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SUPEREXTENSIONS OF CYCLIC SEMIGROUPS

Given a cyclic semigroup S we study right and left zeros, singleton left ideals, the minimal ideal, left cancelable and right cancelable elements of superextensions $\lambda(S)$ and characterize cyclic semigroups whose superextensions are commutative.

Key words and phrases: cyclic semigroup, maximal linked system, superextensions.

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INTRODUCTION

This paper is devoted to describing the structure of superextensions of cyclic semigroups. The thorough study of algebraic properties of superextensions of semigroups was started in [1, 2, 3, 4, 10], where we focused at describing of superextensions of groups, and continued in [5, 6], where we studied the structure of superextensions of semilattices and inverse semigroups.

A family $\mathcal{F}$ of nonempty subsets of a set X that is closed under taking supersets and finite intersections is called a *filter*. A filter $\mathcal{U}$ is called an *ultrafilter* if $\mathcal{U} = \mathcal{F}$ for any filter $\mathcal{F}$ containing $\mathcal{U}$. A family of subsets of a set X is called a *linked system* if intersection of any two elements is nonempty. A linked system $\mathcal{M}$ is said to be a *maximal linked system* if $\mathcal{M} = \mathcal{L}$ for any linked system $\mathcal{L}$ containing $\mathcal{M}$. The family $\beta(X)$ of all ultrafilters on a set X is called the *Stone-Čech compactification*, and the family $\lambda(X)$ of all maximal linked systems is well-known [11, 12] as the *superextension* of a set X.

Each map $f : X \to Y$ induces a map (see [8])

$$\lambda f : \lambda(X) \to \lambda(Y), \quad \lambda f : \mathcal{M} \longmapsto \langle f(M) \subset Y : M \in \mathcal{M} \rangle.$$

Here for a family $\mathcal{B}$ of nonempty subsets of a set Y by $\langle B \subset Y : B \in \mathcal{B} \rangle$ we denote the family $\langle B \subset Y : B \in \mathcal{B} \rangle = \{A \subset Y : \exists B \in \mathcal{B}\ (B \subset A)\}$. An ultrafilter $\langle \{x\} \rangle$, generated by a singleton $\{x\}$, $x \in X$, is called *principal*. We consider $X \subset \beta(X) \subset \lambda(X)$ if each point $x \in X$ is identified with the principal ultrafilter $\langle \{x\} \rangle$ generated by the singleton $\{x\}$.

It was shown in [9] that any associative binary operation $* : S \times S \to S$ can be extended to an associative binary operation $\circ : \lambda(S) \times \lambda(S) \to \lambda(S)$ by the formula

$$\mathcal{L} \circ \mathcal{M} = \Big\langle \bigcup_{a \in L} a * M_a : L \in \mathcal{L},\ \ \{M_a\}_{a \in L} \subset \mathcal{M} \Big\rangle$$

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for maximal linked systems $\mathcal{L}, \mathcal{M} \in \lambda(S)$. In this case the Stone-Čech compactification $\beta(S)$ is a subsemigroup of the superextension $\lambda(S)$.

A nonempty subset I of a semigroup $(S,*)$ is called *an ideal* (resp. a *right ideal*, a *left ideal*) if $I * S \cup S * I \subset I$ (resp. $I * S \subset I$, $S * I \subset I$). An element z of a semigroup $(S,*)$ is called a *zero* (resp. a *left zero*, a *right zero*) in S if $a * z = z * a = z$ (resp. $z * a = z$, $a * z = z$) for any $a \in S$. It is clear that $z \in S$ is a zero (resp. a left zero, a right zero) in S if and only if the singleton $\{z\}$ is an ideal (resp. a right ideal, a left ideal) in S. An ideal $I \subset S$ is called *minimal* if any ideal of S that lies in I coincides with I. By analogy we define minimal left and minimal right ideals of S. The union $K(S)$ of all minimal left (right) ideals of S coincides with the minimal ideal of S, see [11, theorem 2.8]. A semigroup $(S,*)$ is said to be a *right zeros semigroup* if $a * b = b$ for any $a, b \in S$. A map $\varphi : S \to T$ between semigroups $(S,*)$ and $(T,\circ)$ is called a *homomorphism* if $\varphi(a * b) = \varphi(a) \circ \varphi(b)$ for any $a, b \in S$. A homomorphism $\varphi : S \to I$ from a semigroup S into an ideal $I \subset S$ is called a *retraction* if $\varphi(a) = a$ for any element $a \in I$. An element a of a semigroup S is called *left cancelable* (resp. *right cancelable*) if for any points $x, y \in S$ the equation $ax = ay$ (resp. $xa = ya$) implies $x = y$. This is equivalent to saying that the left (resp. right) shift $l_a : S \to S$, $l_a : x \mapsto a * x$, (resp. $r_a : S \to S$, $r_a : x \mapsto x * a$) is injective. A semigroup S is called *left (right) cancellative* if all elements of S are left (right) cancelable. A semigroup that is both left and right cancellative is said to be *cancellative*.

A semigroup $\langle a \rangle = \{a^n\}_{n \in \mathbb{N}}$ generated by a single element a is called *cyclic*. If a cyclic semigroup is infinite, then it is isomorphic to the additive semigroup $\mathbb{N}$. A finite cyclic semigroup $S = \langle a \rangle$ also has very simple structure (see [7]). There are positive integer numbers r and m called the *index* and the *period* of S such that: (i) $S = \{a, a^2, \ldots, a^{m+r-1}\}$ and $m + r - 1 = |S|$; (ii) for any $i, j \in \omega$ the equality $a^{r+i} = a^{r+j}$ holds if and only if $i \equiv j \mod m$; (iii) $C_m = \{a^r, a^{r+1}, \ldots, a^{m+r-1}\}$ is the minimal ideal, a cyclic and maximal subgroup of S with the neutral element $e = a^n \in C_m$, where m divides n.

From now on we denote by $C_{r,m}$ a finite cyclic semigroup of index r and period m, and maximal subgroup of $C_{r,m}$ is denoted by C_m.

1 HOMOMORPHISMS, RIGHT, LEFT ZEROS AND MINIMAL (LEFT) IDEALS

Proposition 1.1. *For any homomorphism $\varphi : S \to T$ between semigroups $(S, *_1)$ and $(T, *_2)$ the induced map $\lambda\varphi : \lambda(S) \to \lambda(T)$ is a homomorphism of the semigroups $(\lambda(S), \circ_1)$ and $(\lambda(T), \circ_2)$.*

Proof. Given two maximal linked systems $\mathcal{L}, \mathcal{M} \in \lambda(S)$ observe that

$$\begin{aligned}
\lambda\varphi(\mathcal{L} \circ_1 \mathcal{M}) &= \lambda\varphi\big(\big\langle \bigcup_{x \in L} x *_1 M_x : L \in \mathcal{L},\ \{M_x\}_{x \in L} \subset \mathcal{M} \big\rangle\big) \\
&= \big\langle \varphi\big(\bigcup_{x \in L} x *_1 M_x \big) : L \in \mathcal{L},\ \{M_x\}_{x \in L} \subset \mathcal{M} \big\rangle \\
&= \big\langle \bigcup_{x \in L} \varphi(x) *_2 \varphi(M_x) : L \in \mathcal{L},\ \{M_x\}_{x \in L} \subset \mathcal{M} \big\rangle \\
&= \big\langle \bigcup_{x \in \varphi(L)} x *_2 \varphi(M_x) : L \in \mathcal{L},\ \{\varphi(M_x)\}_{x \in \varphi(L)} \subset \lambda\varphi(\mathcal{M}) \big\rangle \\
&= \big\langle \varphi(L) : L \in \mathcal{L} \big\rangle \circ_2 \big\langle \varphi(M) : M \in \mathcal{M} \big\rangle = \lambda\varphi(\mathcal{L}) \circ_2 \lambda\varphi(\mathcal{M}).
\end{aligned}$$

$\square$

Let us note that for a subsemigroup T of a semigroup S the homomorphism $i : \lambda(T) \to \lambda(S)$, $i : \mathcal{A} \to \langle \mathcal{A} \rangle_S$ is injective, and thus we can identify the semigroup $\lambda(T)$ with the subsemigroup $i(\lambda(T)) \subset \lambda(S)$.

Lemma 1.1. *Let I be an ideal of a semigroup S. If a map $\varphi : S \to I$ is a retraction, then the map $\lambda\varphi : \lambda(S) \to \lambda(I)$ is a retraction too.*

Proof. Indeed, let $\mathcal{A} \in \lambda(I)$, $\mathcal{M} \in \lambda(S)$, then $\mathcal{A} \circ \mathcal{M} = \big\langle \bigcup_{a \in A} a * M_a : A \in \mathcal{A}, A \subset I, \{M_a\}_{a \in A} \subset \mathcal{M} \big\rangle = \big\langle \bigcup_{a \in A} a * M_a : A \in \mathcal{A}, \{M_a\}_{a \in A} \subset \mathcal{M}, \bigcup_{a \in A} a * M_a \subset I \big\rangle \in \lambda(I)$. By analogy $\mathcal{M} \circ \mathcal{A} \in \lambda(I)$, and therefore $\lambda(I)$ is an ideal of the semigroup $\lambda(S)$. If $\mathcal{A} \in \lambda(I)$, then $\lambda\varphi(\mathcal{A}) = \langle \varphi(A) : A \subset I, A \in \mathcal{A} \rangle = \langle A \subset I : A \in \mathcal{A} \rangle = \{A \subset I : A \in \mathcal{A}\} = \mathcal{A}$ and hence $\lambda\varphi$ is a retraction. □

Lemma 1.2. *Let I be an ideal of a semigroup S and a map $\varphi : S \to I$ is a retraction. The semigroup S has a right (left) zero if and only if the semigroup I has a right (left) zero, and all right and left zeros of the semigroup S are contained in I.*

Proof. Let z be a right (left) zero of the semigroup S, that is $sz = z$ ($zs = z$) for any $s \in S$. Since φ is a homomorphism, $\varphi(s)\varphi(z) = \varphi(z)$ ($\varphi(z)\varphi(s) = \varphi(z)$). Specifically for any $s \in I$ the equality $\varphi(s) = s$ holds, and then $s\varphi(z) = \varphi(s)\varphi(z) = \varphi(z)$ ($\varphi(z)s = \varphi(z)\varphi(s) = \varphi(z)$). Consequently, $\varphi(z)$ is a right (left) zero of the semigroup I.

Let $z \in I$ be a right (left) zero of the semigroup I. Since I is an ideal, then for any $s \in S$ we have that $sz, zs \in I$, and hence $sz = \varphi(sz) = \varphi(s)\varphi(z) = \varphi(s)z = z$ ($zs = \varphi(zs) = \varphi(z)\varphi(s) = z\varphi(s) = z$). Consequently, z is a right (left) zero of the semigroup S.

If z is a right (left) zero of the semigroup S, then $z = sz \in I$ ($z = zs \in I$), where $s \in I$. Therefore, all right (left) zeros of the semigroup S are contained in I. □

Let e be the neutral element of the maximal subgroup C_m of a cyclic semigroup $C_{r,m}$.

Lemma 1.3. *The map $\varphi : C_{r,m} \to C_m$, $\varphi(x) = ex$ is a retraction and $\varphi(x)y = xy$ for any $x \in C_{r,m}$ and $y \in C_m$.*

Proof. Since the semigroup C_m is an ideal of the semigroup $C_{r,m}$, $\varphi(x) = ex \in C_m$. Consequently, $\varphi(xy) = exy = eexy = exey = \varphi(x)\varphi(y)$ for any $x, y \in C_{r,m}$ and $\varphi(x) = ex = x$ for $x \in C_m$. Hence the map $\varphi : C_{r,m} \to C_m$ is a retraction. Further for any $x \in C_{r,m}$ and $y \in C_m$ we have that $xy \in C_m$, and therefore $\varphi(xy) = xy$. On the other hand, $\varphi(xy) = \varphi(x)\varphi(y) = \varphi(x)y$, since $y \in C_m$. □

Combining Lemmas 1.1–1.3 we get

Proposition 1.2. *The semigroup $\lambda(C_{r,m})$ contains a right (left) zero if and only if its subgroup $\lambda(C_m)$ contains a right (left) zero. Each right (left) zero of $\lambda(C_{r,m})$ belongs to $\lambda(C_m)$.*

It was proved in [1] that the semigroup $\lambda(G)$ possesses a right zero if and only if the group G is periodic and each element of G has odd order. Since each element of a finite group G has odd order if and only if the group G has odd order, Proposition 1.2 implies the following characterization of superextensions of finite cyclic semigroups that have right zeros.

Theorem 1. *The superextension $\lambda(C_{r,m})$ of a finite cyclic semigroup $C_{r,m}$ has a right zero if and only if the period m of the cyclic semigroup $C_{r,m}$ is an odd number.*

Proposition 1.3. *The superextension of the infinite cyclic semigroup has neither right nor left zeros.*

Proof. Let $\langle a\rangle = \{a, a^2, \dots, a^n \dots\}$ be the infinite cyclic semigroup and $\mathcal{M} \in \lambda(\langle a\rangle)$. First observe that if $\langle a\rangle = A \cup B$ is any partition of the set $\langle a\rangle$, then either $A \in \mathcal{M}$ or $B \in \mathcal{M}$. Indeed, if $A \notin \mathcal{M}$, then $M \cap B \neq \varnothing$ for any $M \in \mathcal{M}$, and thus the maximality of $\mathcal{M}$ implies that $B \in \mathcal{M}$. Consider the partition $\langle a\rangle = A \cup B$, where $A = \{a, a^3, \dots, a^{2k-1}, \dots\}$, $B = \{a^2, a^4, \dots, a^{2k}, \dots\}$. Assume that a maximal linked system $\mathcal{M}$ is a right (left) zero of the semigroup $\langle a\rangle$. Then for any $x \in \langle a\rangle$ we have $\langle\{x\}\rangle \circ \mathcal{M} = \mathcal{M}$ ($\mathcal{M} \circ \langle\{x\}\rangle = \mathcal{M}$), and therefore $xM \in \mathcal{M}$ ($Mx \in \mathcal{M}$) for any $M \in \mathcal{M}$. If $A \in \mathcal{M}$, then $B = aA = Aa \in \mathcal{M}$, that is impossible, since $A \cap B = \varnothing$. By analogy, if $B \in \mathcal{M}$, then $A \supset aB = Ba \in \mathcal{M}$. This contradiction implies that the superextension of the infinite cyclic semigroup contains neither right nor left zeros. □

It was proved in [1] that for the semigroup $\lambda(G)$ has a (left) zero if and only if a group G is of order $|G| \in \{1, 3, 5\}$.

Consequently, Proposition 1.2 implies the following characterization of superextensions of finite cyclic semigroups that have (left) zeros.

Theorem 2. *The superextension $\lambda(C_{r,m})$ of a cyclic semigroup $C_{r,m}$ has a (left) zero if and only if $m \in \{1, 3, 5\}$.*

Now we shall characterize cyclic semigroups whose superextensions have one-point minimal left ideals.

If $C_{r,m}$ is a finite cyclic semigroup of odd period m and C_m is the maximal subgroup of $C_{r,m}$, then the superextension $\lambda(C_{r,m})$ contains a right zero, in particular the maximal linked system

$$\mathcal{L} = \langle A \subset C_m : |A| > m/2\rangle$$

is a right zero of the semigroup $\lambda(C_{r,m})$. A maximal linked system $\mathcal{Z} \in \lambda(C_{r,m})$ is a right zero of the semigroup $\lambda(C_{r,m})$ if and only if the one-point set $\{\mathcal{Z}\}$ is a minimal left ideal of $\lambda(C_{r,m})$. Taking into account that all minimal left ideals are isomorphic and the union $K(\lambda(C_{r,m}))$ of all minimal left ideals in $\lambda(C_{r,m})$ coincides with the minimal ideal of $\lambda(C_{r,m})$ (see [11, Theorem 2.8]), Theorem 1 and Proposition 1.3 imply the following theorem.

Theorem 3. *A finite cyclic semigroup $C_{r,m}$ has odd period m if and only if all minimal left ideals of the semigroup $\lambda(C_{r,m})$ are singletons. In this case the minimal ideal $K(\lambda(C_{r,m}))$ of the semigroup $\lambda(C_{r,m})$ is the subsemigroup of right zeros of $\lambda(C_{r,m})$. The infinite cyclic semigroup has no one-point minimal left (right) ideals.*

2 COMMUTATIVITY OF SUPEREXTENSIONS OF CYCLIC SEMIGROUPS

Theorem 4. *A finite cyclic semigroup $C_{r,m} = \{a, a^2, \dots, a^r, \dots, a^{m+r-1} | a^{r+m} = a^r\}$ of order $m + r - 1$ has commutative superextension if and only if*

$$(r, m) \in \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 1), (2, 2), (2, 3), (2, 4), (3, 1), (3, 2), (4, 1)\}.$$

The superextension of the infinite cyclic semigroup is not commutative.

Proof. It was proved in the paper [1] that the superextension of a group G is commutative if and only if $|G| \leq 4$. Since for $m > 4$ the superextension $\lambda(C_{r,m})$ contains a noncommutative subsemigroup $\lambda(C_m)$, $\lambda(C_{r,m})$ is not commutative. So it is sufficient to consider only cyclic semigroups of period $m \leq 4$.

If index $r = 1$, then $C_{r,m}$ is a cyclic group of order m, and thus for $r = 1$ the semigroup $\lambda(C_{r,m})$ is commutative if and only if $m \leq 4$.

If $|C_{r,m}| \in \{1, 2\}$, then the superextension $\lambda(C_{r,m})$ is isomorphic to the semigroup $C_{r,m}$, and $\lambda(C_{r,m})$ is commutative. In the case $|C_{r,m}| = 3$ the superextension $\lambda(C_{r,m})$ contains only one maximal linked system, which is not a principal ultrafilter. Since all principal ultrafilters commute with maximal linked systems, the superextension $\lambda(C_{r,m})$ is commutative.

It follows that for

$$(r, m) \in \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 1), (2, 2), (3, 1)\}$$

the superextension $\lambda(C_{r,m})$ is commutative.

If $r = 2$, $m \in \{3, 4\}$, then the product xy of any two elements $x, y \in C_{r,m}$ is contained in the maximal subgroup C_m, and thus $xy = \varphi(xy) = \varphi(x)\varphi(y)$, where $\varphi : C_{r,m} \to C_m$ is the retraction $\varphi : s \to es$. Since superextensions of groups of order 3 and 4 are commutative,

$\mathcal{A} \circ \mathcal{B} = \lambda\varphi(\mathcal{A}) \circ \lambda\varphi(\mathcal{B}) = \lambda\varphi(\mathcal{B}) \circ \lambda\varphi(\mathcal{A}) = \mathcal{B} \circ \mathcal{A}$ for any $\mathcal{A}, \mathcal{B} \in \lambda(C_{r,m})$. Consequently, the semigroups $\lambda(C_{2,3})$ and $\lambda(C_{2,4})$ are commutative.

Let $r = 3$. The case $m = 1$ was considered before.

For the semigroup $C_{3,2} = \{a, a^2, a^3, a^4 | a^5 = a^3\}$ the semigroup $\lambda(C_{3,2})$ contains 12 elements:

$$\mathcal{U}_k = \langle\{a^k\}\rangle, \quad \Delta_k = \langle A \subset C_{3,2} : |A| = 2,\ a^k \notin A\rangle$$

and

$$\Box_k = \langle C_{3,2} \setminus \{a^k\}, A : A \subset C_{3,2},\ |A| = 2,\ a^k \in A\rangle, \text{ where } k \in \{1, 2, 3, 4\}.$$

The following table implies the commutativity of $\lambda(C_{3,2})$:

$\circ$	Δ_1	Δ_2	Δ_3	Δ_4	$\Box_1$	$\Box_2$	$\Box_3$	$\Box_4$
Δ_1	$\mathcal{U}_4$	$\mathcal{U}_3$	$\mathcal{U}_4$	$\mathcal{U}_3$	$\mathcal{U}_3$	$\mathcal{U}_4$	$\mathcal{U}_3$	$\mathcal{U}_4$
Δ_2	$\mathcal{U}_3$	Δ_1	$\mathcal{U}_3$	Δ_1	Δ_1	$\mathcal{U}_3$	Δ_1	$\mathcal{U}_3$
Δ_3	$\mathcal{U}_4$	$\mathcal{U}_3$	$\mathcal{U}_4$	$\mathcal{U}_3$	$\mathcal{U}_3$	$\mathcal{U}_4$	$\mathcal{U}_3$	$\mathcal{U}_4$
Δ_4	$\mathcal{U}_3$	Δ_1	$\mathcal{U}_3$	Δ_1	Δ_1	$\mathcal{U}_3$	Δ_1	$\mathcal{U}_3$
$\Box_1$	$\mathcal{U}_3$	Δ_1	$\mathcal{U}_3$	Δ_1	Δ_1	$\mathcal{U}_3$	Δ_1	$\mathcal{U}_3$
$\Box_2$	$\mathcal{U}_4$	$\mathcal{U}_3$	$\mathcal{U}_4$	$\mathcal{U}_3$	$\mathcal{U}_3$	$\mathcal{U}_4$	$\mathcal{U}_3$	$\mathcal{U}_4$
$\Box_3$	$\mathcal{U}_3$	Δ_1	$\mathcal{U}_3$	Δ_1	Δ_1	$\mathcal{U}_3$	Δ_1	$\mathcal{U}_3$
$\Box_4$	$\mathcal{U}_4$	$\mathcal{U}_3$	$\mathcal{U}_4$	$\mathcal{U}_3$	$\mathcal{U}_3$	$\mathcal{U}_4$	$\mathcal{U}_3$	$\mathcal{U}_4$

If $m \in \{3, 4\}$, then $C_{3,m} = \{a, a^2, \ldots, a^{m+2} | a^{m+3} = a^3\}$. Consider maximal linked systems $\mathcal{A} = \langle\{a, a^2\}, \{a, a^3\}, \{a^2, a^3\}\rangle$ and $\mathcal{B} = \langle\{a, a^2\}, \{a, a^{m+1}\}, \{a^2, a^{m+1}\}\rangle$. Observe that $\{a^2, a^3\} = a\{a, a^2\} \cup a^2\{a, a^{m+1}\} \in \mathcal{A} \circ \mathcal{B}$, but $\{a^2, a^3\} \notin \mathcal{B} \circ \mathcal{A}$. Therefore, $\mathcal{A} \circ \mathcal{B} \neq \mathcal{B} \circ \mathcal{A}$ and the semigroup $C_{3,m}$ is not commutative.

Let $r \geq 4$. First consider the case of the semigroup $C_{4,1} = \{a, a^2, a^3, a^4 | a^5 = a^4\}$. Each maximal linked system different from the principal ultrafilter $\langle\{a\}\rangle$ contains the set $\{a^2, a^3, a^4\}$.

Since $\{a^2, a^3, a^4\}\{a^2, a^3, a^4\} = \{a^4\}$, the product of such maximal linked systems is the principal ultrafilter $\langle\{a^4\}\rangle$. The fact that the principal ultrafilter $\langle\{a\}\rangle$ commutes with all maximal linked systems implies the commutativity of the semigroup $\lambda(C_{4,1})$.

Put $\mathcal{A} = \langle\{a, a^2\}, \{a, a^3\}, \{a^2, a^3\}\rangle$, $\mathcal{B} = \langle\{a, a^2\}, \{a, a^{m+r-2}\}, \{a^2, a^{m+r-2}\}\rangle$. We have that $\{a^3, a^4\} = a\{a^2, a^3\} \cup a^2\{a, a^2\} \in \mathcal{B} \circ \mathcal{A}$, but $\{a^3, a^4\} \notin \mathcal{A} \circ \mathcal{B}$, since the equality $a^{m+r+1} = a^4$ holds only if $r = 4$ and $m = 1$, which we considered before. Consequently, $\mathcal{A} \circ \mathcal{B} \neq \mathcal{B} \circ \mathcal{A}$ and a semigroup $\lambda(C_{r,m})$ for $(r, m) \neq (4, 1)$ is not commutative.

Let $\langle a\rangle = \{a, \dots, a^n, \dots\}$ be the infinite cyclic semigroup. Put $\mathcal{A} = \langle\{a, a^2\}, \{a, a^3\}, \{a^2, a^3\}\rangle$, $\mathcal{B} = \langle\{a, a^2\}, \{a, a^4\}, \{a^2, a^4\}\rangle$. Let us observe that $\{a^3, a^4\} = a\{a^2, a^3\} \cup a^2\{a, a^2\} \in \mathcal{B} \circ \mathcal{A}$, but $\{a^3, a^4\} \notin \mathcal{A} \circ \mathcal{B}$. Therefore, $\mathcal{A} \circ \mathcal{B} \neq \mathcal{B} \circ \mathcal{A}$ and the semigroup $\lambda(\langle a\rangle)$ is not commutative. $\square$

3 RIGHT (LEFT) CANCELABLE ELEMENTS

In this section we shall detect right (left) cancelable elements of superextensions of cyclic semigroups.

Proposition 3.1. *The superextension $\lambda(C_{r,m})$ has (left, right) cancelable elements if and only if index r of a cyclic semigroup $C_{r,m}$ is equal to 1.*

Proof. Let $r > 1$ and a be the generator of a semigroup $C_{r,m}$. Consider the map $\varphi : C_{r,m} \to C_m$, $\varphi : x \to ex$, where e is the neutral element of the cyclic group C_m. According to Lemma 1.3 this map is a retraction. Since $a^{r-1}x \in C_m = \{a^r, \dots a^{r+m-1}\}$ for any $x \in C_{r,m}$, $a^{r-1}x = \varphi(a^{r-1}x) = \varphi(a^{r-1})\varphi(x)$. On the other hand, since C_m is an ideal of $C_{r,m}$, $\varphi(a^{r-1})x \in C_m$ and $\varphi(a^{r-1})x = \varphi(\varphi(a^{r-1})x) = \varphi(\varphi(a^{r-1}))\varphi(x) = \varphi(a^{r-1})\varphi(x)$. Consequently, $\varphi(a^{r-1})x = a^{r-1}x$ for any $x \in C_{r,m}$.

Let $\mathcal{M}$ be a maximal linked system on a semigroup $C_{r,m}$. Then we obtain $\langle\{a^{r-1}\}\rangle \circ \mathcal{M} = \left\langle \bigcup_{a\in\{a^{r-1}\}} a * M_a : \quad \{M_a\}_{a\in L} \subset \mathcal{M}\right\rangle = \left\langle a^{r-1}M : M \in \mathcal{M}\right\rangle = \left\langle \varphi(a^{r-1})M : M \in \mathcal{M}\right\rangle = \langle\{\varphi(a^{r-1})\}\rangle \circ \mathcal{M}$ and $\mathcal{M} \circ \langle\{a^{r-1}\}\rangle = \left\langle \bigcup_{a\in M} a * \{a^{r-1}\} : \quad M \in \mathcal{M}\right\rangle = \left\langle Ma^{r-1} : M \in \mathcal{M}\right\rangle = \left\langle M\varphi(a^{r-1}) : M \in \mathcal{M}\right\rangle = \mathcal{M} \circ \langle\{\varphi(a^{r-1})\}\rangle$. Since $a^{r-1} \neq \varphi(a^{r-1})$, the maximal linked system $\mathcal{M}$ is neither left nor right cancelable.

If $r = 1$, then a cyclic semigroup $C_{1,m} = C_m$ is a group. Let e be the neutral element of the group C_m. Then $\langle\{e\}\rangle \circ \mathcal{M} = \mathcal{M} = \mathcal{M} \circ \langle\{e\}\rangle$ for any $\mathcal{M} \in \lambda(C_m)$, and equalities $\mathcal{X} \circ \langle\{e\}\rangle = \mathcal{Y} \circ \langle\{e\}\rangle$, $\langle\{e\}\rangle \circ \mathcal{X} = \langle\{e\}\rangle \circ \mathcal{Y}$ imply that $\mathcal{X} = \mathcal{Y}$. Consequently, the principal ultrafilter $\langle\{e\}\rangle$ is a cancelable element of the semigroup $\lambda(C_{1,m})$. $\square$

If G is a group, then the formula

$$\mathcal{L} \circ \mathcal{M} = \Big\langle \bigcup_{a\in L} a * M_a : L \in \mathcal{L},\ \ \{M_a\}_{a\in L} \subset \mathcal{M}\Big\rangle$$

implies that the product $\mathcal{L} \circ \mathcal{M}$ of any two maximal linked systems $\mathcal{L}$ and $\mathcal{M}$ is a principal ultrafilter if and only if both $\mathcal{L}$ and $\mathcal{M}$ are principal ultrafilters. Therefore, we deduce the following proposition.

Proposition 3.2. *For a group G the set $\lambda(G) \setminus \{\langle\{g\}\rangle : g \in G\}$ is an ideal in $\lambda(G)$.*

Lemma 3.1. *A semigroup S is a left (right) cancellative semigroup if and only if all principal ultrafilters are left (right) cancelable elements in the superextension $\lambda(S)$.*

Proof. If an element $a \in S$ is not left (right) cancelable in the semigroup S, then it is clear that the principal ultrafilter generated by the element a is not cancelable in $\lambda(S)$.

Let S be a left (right) cancellative semigroup, $a \in S$ and $\mathcal{X}, \mathcal{Y} \in \lambda(S)$, $\mathcal{X} \neq \mathcal{Y}$, then $X \cap Y = \varnothing$ for some $X \in \mathcal{X}, Y \in \mathcal{Y}$. Since each element of S is left (right) cancelable, then $aX \cap aY = \varnothing$ ($Xa \cap Ya = \varnothing$), and thus $\langle\{a\}\rangle \circ \mathcal{X} \neq \langle\{a\}\rangle \circ \mathcal{Y}$ ($\mathcal{X} \circ \langle\{a\}\rangle \neq \mathcal{Y} \circ \langle\{a\}\rangle$). Consequently, the left $l_{\langle\{a\}\rangle}$ (right $r_{\langle\{a\}\rangle}$) shift is injective and the principal ultrafilter $\langle\{a\}\rangle$ is left (right) cancelable. $\square$

Proposition 3.3. *An element $\mathcal{M} \in \lambda(C_{1,m})$ is left (right) cancelable if and only if $\mathcal{M}$ is a principal ultrafilter.*

Proof. Since in any group, in particular in the cyclic group $C_{1,m}$, all elements are cancelable, according to Lemma 3.1 all principal ultrafilters are right cancelable in the superextension $\lambda(C_{1,m})$.

Assume that some maximal linked system $\mathcal{M} \in \lambda(C_{1,m}) \setminus \{\langle\{g\}\rangle : g \in C_{1,m}\}$ is left cancelable. This means that the left shift $l_{\mathcal{M}} : \lambda(C_{1,m}) \to \lambda(C_{1,m})$, $l_{\mathcal{M}} : \mathcal{A} \mapsto \mathcal{M} \circ \mathcal{A}$, is injective. According to Proposition 3.2, the set $\lambda(C_{1,m}) \setminus \{\langle\{g\}\rangle : g \in C_{1,m}\}$ is an ideal in $\lambda(C_{1,m})$. Consequently, $l_{\mathcal{M}}(\lambda(C_{1,m})) = \mathcal{M} \circ \lambda(C_{1,m}) \subset \lambda(C_{1,m}) \setminus \{\langle\{g\}\rangle : g \in C_{1,m}\}$. Since $\lambda(C_{1,m})$ is finite, $l_{\mathcal{M}}$ cannot be injective.

For the right cancelable elements the proof is analogous. $\square$

Since the infinite cyclic semigroup is a cancellative semigroup, then Lemma 3.1 implies the following proposition.

Proposition 3.4. *All principal ultrafilters are cancelable elements in the superextension of the infinite cyclic semigroup.*

Proposition 3.5. *Let S be the infinite cyclic semigroup and $\mathcal{L} \in \lambda(S)$. A maximal linked system $\mathcal{L}$ is right cancelable in $\lambda(S)$ provided for every $s \in S$ there is a set $L_s \in \mathcal{L}$ such that the family $\{s * L_s : s \in S\}$ is disjoint.*

Proof. Assume that $\{L_s\}_{s\in S} \subset \mathcal{L}$ is a family such that $\{s * L_s : s \in S\}$ is disjoint. To prove that $\mathcal{L}$ is right cancelable, take two maximal linked systems $\mathcal{A}, \mathcal{B} \in \lambda(S)$ with $\mathcal{A} \circ \mathcal{L} = \mathcal{B} \circ \mathcal{L}$. It is sufficient to show that $\mathcal{A} \subset \mathcal{B}$. Take any set $A \in \mathcal{A}$ and observe that the set $\bigcup_{a\in A} a * L_a$ belongs to $\mathcal{A} \circ \mathcal{L} = \mathcal{B} \circ \mathcal{L}$. Consequently, there is a set $B \in \mathcal{B}$ and a family of sets $\{M_b\}_{b\in B} \subset \mathcal{L}$ such that

$$\bigcup_{b\in B} b * M_b \subset \bigcup_{a\in A} a * L_a.$$

It follows from $L_b \in \mathcal{L}$ that $M_b \cap L_b$ is not empty for every $b \in B$.

Since the sets $a * L_a$ i $b * L_b$ are disjoint for different $a, b \in S$, the inclusion

$$\bigcup_{b\in B} b * (M_b \cap L_b) \subset \bigcup_{b\in B} b * M_b \subset \bigcup_{a\in A} a * L_a$$

implies $B \subset A$ and hence $A \in \mathcal{B}$. $\square$

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У статті вивчаються праві і ліві нулі, одноточкові ліві ідеали, мінімальний ідеал, скоротні зліва і скоротні справа елементи суперрозширення $\lambda(S)$ циклічної напівгрупи S, а також характеризуються циклічні напівгрупи, суперрозширення яких є комутативними.

Ключові слова і фрази: циклічна напівгрупа, максимальна зчеплена система, суперрозширення.

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В работе изучаются правые и левые нули, одноточечные левые идеалы, минимальный идеал, сократимые слева и сократимые справа элементы суперрасширения $\lambda(S)$ циклической полугруппы S, а также характеризуются циклические полугруппы, суперрасширения которых коммутативны.

Ключевые слова и фразы: циклическая полугруппа, максимальная сцепленная система, суперрасширение.