$|a_{i(2n-1)}| \leq \alpha/N, \quad i_p = \overline{1, N}, \quad p = \overline{1, 2n-1}, \quad n \geq 1, \tag{3}$$

and for each multiindex $i(2n - 1)$ there is a single index j_{2n} , $1 \leq j_{2n} \leq N$, such that

$$|a_{i(2n-1), j_{2n}}| \geq R, \quad i_p = \overline{1, N}, \quad p = \overline{1, 2n-1}, \quad n \geq 1, \tag{4}$$

$$|a_{i(2n)}| \leq r/(N - 1), \quad i_{2n} \neq j_{2n}, \quad i_p = \overline{1, N}, \quad p = \overline{1, 2n}, \quad n \geq 1. \tag{5}$$

Then the BCF (1) converges.

This is analog of result by Leighton–Wall [13] on twin convergence regions, dealing with continued fractions. In the present paper, we shall study what happens to conditions on numbers α, r and R , and convergence speed estimates, when the conditions (3)–(5) are replaced by

$$\sum_{i_1=1}^N |a_{i(1)}| \leq \alpha(1 - \varepsilon), \quad \sum_{i_{2n+1}=1}^N |a_{i(2n+1)}| \leq \alpha(1 - \varepsilon), \quad i_p = \overline{1, N}, \quad p = \overline{1, 2n}, \quad n \geq 1, \tag{6}$$

where $0 < \varepsilon < 1$, and

$$|a_{i(2n-1), j_{2n}}| \geq R, \quad \sum_{i_{2n} \in \{1, 2, \dots, N\} \setminus \{j_{2n}\}} |a_{i(2n)}| \leq r, \quad i_p = \overline{1, N}, \quad p = \overline{1, 2n-1}, \quad n \geq 1. \tag{7}$$

The same type of problem of convergence regions for BCF is discussed in [2–6, 14]. Application of the value regions to the study of the convergence of functional BCF may be found in [5, 8, 10]. Expansions of certain analytic functions in some classes of BCF are given in [1, 8, 11, 12].

We give here a few facts (see [7]) that are used. Let $Q_{i(k)}^{(n)}$ denotes the “tails” of (1), that is $Q_{i(s)}^{(s)} = 1, i_p = \overline{1, N}, p = \overline{1, s}, s \geq 1$, and

$$Q_{i(k)}^{(n)} = 1 + \sum_{i_{k+1}=1}^{i_k} \frac{a_{i(k+1)}}{1} + \sum_{i_{k+2}=1}^{i_{k+1}} \frac{a_{i(k+2)}}{1} + \dots + \sum_{i_n=1}^{i_{n-1}} \frac{a_{i(n)}}{1},$$

where $i_p = \overline{1, N}, p = \overline{1, k}, k = \overline{1, n-1}, n \geq 2$. It is clear that the following recurrence relations hold

$$Q_{i(k)}^{(n)} = 1 + \sum_{i_{k+1}=1}^{i_k} \frac{a_{i(k+1)}}{Q_{i(k+1)}^{(n)}}, \quad i_p = \overline{1, N}, \quad p = \overline{1, k}, \quad k = \overline{1, n-1}, \quad n \geq 2.$$

If f_n denotes the n -th approximant of (1), then $f_n = \sum_{i_1=1}^N (a_{i(1)} / Q_{i(1)}^{(n)})$, $n \geq 1$, and if all $Q_{i(k)}^{(n)} \neq 0$, then

$$f_m - f_n = (-1)^n \sum_{i_1=1}^N \sum_{i_2=1}^N \dots \sum_{i_{n+1}=1}^N \frac{\prod_{k=1}^{n+1} a_{i(k)}}{\prod_{k=1}^{n+1} Q_{i(k)}^{(m)} \prod_{k=1}^n Q_{i(k)}^{(n)}}, \quad m > n \geq 1. \tag{8}$$

2 CONVERGENCE CRITERIA

We shall prove the auxiliary lemma.

Lemma. *Let there exist real numbers α, r and R such that*

$$0 \leq \alpha < 1, 0 \leq r < \infty, R(1 - \alpha) \geq (1 + \alpha)(r + 2 - 2\alpha), \quad (9)$$

and such that BCF (1) with elements $a_{i(n)}$ satisfying

$$\sum_{i_1=1}^N |a_{i(1)}| \leq \alpha, \quad \sum_{i_{2n+1}=1}^N |a_{i(2n+1)}| \leq \alpha, \quad i_p = \overline{1, N}, \quad p = \overline{1, 2n}, \quad n \geq 1, \quad (10)$$

and for each multiindex $i(2n - 1)$ there is a single index j_{2n} , $1 \leq j_{2n} \leq N$, such that the inequalities (7) hold. If $Q_{i(k)}^{(n)}$ denotes the "tails" of BCF (1), the following inequalities hold

$$1 - \alpha \leq |Q_{i(2k)}^{(n)}| \leq 1 + \alpha, \quad i_p = \overline{1, N}, \quad 1 \leq p \leq 2k \leq n, \quad n \geq 2, \quad (11)$$

$$|Q_{i(2k-1)}^{(n)}| \geq \frac{R}{1 + \alpha} - \frac{r}{1 - \alpha} - 1 \geq 1, \quad i_p = \overline{1, N}, \quad 1 \leq p \leq 2k - 1 \leq n - 1, \quad n \geq 2. \quad (12)$$

Proof. Let n be an arbitrary natural number. By induction on k for each $i(k)$ we show that the inequalities (11) and (12) are valid.

If n is even number and $k = n/2$, then for each $i(n)$ relations (11) are obvious. If n is odd number and $k = (n - 1)/2$, then for arbitrary $i(n - 1)$ use of relation (10) leads to

$$|Q_{i(n-1)}^{(n)}| \geq 1 - \sum_{i_n=1}^N |a_{i(n)}| \geq 1 - \alpha \quad \text{and} \quad |Q_{i(n-1)}^{(n)}| \leq 1 + \sum_{i_n=1}^N |a_{i(n)}| \leq 1 + \alpha.$$

By induction hypothesis that (11) hold for $k = r$ and for each $i(2r)$, where $2r \leq n$, we prove the inequalities (12) for $k = r$ and for each $i(2r - 1)$ and the inequalities (11) for $2k = 2r - 2$ for each $i(2r - 2)$. Indeed, use of relations (7), (9), (10) for arbitrary $i(2r - 1)$ leads to

$$\begin{aligned} |Q_{i(2r-1)}^{(n)}| &= \left| 1 + \frac{a_{i(2r-1), j_{2r}}}{Q_{i(2r-1), j_{2r}}^{(n)}} + \sum_{i_{2r} \in \{1, 2, \dots, N\} \setminus \{j_{2r}\}} \frac{a_{i(2r)}}{Q_{i(2r)}^{(n)}} \right| \\ &\geq \frac{|a_{i(2r-1), j_{2r}}|}{|Q_{i(2r-1), j_{2r}}^{(n)}|} - \sum_{i_{2r} \in \{1, 2, \dots, N\} \setminus \{j_{2r}\}} \frac{|a_{i(2r)}|}{|Q_{i(2r)}^{(n)}|} - 1 \geq \frac{R}{1 + \alpha} - \frac{r}{1 - \alpha} - 1 \geq 1 \end{aligned}$$

and for arbitrary $i(2r - 2)$

$$|Q_{i(2r-2)}^{(n)}| \geq 1 - \sum_{i_{2r-1}=1}^N \frac{|a_{i(2r-1)}|}{|Q_{i(2r-1)}^{(n)}|} \geq 1 - \alpha \quad \text{and} \quad |Q_{i(2r-2)}^{(n)}| \leq 1 + \sum_{i_{2r-1}=1}^N \frac{|a_{i(2r-1)}|}{|Q_{i(2r-1)}^{(n)}|} \leq 1 + \alpha.$$

This completes the proof of the lemma. \square

Our main result is the following theorem.

Theorem 2. *Let there exist real numbers α, ε, r and R such that $0 \leq \alpha < 1, 0 < \varepsilon < 1, 0 < r < \infty, R(1 - \alpha) \geq (1 + \alpha)(r + 2 - 2\alpha)$ and such that BCF (1) with elements $a_{i(n)}$ satisfying the inequalities (6) and for each multiindex $i(2n - 1)$ there is a single index j_{2n} , $1 \leq j_{2n} \leq N$, such that the inequalities (7) hold. Then the following statements hold.*

- (A) The BCF (1) converges to a value f .
- (B) If f_n denotes the n -th approximant of the BCF (1) and

$$q = \frac{\alpha(1+\alpha)(R(1-\alpha) + r(1+\alpha))}{(R(1-\alpha) - r(1+\alpha) - 1 + \alpha^2)^2} \leq 1, \quad (13)$$

then

$$|f - f_{2n}| \leq \frac{\alpha(1-\varepsilon)^{n+1}q^n}{R/(1+\alpha) - r/(1-\alpha) - 1}, \quad n \geq 1. \quad (14)$$

- (C) The values of the BCF (1) and of its approximants are in the region $|z| \leq \alpha(1-\varepsilon)$.

Proof. At first, we prove (B). Let $m > 2n + 1$ and $n \geq 1$. From the formula (8) one obtains

$$\begin{aligned} |f_m - f_{2n}| &\leq \sum_{i_1=1}^N \sum_{i_2=1}^N \cdots \sum_{i_{2n+1}=1}^N \frac{|a_{i(1)}|}{|Q_{i(1)}^{(m)}|} \frac{\prod_{k=2}^{2n+1} |a_{i(k)}|}{\prod_{k=2}^{2n+1} |Q_{i(k)}^{(m)}|} \frac{\prod_{k=1}^{2n} |Q_{i(k)}^{(2n)}|}{\prod_{k=1}^{2n} |Q_{i(k)}^{(2n)}|} \\ &= \sum_{i_1=1}^N \sum_{i_2=1}^N \cdots \sum_{i_{2n+1}=1}^N \frac{|a_{i(1)}|}{|Q_{i(1)}^{(m)}|} \frac{\prod_{k=1}^n |a_{i(2k)}|}{\prod_{k=1}^n |Q_{i(2k-1)}^{(2n)} Q_{i(2k)}^{(2n)}|} \frac{\prod_{k=1}^n |a_{i(2k+1)}|}{\prod_{k=1}^n |Q_{i(2k)}^{(m)} Q_{i(2k+1)}^{(m)}|}. \end{aligned}$$

Obviously, the conditions of lemma hold. Let k be an arbitrary natural number. Applying (11) and (12) we have for arbitrary $i(2k-1)$

$$\begin{aligned} &\sum_{i_{2k}=1}^N \frac{|a_{i(2k)}|}{|Q_{i(2k-1)}^{(2n)} Q_{i(2k)}^{(2n)}|} \sum_{i_{2k+1}=1}^N \frac{|a_{i(2k+1)}|}{|Q_{i(2k)}^{(m)} Q_{i(2k+1)}^{(m)}|} \\ &\leq \frac{\alpha(1-\varepsilon)}{(1-\alpha)(R/(1+\alpha) - r/(1-\alpha) - 1)} \sum_{i_{2k}=1}^N \frac{|a_{i(2k)}|}{|Q_{i(2k-1)}^{(2n)} Q_{i(2k)}^{(2n)}|} \\ &= \frac{\alpha(1-\varepsilon)}{(1-\alpha)(R/(1+\alpha) - r/(1-\alpha) - 1)} \\ &\quad \times \left(\frac{|a_{i(2k-1),j_{2k}}|}{|Q_{i(2k-1)}^{(2n)} Q_{i(2k-1),j_{2k}}^{(2n)}|} + \sum_{i_{2k} \in \{1,2,\dots,N\} \setminus \{j_{2k}\}} \frac{|a_{i(2k)}|}{|Q_{i(2k-1)}^{(2n)} Q_{i(2k)}^{(2n)}|} \right). \end{aligned}$$

Since

$$\begin{aligned} \sum_{i_{2k} \in \{1,\dots,N\} \setminus \{j_{2k}\}} \frac{|a_{i(2k)}|}{|Q_{i(2k-1)}^{(2n)} Q_{i(2k)}^{(2n)}|} &\leq \frac{1}{(1-\alpha)(R/(1+\alpha) - r/(1-\alpha) - 1)} \sum_{i_{2k} \in \{1,2,\dots,N\} \setminus \{j_{2k}\}} |a_{i(2k)}| \\ &\leq \frac{r}{(1-\alpha)(R/(1+\alpha) - r/(1-\alpha) - 1)} \end{aligned}$$

and

$$\begin{aligned} \frac{|a_{i(2k-1),j_{2k}}|}{|Q_{i(2k-1)}^{(2n)} Q_{i(2k-1),j_{2k}}^{(2n)}|} &= \left| \frac{a_{i(2k-1),j_{2k}}/Q_{i(2k-1),j_{2k}}^{(2n)}}{1 + a_{i(2k-1),j_{2k}}/Q_{i(2k-1),j_{2k}}^{(2n)} + \sum_{i_{2k} \in \{1,2,\dots,N\} \setminus \{j_{2k}\}} (a_{i(2k)}/Q_{i(2k)}^{(2n)})} \right| \\ &= \left| 1 - \frac{1 + \sum_{i_{2k} \in \{1,2,\dots,N\} \setminus \{j_{2k}\}} (a_{i(2k)}/Q_{i(2k)}^{(2n)})}{1 + a_{i(2k-1),j_{2k}}/Q_{i(2k-1),j_{2k}}^{(2n)} + \sum_{i_{2k} \in \{1,2,\dots,N\} \setminus \{j_{2k}\}} (a_{i(2k)}/Q_{i(2k)}^{(2n)})} \right| \\ &\leq 1 + \left| \frac{1 + \sum_{i_{2k} \in \{1,2,\dots,N\} \setminus \{j_{2k}\}} (a_{i(2k)}/Q_{i(2k)}^{(2n)})}{Q_{i(2k-1)}^{(2n)}} \right| \\ &\leq 1 + \frac{1 + r/(1 - \alpha)}{R/(1 + \alpha) - r/(1 - \alpha) - 1} = \frac{R/(1 + \alpha)}{R/(1 + \alpha) - r/(1 - \alpha) - 1}' \end{aligned}$$

then

$$\sum_{i_{2k}=1}^N \frac{|a_{i(2k)}|}{|Q_{i(2k-1)}^{(2n)} Q_{i(2k)}^{(2n)}|} \sum_{i_{2k+1}=1}^N \frac{|a_{i(2k+1)}|}{|Q_{i(2k)}^{(m)} Q_{i(2k+1)}^{(m)}|} \leq \frac{\alpha(1 - \varepsilon)(R/(1 + \alpha) + r/(1 - \alpha))}{(1 - \alpha)(R/(1 + \alpha) - r/(1 - \alpha) - 1)^2}.$$

Thus, for $m > 2n + 1$ and $n \geq 1$

$$|f_m - f_{2n}| \leq \frac{\alpha^{n+1}(1 - \varepsilon)^{n+1}(R/(1 + \alpha) + r/(1 - \alpha))^n}{(1 - \alpha)^n(R/(1 + \alpha) - r/(1 - \alpha) - 1)^{2n+1}} = \frac{\alpha(1 - \varepsilon)^{n+1}q^n}{R/(1 + \alpha) - r/(1 - \alpha) - 1}' \quad (15)$$

where q is defined by (13). If in (15) we pass to the limit as $n \rightarrow \infty$, then from (13) it follows that BCF (1) converges. On the other hand, if in (15) we pass to the limit as $m \rightarrow \infty$, we obtain the estimate (14). This proves (B).

To prove (A) we consider the following equation

$$F_1(x) = F_2(x), \quad (16)$$

where

$$F_1(x) = \frac{x}{1 - x} \left(\frac{R}{1 + x} + \frac{r}{1 - x} \right), \quad F_2(x) = \left(\frac{R}{1 + x} - \frac{r}{1 - x} - 1 \right)^2.$$

It is clear that $F_1(0) < F_2(0)$, and $F_1(x) > 0$ and $F_2(x) \geq 0$ for all $x \in (0; 1)$. It follows from $F_1'(x) = R(1 + x^2)/(1 - x^2)^2 + r(1 + x)/(1 - x)^3$ that $F_1'(x) > 0$ for all $x \in (0; 1)$. Let us write the function $F_2(x)$ in the form $F_2(x) = (x^2 - (R + r)x + R - r - 1)^2/(1 - x^2)^2$ and consider the following equation

$$x^2 - (R + r)x + R - r - 1 = 0. \quad (17)$$

If $r > 0$, then $x^* = (R + r - \sqrt{(R + r)^2 - 4(R - r - 1)})/2$ is the only root of equation (17) on $(0; 1)$ and, if $r = 0$, then $x^* = 1$ is the only root of (17). Now from

$$F_2'(x) = -2 \frac{x^2 - (R + r)x + R - r - 1}{1 - x^2} \left(\frac{R}{(1 + x)^2} + \frac{r}{(1 - x)^2} \right)$$

we have $F_2'(x) < 0$ for all $x \in (0; x^*)$. It follows that there exists the only root α^* of equation (16) on $(0; x^*)$. If $0 < \alpha \leq \alpha^*$, then $F_1(\alpha) \leq F_2(\alpha)$, that is, the condition (13) holds. In the case when $\alpha^* < \alpha < 1$ we consider the following BCF

$$\sum_{i_1=1}^N \frac{a_{i(1)}z}{1} + \sum_{i_2=1}^N \frac{a_{i(2)}}{1} + \dots + \sum_{i_{2k-1}=1}^N \frac{a_{i(2k-1)}z}{1} + \sum_{i_{2k}=1}^N \frac{a_{i(2k)}}{1} + \dots, \quad (18)$$

where $z \in \mathbb{C}$. It is clear that the elements of BCF (18) satisfy the conditions of lemma in domain $D_\varepsilon = \{z \in \mathbb{C} : |z| < 1/(1 - \varepsilon)\}$. It follows from (11) and (12) that, if $f_n(z)$ denotes the n -th approximant of the BCF (18), for all $z \in D_\varepsilon$

$$|f_n(z)| \leq \sum_{i_1=1}^N |a_{i_1(1)}z| \leq \alpha(1 - \varepsilon)|z| < \alpha,$$

i.e. the sequence $\{f_n(z)\}$ is uniformly bounded in the domain D_ε . If $z \in D_{\alpha^*}$, where $D_{\alpha^*} = \{z \in \mathbb{C} : |z| < \alpha^*/\alpha\}$, then according to the above BCF (18) converges. Obviously, $D_{\alpha^*} \subset D_\varepsilon$. Hence, by [16, Theorem 24.2, p. 108] BCF (18) converges uniformly on each compact subset of the domain D_ε , in particular, for $z = 1$. It follows that BCF (1) converges.

Finally, from

$$|f_n| \leq \sum_{i_1=1}^N \frac{|a_{i_1(1)}|}{|Q_{i_1(1)}^{(n)}|} \leq \sum_{i_1=1}^N |a_{i_1(1)}| \leq \alpha(1 - \varepsilon)$$

follows proof of (C). □

Remark. If the conditions (3)–(5) are replaced by the conditions (6) and (7), then the condition (2) is replaced by the condition (13) and the $0 \leq \alpha \leq 1/4$ is replaced by the $0 \leq \alpha < 1$. It is clear that $Q > q$, and, thus, the estimates (14) are better than similar estimates obtained in the proof of Theorem 1. In addition, if $q < 1$, then ε can be zero.

Corollary. Let there exist real numbers β and ε such that $0 \leq \beta < 1/N$, $0 < \varepsilon < 1$, and such that BCF (1) with elements $a_{i(n)}$ satisfying $|a_{i(2n-1)}| \leq \beta(1 - \varepsilon)$, where $i_p = \overline{1, N}$, $p = \overline{1, 2n - 1}$, $n \geq 1$, and for each multiindex $i(2n - 1)$ there is a single index j_{2n} , $1 \leq j_{2n} \leq N$, such that

$$\begin{aligned} |a_{i(2n-1), j_{2n}}| &\geq (1 + N\beta)(2 - (1 + N\beta)) / (1 - N\beta), \quad i_p = \overline{1, N}, \quad p = \overline{1, 2n - 1}, \quad n \geq 1, \\ |a_{i(2n)}| &\leq \beta, \quad i_{2n} \neq j_{2n}, \quad i_p = \overline{1, N}, \quad p = \overline{1, 2n}, \quad n \geq 1. \end{aligned}$$

Then BCF (1) converges, and its values and its approximants are in the region $|z| \leq N\beta(1 - \varepsilon)$.

Proof. We set $\alpha = N\beta$, $r = (N - 1)\beta$, $R = (1 + N\beta)(2 - \beta(1 + N)) / (1 - N\beta)$. Then

$$R = \frac{1 + N\beta}{1 - N\beta}(2 - 2N\beta + (N - 1)\beta) = (1 + N\beta) \left(2 + \frac{N - 1}{1 - N\beta}\beta \right) = (1 + \alpha) \left(2 + \frac{r}{1 - \alpha} \right).$$

It follows that the conditions of Theorem 2 hold, and, therefore, the corollary is an immediate consequence of this theorem. □

3 EXAMPLE

Let β, r and R be some positive numbers. We consider the periodical BCF

$$\sum_{i_1=1}^2 \frac{a_{i_1(1)}}{1} + \sum_{i_2=1}^2 \frac{a_{i_2(2)}}{1} + \dots + \sum_{i_n=1}^2 \frac{a_{i_n(n)}}{1} + \dots, \tag{19}$$

where $a_{i(1)} = \beta$, $a_{i(2n-1)} = (-1)^{i_{2n-2}-1}\beta$, $a_{i(2n-1),1} = (-1)^{i_{2n-1}-1}R$, $a_{i(2n-1),2} = (-1)^{i_{2n-1}}r$, which form by the following fractional transformation

$$s(w) = \frac{\beta}{1 + \frac{R}{1+w} - \frac{r}{1-w}} + \frac{\beta}{1 - \frac{R}{1+w} + \frac{r}{1-w}}.$$

It follows that BCF (19) can be converged only to the real root of the following equation

$$(w - 2\beta)(1 - w^2)^2 - w(R - r - w(R + r))^2 = 0. \quad (20)$$

We choose $\beta = \alpha(1 - \varepsilon)/2$, $\alpha = 1/3$, $\varepsilon = 1/4$, $r = 2/3$ and $R = 5$. Then it is clear that the conditions of Theorem 2 are satisfied and the inequalities $|w| \leq 2\beta$ are valid. Thus, BCF (19) converges. On the other hand the equation (20) we write in the form

$$9(4w - 1)(1 - w^2)^2 - 4w(13 - 17w)^2 = 0. \quad (21)$$

Let $F(w) = 9(4w - 1)(1 - w^2)^2 - 4w(13 - 17w)^2$. Then $F(0) < 0$ and $F(-1/4) > 0$. Thus, on the interval $[-1/4; 0]$ there is root of the equation (21). The following recurrent formula

$$f_{k+2} = \frac{2\beta(1 - f_k^2)^2}{(1 - f_k^2)^2 - (R - r - f_k(R + r))^2}, \quad k \geq 1,$$

with initial conditions $f_1 = 2\beta$ and $f_2 = 2\beta/(1 - (R - r)^2)$ can be used to find of the above mentioned root.

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Основою цієї роботи є результат Є.А. Болтаровича (1989) про множини збіжності для гіллястого ланцюгового дроби

$$\sum_{i_1=1}^N \frac{a_{i(1)}}{1} + \sum_{i_2=1}^N \frac{a_{i(2)}}{1} + \dots + \sum_{i_n=1}^N \frac{a_{i(n)}}{1} + \dots,$$

де $|a_{i(2n-1)}| \leq \alpha/N$, $i_p = \overline{1, N}$, $p = \overline{1, 2n-1}$, $n \geq 1$, і для кожного мультиіндексу $i(2n-1)$ існує єдиний індекс j_{2n} , $1 \leq j_{2n} \leq N$, такий, що $|a_{i(2n-1), j_{2n}}| \geq R$, $i_p = \overline{1, N}$, $p = \overline{1, 2n-1}$, $n \geq 1$, та $|a_{i(2n)}| \leq r/(N-1)$, $i_{2n} \neq j_{2n}$, $i_p = \overline{1, N}$, $p = \overline{1, 2n}$, $n \geq 1$, де $N > 1$, α , r та R – дійсні числа, що задовольняють певні умови. У цій роботі умови для цих множин замінено на $\sum_{i_1=1}^N |a_{i(1)}| \leq \alpha(1-\varepsilon)$, $\sum_{i_{2n+1}=1}^N |a_{i(2n+1)}| \leq \alpha(1-\varepsilon)$, $i_p = \overline{1, N}$, $p = \overline{1, 2n}$, $n \geq 1$, і для кожного мультиіндексу $i(2n-1)$ існує єдиний індекс j_{2n} , $1 \leq j_{2n} \leq N$, такий, що $|a_{i(2n-1), j_{2n}}| \geq R$ та $\sum_{i_{2n} \in \{1, 2, \dots, N\} \setminus \{j_{2n}\}} |a_{i(2n)}| \leq r$, $i_p = \overline{1, N}$, $p = \overline{1, 2n-1}$, $n \geq 1$, де ε , α , r та R – дійсні числа, що задовольняють певні умови, і, отримано кращі оцінки швидкості збіжності для цього гіллястого ланцюгового дроби.

Ключові слова і фрази: збіжність, множина збіжності, оцінка швидкості збіжності, гіллястий ланцюговий дріб.