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EXTENDING BINARY OPERATIONS TO FUNTOR-SPACES

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Given a continuous monadic functor $T : \mathbf{Comp} \to \mathbf{Comp}$ in the category of compacta and a discrete topological semigroup X we extend the semigroup operation $\varphi : X \times X \to X$ to a right-topological semigroup operation $\Phi : T\beta X \times T\beta X \to T\beta X$, whose topological center Λ_Φ contains the dense subsemigroup $T_f X$ consisting of elements $a \in T\beta X$ that have finite support in X.

Introduction

One of powerful tools in the modern Combinatorics of Numbers is the method of ultrafilters based on the fact that each binary operation $\varphi : X \times X \to X$ defined on a discrete topological space X can be extended to a right-topological operation $\Phi : \beta X \times \beta X \to \beta X$ on the Stone-Čech compactification βX of X, see [13], [16]. The extension of φ is constructed in two step. First, for every $x \in X$ extend the left shift $\varphi_x : X \to X$, $\varphi_x : y \mapsto \varphi(x, y)$, to a continuous map $\overline{\varphi}_x : \beta X \to \beta X$. Next, for every $b \in \beta X$ extend the right shift $\bar{\varphi}^b : X \to \beta X$, $\bar{\varphi}^b : x \mapsto \bar{\varphi}_x(b)$, to a continuous map $\Phi^b : \beta X \to \beta X$ and put $\Phi(a, b) = \Phi^b(a)$ for every $a \in \beta X$. The Stone-Čech extension βX is the space of ultrafilters on X. In [11] it was observed that the binary operation φ extends not only to βX but also to the superextension λX of X and to the space GX of all inclusion hyperspaces on X. If X is a semigroup, then GX is a compact Hausdorff right-topological semigroup containing λX and βX as closed subsemigroups.

In this note we show that an (associative) binary operation $\varphi : X \times X \to X$ on a discrete topological space X can be extended to an (associative) right-topological operation $\Phi : T\beta X \times T\beta X \to T\beta X$ for any monadic functor T in the category $\mathbf{Comp}$ of compact Hausdorff spaces. So, for the functors β, λ or G we get the extensions of the operation φ discussed above.

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1 Monadic functors and their algebras

Let us recall [14, VI], [17, §1.2] that a functor $T : \mathcal{C} \to \mathcal{C}$ in a category $\mathcal{C}$ is called *monadic* if there are natural transformations $\eta : \mathrm{Id} \to T$ and $\mu : T^2 \to T$ making the following diagrams

$$\begin{tikzcd} T \arrow[r, "\eta T"] \arrow[d, "T\eta"'] \arrow[dr, "1_T"] & T^2 \arrow[d, "\mu"] \\ T^2 \arrow[r, "\mu"'] & T \end{tikzcd} \qquad \begin{tikzcd} T^3 \arrow[r, "\mu T"] \arrow[d, "T\mu"'] & T^2 \arrow[d, "\mu"] \\ T^2 \arrow[r, "\mu"'] & T \end{tikzcd}$$

commutative. In this case the triple $\mathbb{T} = (T, \eta, \mu)$ is called a *monad*, the natural transformations $\eta : \mathrm{Id} \to T$ and $\mu : T^2 \to T$ are called the *unit* and *multiplication* of the monad $\mathbb{T}$, and the functor T is the *functorial part* of the monad $\mathbb{T}$.

A pair (X, ξ) consisting of an object X and a morphism $\xi : TX \to X$ of the category $\mathcal{C}$ is called a $\mathbb{T}$-*algebra*, if $\xi \circ \eta_X = \mathrm{id}_X$ and the square

$$\begin{tikzcd} T^2X \arrow[r, "T\xi"] \arrow[d, "\mu"'] & TX \arrow[d, "\xi"] \\ TX \arrow[r, "\xi"'] & X \end{tikzcd}$$

is commutative. For every object X of the category $\mathcal{C}$ the pair (TX, μ) is a $\mathbb{T}$-algebra called the *free* $\mathbb{T}$-*algebra over* X.

For two $\mathbb{T}$-algebras (X, ξ_X) and (Y, ξ_Y) a morphism $h : X \to Y$ is called a *morphism of* $\mathbb{T}$-*algebras*, if the following diagram

$$\begin{tikzcd} TX \arrow[r, "Th"] \arrow[d, "\xi_X"'] & TY \arrow[d, "\xi_Y"] \\ X \arrow[r, "h"'] & Y \end{tikzcd}$$

is commutative. The naturality of the multiplication $\mu : T^2 \to T$ of the monad $\mathbb{T}$ implies that for any morphism $f : X \to Y$ in $\mathcal{C}$ the morphism $Tf : TX \to TY$ is a morphism of free $\mathbb{T}$-algebras.

Each morphism $h : TX \to Y$ from the free $\mathbb{T}$-algebra into a $\mathbb{T}$-algebra (Y, ξ) is uniquely determined by the composition $h \circ \eta$.

Lemma 1.1. *If $h : TX \to Y$ is a morphism of a free $\mathbb{T}$-algebra TX into a $\mathbb{T}$-algebra (Y, ξ), then $h = \mu \circ T(h \circ \eta) = \mu \circ Th \circ T\eta$.*

Proof. Consider the commutative diagram

$$\begin{tikzcd} X \arrow[r, "\eta"] \arrow[d, "\eta"'] & TX \arrow[r, "h"] \arrow[d, shift left, "\eta_T"] & Y \\ TX \arrow[r, shift left, "T\eta"] \arrow[rr, bend right, "T(h\circ\eta)"'] & T^2X \arrow[l, shift left, "\mu"] \arrow[u, shift left, "\mu"] \arrow[r, "Th"] & TY \arrow[u, "\xi"'] \end{tikzcd}$$

and observe that

$$h = h \circ \mu \circ \eta_T = \xi \circ Th \circ \eta_T = \xi \circ Th \circ T\eta \circ \mu \circ \eta_T = \xi \circ T(h \circ \eta).$$

$\square$

By a *topological category* we shall understand a subcategory of the category **Top** of topological spaces and their continuous maps such that:

- for any objects X, Y of the category $\mathcal{C}$ each constant map $f : X \to Y$ is a morphism of $\mathcal{C}$;
- for any objects X, Y of the category $\mathcal{C}$ the product $X \times Y$ is an object of $\mathcal{C}$, and for any object Z of $\mathcal{C}$ and morphisms $f_X : Z \to X$, $f_Y : Z \to Y$ the map $(f_X, f_Y) : Z \to X \times Y$ is a morphism of the category $\mathcal{C}$.

A discrete topological space X is called *discrete in* $\mathcal{C}$, if X is an object of $\mathcal{C}$ and each function $f : X \to Y$ into an object Y of the category $\mathcal{C}$ is a morphism of $\mathcal{C}$. It is clear that any bijection $f : X \to Y$ between discrete objects of the category $\mathcal{C}$ is an isomorphism in $\mathcal{C}$.

From now on we shall assume that $(\mathbb{T}, \eta, \mu)$ is a monad in a topological category $\mathcal{C}$ such that for any discrete objects X, Y in $\mathcal{C}$ the product $X \times Y$ is discrete in $\mathcal{C}$.

2 BINARY OPERATIONS AND THEIR $\mathbb{T}$-EXTENSIONS

By a *binary operation in the category* $\mathcal{C}$ we understand any function $\varphi : X \times Y \to Z$, where X, Y, Z are objects of the category $\mathcal{C}$. For any $a \in X$ and $b \in Y$ the functions

$$\varphi_a : Y \to Z, \;\; \varphi_a : y \mapsto \varphi(a, y)$$

and

$$\varphi^b : X \to Z, \;\; \varphi^b : x \mapsto \varphi(x, b),$$

are called the *left* and *right shifts*, respectively.

A binary operation $\varphi : X \times Y \to Z$ is called *right-topological*, if for every $y \in Y$ the right shift $\varphi^y : X \to Z$, $\varphi^y : x \mapsto \varphi(x, y)$, is continuous. The *topological center* of a right-topological binary operation $\varphi : X \times Y \to Z$ is the set Λ_φ of all elements $x \in X$ such that the left shift $\varphi_x : Y \to Z$ is continuous.

Definition 2.1. *Let $\varphi : X \times Y \to Z$ be a binary operation in the category $\mathcal{C}$. A binary operation $\Phi : TX \times TY \to TZ$ is defined to be a* $\mathbb{T}$-extension *of φ if:*

1. *$\Phi(\eta_X(x), \eta_Y(y)) = \eta_Z(\varphi(x, y))$ for any $x \in X$ and $y \in Y$;*
2. *for every $b \in TY$ the right shift $\Phi^b : TX \to TZ$, $\Phi^b : x \mapsto \Phi(x, b)$, is a morphism of the free $\mathbb{T}$-algebras TY, TZ;*
3. *for every $x \in X$ the left shift $\Phi_{\eta(x)} : TY \to TZ$, $\Phi_{\eta(x)} : y \mapsto \Phi(\eta(x), y)$, is a morphism of the free $\mathbb{T}$-algebras TX, TZ.*

This definition implies that for any binary operation $\varphi : X \times Y \to Z$ its $\mathbb{T}$-extension $\Phi : TX \times TY \to TZ$ is a right-topological binary operation, whose topological center Λ_Φ contains the set $\eta(X) \subset TX$.

Theorem 1. *Let $\varphi : X \times Y \to Z$ be a binary operation in the category $\mathcal{C}$.*

1. *The binary operation φ has at most one $\mathbb{T}$-extension $\Phi : TX \times TY \to TZ$.*
2. *If X, Y are discrete in $\mathcal{C}$, then φ has a unique $\mathbb{T}$-extension $\Phi : TX \times TY \to TZ$.*

Proof. 1. Let $\Phi, \Psi : TX \times TY \to TZ$ be two $\mathbb{T}$-extensions of the operation φ. By the condition (3) of Definition 2.1, for every $x \in X$ and $a = \eta_X(x) \in TX$ the left shifts $\Phi_a, \Psi_a : TY \to TZ$ are morphisms of the free $\mathbb{T}$-algebras.

By the condition (1) of Definition 2.1,

$$\Phi_a \circ \eta_Y = \eta_Z \circ \varphi_x = \Psi_a \circ \eta_Y.$$

Then Lemma 1.1 implies that

$$\Phi_a = \mu \circ T(\Phi_a \circ \eta_X) = \mu \circ T(\eta_Z \circ \varphi_x) = \mu \circ T(\Psi_a \circ \eta_X) = \Psi_a.$$

The equality $\Phi = \Psi$ will follow as soon as we check that $\Phi^b = \Psi^b$ for every $b \in TY$. Since $\Phi^b, \Psi^b : TX \to TZ$ are morphisms of the free $\mathbb{T}$-algebras TX and TZ, the equality $\Phi^b = \Psi^b$ follows from the equality

$$\Phi^b \circ \eta(x) = \Phi_{\eta(x)}(b) = \Psi_{\eta(x)}(b) = \Psi^b \circ \eta(x), \quad x \in X,$$

according to Lemma 1.1.

2. Now assuming that the spaces X, Y are discrete in $\mathcal{C}$, we show that the binary operation $\varphi : X \times Y \to Z$ has a $\mathbb{T}$-extension. For every $x \in X$ consider the left shift $\varphi_x : Y \to Z$. Since Y is discrete in $\mathcal{C}$, the function φ_x is a morphism of the category $\mathcal{C}$. Applying the functor T to this morphism, we get a morphism $T\varphi_x : TY \to TZ$. Now for every $b \in TY$ consider the function $\varphi^b : X \to TZ$, $\varphi^b : x \mapsto T\varphi_x(b)$. Since the object X is discrete, the function φ^b is a morphism of the category $\mathcal{C}$. Applying to this morphism the functor T, we get a morphism $T\varphi^b : TX \to T^2Z$. Composing this morphism with the multiplication $\mu : T^2Z \to TZ$ of the monad $\mathbb{T}$, we get the function $\Phi^b = \mu \circ T\varphi^b : TZ \to TZ$. Define a binary operation $\Phi : TX \times TY \to TZ$, letting $\Phi(a, b) = \Phi^b(a)$ for $a \in TX$.

Claim 2.1. $\Phi(\eta(x), b) = T\varphi_x(b)$ *for every $x \in X$ and $b \in TY$.*

Proof. The commutativity of the diagram

$$\begin{tikzcd}
X \arrow[r, "\varphi^b"] \arrow[d, "\eta"'] & TZ \arrow[d, "\eta"'] \\
TX \arrow[r, "T\varphi^b"'] \arrow[ur, "\Phi^b"] & T^2Z \arrow[u, bend right, "\mu"']
\end{tikzcd}$$

implies the desired equality

$$\Phi(\eta(x), b) = \mu \circ T\varphi^b(\eta(x)) = \varphi^b(x) = T\varphi_x(b).$$

□

Now we shall prove that Φ is a $\mathbb{T}$-extension of φ.

i) For every $x \in X$ and $y \in Y$ we need to prove the equality

$$\Phi(\eta_X(x), \eta_Y(y)) = \eta_Z \circ \varphi(x, y).$$

By Claim 2.1,

$$\Phi(\eta_X(x), \eta_Y(y)) = T\varphi_x \circ \eta_Y(y) = \eta_Z \circ \varphi_x(y) = \eta_Z \circ \varphi(x, y).$$

The latter equality follows from the naturality of the transformation $\eta : \mathrm{Id} \to T$.

ii) The definition of Φ implies that for every $b \in TY$ the right shift $\Phi^b = \mu_Z \circ T\varphi^b$ is a morphism of free $\mathbb{T}$-algebras, being the composition of two morphisms $T\varphi^b : TX \to T^2Z$ and $\mu_Z : T^2Z \to TZ$ of free $\mathbb{T}$-algebras.

iii) Claim 2.1 guarantees that for every $x \in X$ the left shift $\Phi_{\eta(x)} = T\varphi_x : TY \to TZ$ is a morphism of the free $\mathbb{T}$-algebras. $\square$

Proposition 2.1. *Let $\varphi : X \times Y \to Z$, $\psi : X' \times Y' \to Z'$ be two binary operations in $\mathcal{C}$, $\Phi : TX \times TY \to TZ$, $\Psi : TX' \times TY' \to TZ'$ be their $\mathbb{T}$-extensions, and $h_X : X \to X'$, $h_Y : Y \to Y'$, $h_Z : Z \to Z'$ be morphisms in $\mathcal{C}$. If $\psi(h_X \times h_Y) = h_Z \circ \varphi$, then $T\Psi(Th_X \times Th_Y) = Th_Z \circ \Phi$.*

Proof. Observe that for any $x \in X$ and $x' = h_X(x)$ the commutativity of the diagrams

$$\begin{tikzcd} Y \arrow[r, "\varphi_x"] \arrow[d, "h_Y"'] & Z \arrow[d, "h_Z"] \\ Y' \arrow[r, "\psi_{x'}"'] & Z' \end{tikzcd} \qquad \begin{tikzcd} TY \arrow[r, "T\varphi_x"] \arrow[d, "Th_Y"'] & TZ \arrow[d, "Th_Z"] \\ TY' \arrow[r, "T\psi_{x'}"'] & TZ' \end{tikzcd}$$

implies that $Th_Z \circ T\varphi_x(b) = T\psi_{x'}(b')$ for every $b \in TY$ and $b' = Th_Y(b) \in TY'$.

It follows from Lemma 2.1 that $\Phi_{\eta(x)} = T\varphi_x : TY \to TZ$ and $\Psi_{\eta(x')} = T\psi_{x'} : TY' \to TZ'$. Consequently,

$$Th_Z \circ \Phi^b(\eta(x)) = Th_Z \circ \Phi_{\eta(x)}(b) = Th_Z \circ T\varphi_x(b) = T\psi_{x'}(b') = \Psi_{\eta(x')}(b') = \Psi^{b'}(\eta(x'))$$

and hence

$$Th_Z \circ \Phi^b \circ \eta = \Psi^{b'} \circ \eta \circ h_X.$$

Applying the functor T to this equality, we get

$$T^2h_Z \circ T(\Phi^b \circ \eta) = T(\Psi^{b'} \circ \eta) \circ Th_X.$$

Since $\Phi^b : TX \to TZ$ and $\Psi^{b'} : TX' \to TZ'$ are homomorphisms of the free $\mathbb{T}$-algebras, we can apply Lemma 1.1 and conclude that $\Phi^b = \mu \circ T(\Phi^b \circ \eta)$, and hence

$$Th_Z \circ \Phi^b = Th_Z \circ \mu_Z \circ T(\Phi^b \circ \eta) = \mu_{Z'} \circ T^2h_Z \circ T(\Phi^b \circ \eta) = \mu_{Z'} \circ T(\Psi^{b'} \circ \eta) \circ Th_X = \Psi^{b'} \circ Th_X.$$

Then for every $a \in TX$ we get

$$Th_Z \circ \Phi(a, b) = Th_Z \circ \Phi^b(a) = \Psi^{b'} \circ Th_X(a) = \Psi(Th_X(a), Th_Y(b)).$$

$\square$

3 Binary operations and tensor products

In this section we shall discuss the relation of $\mathbb{T}$-extensions to tensor products. The tensor product is a function $\otimes : TX \times TY \to T(X \times Y)$ defined for any objects $X, Y \in \mathcal{C}$ such that X is discrete in $\mathcal{C}$.

For every $x \in X$ consider the embedding $i_x : Y \to X \times Y$, $i_x : y \mapsto (x, y)$. The embedding i_x is a morphism of the category $\mathcal{C}$, because the constant map $c_x : Y \to \{x\} \subset X$ and the identity map $\mathrm{id} : Y \to Y$ are morphisms of the category and $\mathcal{C}$ contains products of its objects. Applying the functor T to the morphism i_x, we get a morphism $Ti_x : TY \to T(X \times Y)$ of the category $\mathcal{C}$. Next, for every $b \in TY$ consider the function $Ti^b : X \to T(X \times Y)$, $Ti^b : x \mapsto Ti_x(b)$. Since X is discrete in $\mathcal{C}$, the function Ti^b is a morphism of the category $\mathcal{C}$. Applying the functor T to this morphism, we get a morphism $TTi^b : TX \to T^2(X \times Y)$. Composing this morphism with the multiplication $\mu : T^2(X \times Y) \to T(X \times Y)$ of the monad $\mathbb{T}$, we get the morphism $\otimes^b = \mu \circ TTi^b : TX \to T(X \times Y)$. Finally, define the tensor product $\otimes : TX \times TY \to T(X \times Y)$, letting $a \otimes b = \otimes^b(a)$ for $a \in TX$.

The following proposition describes some basic properties of the tensor product. For monadic functors in the category **Comp** of compact Hausdorff spaces those properties were established in [17, 3.4.2].

Proposition 3.1. *1. The diagram* $\begin{tikzcd} X \times Y \arrow[r, "\eta\times\eta"'] \arrow[rr, bend left, "\eta"] & TX \times TY \arrow[r, "\otimes"'] & T(X \times Y) \end{tikzcd}$ *is commutative for any discrete object X and any object Y of $\mathcal{C}$;*

2. *the tensor product is natural in the sense that for any morphisms $h_X : X \to X'$, $h_Y : Y \to Y'$ of $\mathcal{C}$ with discrete X, Y the following diagram*

$$\begin{tikzcd} TX \times TY \arrow[r, "\otimes"] \arrow[d, "Th_X \times Th_Y"'] & T(X \times Y) \arrow[d, "T(h_X \times h_Y)"] \\ TX' \times TY' \arrow[r, "\otimes"] & T(X' \times Y') \end{tikzcd}$$

is commutative;

3. *the tensor product is associative in the sense that for any discrete objects X, Y, Z of $\mathcal{C}$ the diagram*

$$\begin{tikzcd} TX \times TY \times TZ \arrow[r, "\otimes\times\mathrm{id}"] \arrow[d, "\mathrm{id}\times\otimes"'] & T(X \times Y) \times TZ \arrow[d, "\otimes"] \\ TX \times T(Y \times Z) \arrow[r, "\otimes"'] & T(X \times Y \times Z) \end{tikzcd}$$

is commutative, which means that $(a \otimes b) \otimes c = a \otimes (b \otimes c)$ for any $a \in TX$, $b \in TY$, $c \in TZ$.

Proof. 1. Fix any $y \in Y$ and consider the element $b = \eta_Y(y) \in TY$. The definition of the right shift $\otimes^b$ implies that the following diagram is commutative:

$$\begin{tikzcd} X \arrow[r, "Ti^b"] \arrow[d, "\eta"'] & T(X\times Y) \\ TX \arrow[ur, "\otimes^b"] \arrow[r, "TTi^b"'] & T^2(X\times Y) \arrow[u, "\mu"'] \end{tikzcd}$$

Consequently, for every $x \in X$ we get

$$\eta(x) \otimes \eta(y) = \otimes^b \circ \eta(x) = Ti^b \circ \eta(x) = Ti_x(\eta(y)) = \eta(i_x(y)) = \eta(x,y).$$

The latter equality follows from the diagram

$$\begin{tikzcd} Y \arrow[r, "i_x"] \arrow[d, "\eta"'] & X\times Y \arrow[d, "\eta"] \\ TY \arrow[r, "Ti_x"'] & T(X\times Y) \end{tikzcd},$$

whose commutativity follows from the naturality of the transformation $\eta : \mathrm{Id} \to T$.

2. Let $h_X : X \to X'$ and $h_Y : Y \to Y'$ be any functions between discrete objects of the category $\mathcal{C}$. Let $Z = X \times Y$, $Z' = X' \times Y'$ and $h_Z = h_X \times h_Y : Z \to Z'$. Given any point $b \in TY$, consider the element $b' = Th_Y(b) \in TY'$. The statement (2) will follow as soon as we check that $Th_Z \circ \otimes^b = \otimes^{b'} \circ Th_X$. By Lemma 1.1, this equality will follow as soon as we check that $Th_Z \circ \otimes^b \circ \eta_X = \otimes^{b'} \circ Th_X \circ \eta_X = \otimes^{b'} \circ \eta_{X'} \circ h_X$. The last equality follows from the naturality of the transformation $\eta : \mathrm{Id} \to T$. As we know from the proof of the preceding item, $\otimes^{b'} \circ \eta_{X'}(x') = Ti_{x'}(b')$ for any $x' \in X'$. For every $x \in X$ and $x' = h_X(x)$ we can apply the functor T to the commutative diagram

$$\begin{tikzcd} Y \arrow[r, "i_x"] \arrow[d, "h_Y"'] & Z \arrow[d, "h_Z"] \\ Y' \arrow[r, "i_{x'}"'] & Z' \end{tikzcd}$$

and obtain the equality $Th_Z \circ Ti_x = Ti_{x'} \circ Th_Y$, which implies the desired equality:

$$\otimes^{b'} \circ \eta_{X'} \circ h_X(x) = \otimes^{b'} \circ \eta_{X'}(x') = Ti_{x'}(b') = Th_Z \circ Ti_x(b) = Th_Z \circ \otimes^b \circ \eta(x).$$

3. The proof of the associativity of the tensor product can be obtained by literal rewriting the proof of Proposition 3.4.2(4) of [17]. □

Theorem 2. *Let $\varphi : X\times Y \to Z$ be a binary operation in the category $\mathcal{C}$ and $\Phi : TX\times TY \to TZ$ be its $\mathbb{T}$-extension. If X is a discrete object in $\mathcal{C}$, then $\Phi(a,b) = T\varphi(a\otimes b)$ for any elements $a \in TX$ and $b \in TY$.*

Proof. Our assumptions on the category $\mathcal{C}$ guarantee that the product $X \times Y$ is a discrete object of $\mathcal{C}$ and hence $\varphi : X \times Y \to Z$ is a morphism of the category $\mathcal{C}$. So, it is legal to consider the morphism $T\varphi : T(X \times Y) \to TZ$. We claim that the binary operation

$$\Psi : TX \times TY \to TZ, \quad \Psi(a,b) = T\varphi(a \otimes b),$$

is a $\mathbb{T}$-extension of φ.

1. The first item of Definition 2.1 follows Proposition 3.1(1) and the naturality of the transformation $\eta : \mathrm{Id} \to T$:

$$\Psi(\eta_X(x), \eta_Y(y)) = T\varphi(\eta_X(x) \otimes \eta_Y(y)) = T\varphi \circ \eta_{X\times Y}(x, y) = \eta_Z \circ \varphi(x, y).$$

2. For every $b \in TY$ the morphism

$$\Psi^b = T\varphi \circ \otimes^b = T\varphi \circ \mu \circ TTi^b$$

is a morphism of the free $\mathbb{T}$-algebras TX and TZ.

3. For every $x \in X$ we see that

$$\Psi_{\eta(x)}(b) = T\varphi(\otimes^b(\eta(x))) = T\varphi \circ \mu \circ TTi^b \circ \eta(x) = T\varphi \circ \mu \circ \eta \circ Ti^b(x) = T\varphi \circ Ti^b(x)$$

is a morphism of the free $\mathbb{T}$-algebras TY and TZ.

Thus Ψ is a $\mathbb{T}$-extension of the binary operation φ. By the Uniqueness Theorem 1(1), Ψ coincides with Φ and hence $\Phi(a, b) = \Psi(a, b) = T\varphi(a \otimes b)$. □

4 The topological center of $\mathbb{T}$-extended operation

Definition 2.1 guarantees that for a binary operation $\varphi : X \times Y \to Z$ in $\mathcal{C}$ any $\mathbb{T}$-extension $\Phi : TX \times TY \to TZ$ of φ is a right-topological operation, whose topological center Λ_φ contains the subset $\eta_X(X)$. In this section we shall find conditions on the functor T and the space X guaranteeing that the topological center Λ_Φ is dense in TX.

We shall say that the functor T is *continuous*, if for each compact Hausdorff space K, that belongs to the category $\mathcal{C}$, and any object Z of $\mathcal{C}$ the map $T : \mathrm{Mor}(K, Z) \to \mathrm{Mor}(TK, TZ)$, $T : f \mapsto Tf$, is continuous with respect to the compact-open topology on the spaces of morphisms (which are continuous maps).

Theorem 3. *Let $\varphi : X \times Y \to Z$ be a binary operation in $\mathcal{C}$ and $\Phi : X \times Y \to Z$ be its $\mathbb{T}$-extension. If the object X is finite and discrete in $\mathcal{C}$, TX is locally compact and Hausdorff, and the functor T is continuous, then the operation Φ is continuous.*

Proof. Since the space X is discrete, the condition (2) of Definition 2.1 implies that the map $\Phi_\eta : X \times TY \to TZ$, $\Phi_\eta : (x, b) \mapsto \Phi(\eta(x), b)$, is continuous. Since X is finite, the induced map

$$\Phi_\eta^{(\cdot)} : TY \to \mathrm{Mor}(X, TZ), \ \ \Phi_\eta^{(\cdot)} : b \mapsto \Phi_\eta^b,$$

where $\Phi_\eta^b : x \mapsto \Phi(\eta(x), b)$, is continuous. By the continuity of the functor T, the map $T : \mathrm{Mor}(X, TZ) \to \mathrm{Mor}(TX, T^2Z)$, $T : f \mapsto Tf$, is continuous and so is the composition $T \circ \Phi_\eta^{(\cdot)} : TY \to \mathrm{Mor}(TX, T^2Z)$. Since TX is locally compact and Hausdorff, we can apply [9, 3.4.8] and conclude that the map

$$T\Phi_\eta^{(\cdot)} : TX \times TY \to T^2Z, \ \ T\Phi_\eta^{(\cdot)} : (a, b) \mapsto T\Phi_\eta^b(a),$$

is continuous and so is the composition $\Psi = \mu \circ T\Phi^{(\cdot)}_{\eta} : TX \times TY \to TZ$. Using the Uniqueness Theorem 1(1), we can prove that $\Psi = \Phi$ and hence the binary operation Φ is continuous. $\square$

Let X be an object of the category $\mathcal{C}$. We say that an element $a \in FX$ has *discrete (finite) support*, if there is a morphism $f : D \to X$ from a discrete (and finite) object D of the category $\mathcal{C}$ such that $a \in Ff(FD)$. By T_dX (resp. T_fX) we denote the set of all elements $a \in TX$ that have discrete (finite) support. It is clear that $T_fX \subset T_dX \subset TX$.

Theorem 4. *Let $\varphi : X \times Y \to Z$ be a binary operation and $\Phi : TX \times TY \to TZ$ be a $\mathbb{T}$-extension of φ. If the functor T is continuous, and for every finite discrete object D of $\mathcal{C}$ the space TD is locally compact and Hausdorff, then the topological center Λ_Φ of the binary operation Φ contains the subspace T_fX of TX. If T_fX is dense in TX, then the topological center Λ_Φ of Φ is dense in TX.*

Proof. We need to prove that for every $a \in T_fX$ the left shift $\Phi_a : TY \to TZ$, $\Phi_a : b \mapsto \Phi(a, b)$, is continuous. Since $a \in T_fX$, there is a finite discrete object D of the category $\mathcal{C}$ and a morphism $f : D \to X$ such that $a \in Ff(FD)$. Fix an element $d \in FD$ such that $a = Ff(d)$.

Consider the binary operations

$$\psi : D \times Y \to Z,\ \psi : (x, y) \mapsto \varphi(f(x), y),$$

and

$$\Psi : TD \times TY \to TZ,\ \Psi : (a, b) \mapsto \Phi(Ff(a), b).$$

It can be shown that Ψ is a $\mathbb{T}$-extension of ψ.

By Theorem 3, the binary operation Ψ is continuous. Consequently, the left shift $\Psi_d : TY \to TZ$, $\Psi_d : b \mapsto \Psi(d, b)$, is continuous. Since $\Psi_d = \Phi_a$, the left shift Φ_a is continuous too and hence $a \in \Lambda_\Phi$. $\square$

5 The associativity of $\mathbb{T}$-extensions

In this section we investigate the associativity of the $\mathbb{T}$-extensions. We recall that a binary operation $\varphi : X \times X \to X$ is *associative*, if $\varphi(\varphi(x, y), z) = \varphi(x, \varphi(y, z))$ for any $x, y, z \in X$. In this case we say that X is a *semigroup*.

A subset A of a set X endowed with a binary operation $\varphi : X \times X \to X$ is called a *subsemigroup* of X, if $\varphi(A \times A) \subset A$ and $\varphi(\varphi(x, y), z) = \varphi(x, \varphi(y, z))$ for all $x, y, z \in A$.

Lemma 5.1. *Let $\varphi : X \times X \to X$ be an associative operation in $\mathcal{C}$ and $\Phi : TX \times TX \to TX$ be its $\mathbb{T}$-extension.*

1. *for any morphisms $f_A : A \to X$, $f_B : B \to X$ from discrete objects A, B in $\mathcal{C}$, the map $\varphi_{AB} = \varphi(f_A \times f_B) : A \times B \to X$ is a morphism of $\mathcal{C}$ such that $\Phi(Tf_A(a), Tf_B(b)) = T\varphi_{AB}(a \otimes b)$ for all $a \in TA$ and $b \in TB$;*
2. *$\Phi(T_dX \times T_dX) \subset T_dX$ and $\Phi(T_fX \times T_fX) \subset T_fX$;*

3. $\Phi((a,b),c) = \Phi(a,\Phi(b,c))$ *for any* $a,b,c \in T_dX$.

Proof. 1. Let $f_A : A \to X$, $f_B : B \to X$ be morphisms from discrete objects A, B of $\mathcal{C}$ and $\varphi_{AB} = \varphi(f_A \times f_B) : A \times B \to X$. By our assumption on the category $\mathcal{C}$, the product $A \times B$ is a discrete object in $\mathcal{C}$ and hence φ_{AB} is a morphism in $\mathcal{C}$. Consider the binary operation $\Phi_{AB} : TA \times TB \to TX$ defined by $\Phi_{AB}(a,b) = \Phi(Tf_A(a), Tf_B(b))$. The following diagram

$$\begin{tikzcd}
TX \times TX \arrow[rrr, "\Phi"] & & & TX \arrow[ddd, leftrightarrow, "\mathrm{id}"] \\
& X \times X \arrow[ul, "\eta\times\eta"'] \arrow[r, "\varphi"] & X \arrow[ur, "\eta"] \arrow[d, leftrightarrow, "\mathrm{id}"] & \\
& A \times B \arrow[u, "f_A\times f_B"] \arrow[r, "\varphi_{AB}"] \arrow[dl, "\eta\times\eta"] & X \arrow[dr, "\eta"] & \\
TA \times TB \arrow[uuu, "Tf_A\times Tf_B"] \arrow[rrr, "\Phi_{AB}"'] & & & TX
\end{tikzcd}$$

implies that Φ_{AB} is a $\mathbb{T}$-extension of φ_{AB}. By Theorem 2,

$$\Phi(Tf_A(a), Tf_B(b)) = \Phi_{AB}(a,b) = T\varphi_{AB}(a \otimes b)$$

for all $a \in TA$ and $b \in TB$.

2. Given elements $a, b \in T_dX$, we need to show that the element $\Phi(a,b) \in TX$ has discrete support. Find discrete objects A, B in $\mathcal{C}$ and morphisms $f_A : A \to X$, $f_B : B \to X$ such that $a \in Ff_A(FA)$ and $b \in f_B(FB)$. Fix elements $\tilde{a} \in FA$, $\tilde{b} \in FB$ such that $a = Ff_A(\tilde{a})$ and $b = Ff_B(\tilde{b})$. Our assumption on the category $\mathcal{C}$ guarantees that $A \times B$ is a discrete object in $\mathcal{C}$.

Consider the binary operations $\psi : A \times B \to X$ and $\Psi : FA \times FB \to FZ$ defined by the formulas $\psi = \varphi \circ (f_A \times f_B)$ and $\Psi = \Phi \circ (Tf_A \times Tf_B)$. Let $\tilde{c} = \tilde{a} \otimes \tilde{b} \in T(A \times B)$. By the first statement, $\Phi(a,b) = T\psi(\tilde{a} \otimes \tilde{b}) = T\psi(\tilde{c}) \in T\psi(A \times B)$, witnessing that the element $\Phi(a,b)$ has discrete support and hence belongs to T_dX.

By analogy, we can prove that $\Phi(T_fX \times T_fX) \subset T_fX$.

3. Given any points $a, b, c \in T_dX$, we need to check the equality

$$\Phi(\Phi(a,b),c) = \Phi(a,\Phi(b,c)).$$

Find discrete objects A, B, C in $\mathcal{C}$ and morphisms $f_A : A \to X$, $f_B : B \to X$, $f_C : C \to X$ such that $a \in Tf_A(TA)$, $b \in Tf_B(TB)$ and $c \in Tf_C(TC)$. Fix elements $\tilde{a} \in TA$, $\tilde{b} \in TB$, and $\tilde{c} \in TC$ such that $a = Tf_A(\tilde{a})$, $b = Tf_B(\tilde{b})$ and $c = Tf_C(\tilde{c})$.

Consider the morphisms $\varphi_{AB} = \varphi(f_A \times f_B) : A \times B \to X$, $\varphi_{BC} = \varphi(f_B \times f_C) : B \times C \to X$

and $\varphi_{ABC} = \varphi(\varphi_{AB} \times f_C) = \varphi(f_A \times \varphi_{BC}) : A \times B \times C \to X$. Consider the following diagram:

$$\begin{tikzcd}
TX \times TX \times TX \arrow[rrr, "\Phi\times\mathrm{id}"] \arrow[ddd, "\mathrm{id}\times\Phi"'] & & & TX \times TX \arrow[ddd, "\Phi"] \\
& TA \times TB \times TC \arrow[ul, "Tf_A\times Tf_B\times Tf_C"'] \arrow[r, "\otimes\times\mathrm{id}"] \arrow[d, "\mathrm{id}\times\otimes"'] & T(A\times B)\times TC \arrow[ur, "T\varphi_{AB}\times Tf_C"'] \arrow[d, "\otimes"] & \\
& TA \times T(B\times C) \arrow[dl, "Tf_A\times T\varphi_{BC}"] \arrow[r, "\otimes"'] & T(A\times B\times C) \arrow[dr, "T\varphi_{ABC}"'] & \\
TX \times TX \arrow[rrr, "\Phi"'] & & & TX
\end{tikzcd}$$

In this diagram the central square is commutative because of the associativity of the tensor product $\otimes$. By the item (1) all four margin squares also are commutative. Now we see that

$$\Phi(\Phi(a,b),c)) = \Phi(\Phi(Tf_A(\tilde{a}), Tf_B(\tilde{b})), Tf_C(\tilde{c})) =$$
$$\Phi(T\varphi_{AB}(\tilde{a}\otimes\tilde{b}), Tf_C(\tilde{c})) = T\varphi_{ABC}((\tilde{a}\otimes\tilde{b})\otimes\tilde{c}) = T\varphi_{ABC}(\tilde{a}\otimes(\tilde{b}\otimes\tilde{c})) =$$
$$\Phi(Tf_A(\tilde{a}), T\varphi_{BC}(\tilde{a}\otimes\tilde{b})) = \Phi(Tf_A(\tilde{a}), \Phi(Tf_B(\tilde{b}), Tf_C(\tilde{c}))) = \Phi(a, \Phi(b,c)).$$

□

Combining Lemma 5.1 with Theorem 4, we get the main result of this paper:

Theorem 5. *Assume that the monadic functor T is continuous and for each finite discrete space F in $\mathcal{C}$ the space TF is Hausdorff and locally compact. Let $\varphi : X \times X \to X$ be an associative binary operation in $\mathcal{C}$ and $\Phi : X \times X \to X$ be its $\mathbb{T}$-extension. If the set $T_f X$ of elements with finite support is dense in TX, then the operation Φ is associative.*

Proof. By Theorem 4, the set $T_f X$ lies in the topological center Λ_Φ of the operation Φ and by Lemma 5.1, $T_f X$ is a subsemigroup of (TX, Φ). Now the associativity of Φ follows from the following general fact. □

Proposition 5.1. *A right topological operation $\cdot : X \times X \to X$ on a Hausdorff space X is associative, if its topological center contains a dense subsemigroup S of X.*

Proof. Assume conversely that $(xy)z \neq x(yz)$ for some points $x, y, z \in X$. Since X is Hausdorff, the points $(xy)z$ and $x(yz)$ have disjoint open neighborhoods $O((xy)z)$ and $O(x(yz))$ in X. Since the right shifts in X are continuous, there are open neighborhoods $O(xy)$ and $O(x)$ of the points xy and x such that $O(xy)\cdot z \subset O((xy)z)$ and $O(x)\cdot(yz) \subset O(x(yz))$. We can assume that $O(x)$ is so small that $O(x)\cdot y \subset O(xy)$. Take any point $a \in O(x) \cap S$. It follows that $a(yz) \in O(x(yz))$ and $ay \in O(xy)$. Since the left shift $l_a : \beta S \to \beta S$, $l_a : y \mapsto ay$, is continuous, the points yz and y have open neighborhoods $O(yz)$ and $O(y)$ such that $a\cdot O(yz) \subset O(x(yz))$ and $a\cdot O(y) \subset O(xy)$. We can assume that the neighborhood $O(y)$ is so small that $O(y)\cdot z \subset O(yz)$. Choose a point $b \in O(y) \cap S$ and observe that $bz \in O(y)\cdot z \subset O(yz)$, $ab \in a\cdot O(y) \subset O(xy)$, and thus $(ab)z \in O(xy)\cdot z \subset O((xy)z)$. The continuity of the left shifts l_b and l_{ab} allows us to find an open neighborhood $O(z) \subset \beta S$ of

z such that $b \cdot O(z) \subset O(yz)$ and $ab \cdot O(z) \subset O((xy)z)$. Finally take any point $c \in S \cap O(z)$. Then $(ab)c \in ab \cdot O(z) \subset O((xy)z)$ and $a(bc) \subset a \cdot O(yz) \subset O(x(yz))$ belong to disjoint sets, which is not possible as $(ab)c = a(bc)$. □

6 $\mathbb{T}$-EXTENSION FOR SOME CONCRETE MONADIC FUNCTORS

In this section we consider some examples of monadic functors in topological categories. Let **Tych** denote the category of Tychonov spaces and their continuous maps and **Comp** be the full subcategory of the category **Tych**, consisting of compact Hausdorff spaces.

Discrete objects in the category **Tych** are discrete topological spaces, while discrete objects in the category **Comp** are finite discrete spaces.

Consider the functor β : **Tych** $\to$ **Comp**, assigning to each Tychonov space X its Stone-Čech compactification and to a continuous map $f : X \to Y$ between Tychonov spaces its continuous extension $\beta f : \beta X \to \beta Y$. The functor β can be completed to a monad $\mathbb{T}_\beta = (\beta, \eta, \mu)$, where $\eta : X \to \beta X$ is the canonical embedding and $\mu : \beta(\beta X) \to \beta X$ is the identity map. A pair (X, ξ) is a $\mathbb{T}_\beta$-algebra if and only if X is a compact space and $\xi : \beta X \to X$ is the identity map.

Combining Theorems 1, 5, we get the following well-known corollary.

Corollary 6.1. *Each binary right-topological operation $\varphi : X \times Y \to Z$ in* **Tych** *with discrete X can be extended to a right-topological operation $\Phi : \beta X \times \beta Y \to \beta Z$, containing X in its topological center Λ_Φ. If $X = Y = Z$ and the operation φ is associative, then so is the operation Φ.*

Now let $\mathbb{T} = (T, \eta, \mu)$ be a monad in the category **Comp**. Taking the composition of the functors β : **Tych** $\to$ **Comp** and T : **Comp** $\to$ **Comp**, we obtain a monadic functor $T\beta$: **Tych** $\to$ **Comp**.

Theorem 6. *Each binary right-topological operation $\varphi : X \times Y \to Z$ in the category* **Tych** *with discrete X can be extended to a right-topological operation $\Phi : T\beta X \times T\beta Y \to T\beta Z$ that contain the set $\eta(X) \subset T\beta X$ in its topological center Λ_Φ. If the functor T is continuous, then the set $T_f X$ of elements $a \in T\beta X$ with finite support is dense in $T\beta X$ and lies in the topological center Λ_Φ of the operation Φ. Moreover, if $X = Y = Z$ and the operation φ is associative, the so is the operation Φ.*

Proof. By Theorem 1, the binary operation φ has a unique $\mathbb{T}$-extension $\Phi : TX \times TY \to TZ$. By Definition 2.1, the set $\eta(X) \subset T\beta X$ lies in the topological center Λ_φ of φ.

Now assume that the functor T is continuous. First we show that the set $T_f X$ is dense in $T\beta X$. Fix any point $a \in F\beta X$ and an open neighborhood $U \subset T\beta X$ of a. Then $[a, U] = \{f \in \mathrm{Mor}(F\beta X, F\beta X) : f(a) \in U\}$ is an open neighborhood of the identity map $\mathrm{id} : F\beta X \to F\beta X$ in the function space $\mathrm{Mor}(F\beta X, F\beta X)$ endowed with the compact-open topology. The continuity of the functor T yields a neighborhood $\mathcal{U}(\mathrm{id}_{\beta X})$ of the identity map $\mathrm{id}_{\beta X} \in \mathrm{Mor}(\beta X, \beta X)$ such that $Tf \in [a, U]$ for any $f \in \mathcal{U}(\mathrm{id}_{\beta X})$. It follows from the definition of the compact-open topology, that there is an open cover $\mathcal{U}$ of βX such that a map $f : \beta X \to \beta X$ belongs to $\mathcal{U}(\mathrm{id}_{\beta X})$, if f is $\mathcal{U}$-near to $\mathrm{id}_{\beta X}$ in the sense that for every

$x \in \beta X$ there is a set $U \in \mathcal{U}$ with $\{x, f(x)\} \subset U$. Since βX is compact, we can assume that the cover $\mathcal{U}$ is finite. Since X is discrete, the space βX has covering dimension zero [9, 7.1.17]. So, we can assume that the finite cover $\mathcal{U}$ is disjoint. For every $U \in \mathcal{U}$ choose an element $x_U \in U \cap X$. Those elements compose a finite discrete subspace $A = \{x_U : U \in \mathcal{U}\}$ of X. Let $i : A \to X$ be the identity embedding and $f : X \to A$ be the map defined by $f^{-1}(x_U) = U$ for $U \in \mathcal{U}$. It follows that $i \circ f \in \mathcal{U}(\mathrm{id}_{\beta X})$ and thus $T(i \circ f) \in [a, U]$ and $Ti \circ Tf(a) \in U$. Now we see that $b = Tf(a) \in TA$ and $c = Ti(b) \in T_f X \cap U$, so $T_f X$ is dense in βX.

By Theorem 4, the set $T_f X$ lies in the topological center Λ_Φ of Φ.

Now assume that the operation φ is associative. By Lemma 5.1, $T_f X$ is a subsemigroup of (X, Φ). Since $T_f X$ is dense and lies in the topological center Λ_Φ, we may derive the associativity of Φ from Proposition 5.1. □

Problem 1. *Given a discrete semigroup X investigate the algebraic and topological properties of the compact right-topological semigroup $T\beta X$ for some concrete continuous monadic functors T :* **Comp** $\to$ **Comp**.

This problem was addressed in [10], [11] for the monadic functor G of inclusion hyperspaces, in [2]–[5] for the functor of superextension λ, in [1], [12], [15] for the functor P of probability measures and in [6], [7], [8], [18] for the hyperspace functor exp.

In [19] it was shown that for each continuous monadic functor T : **Comp** $\to$ **Comp** any continuous (associative) operation $\varphi : X \times Y \to Z$ in **Comp** extends to a continuous (associative) operation $\Phi : TX \times TX \to TX$.

Problem 2. *For which monads $\mathbb{T} = (T, \eta, \mu)$ in the category* **Comp** *each right-topological (associative) binary operation $\varphi : X \times Y \to Z$ in* **Comp** *extends to a right-topological (associative) binary operation $\Phi : TX \times TY \to TZ$? Are all such monads power monads?*

REFERENCES

1. Banakh T., Cencelj M., Hryniv O., Repovs D. *Characterizing compact Clifford semigroups that embed into convolution and functor-semigroups*, preprint (http://arxiv.org/abs/0811.1026).

2. Banakh T., Gavrylkiv V., Nykyforchyn O. *Algebra in superextensions of groups, I: zeros and commutativity*, Algebra Discrete Math, 3 (2008), 1-29.

3. Banakh T., Gavrylkiv V. *Algebra in superextension of groups, II: cancelativity and centers*, Algebra Discrete Math, 4 (2008), 1-14.

4. Banakh T., Gavrylkiv V. *Algebra in the superextensions of groups, III: minimal left ideals*, Mat. Stud., **31**, 2 (2009), 142-148.

5. Banakh T., Gavrylkiv V. *Algebra in the superextensions of groups, IV: representation theory*, preprint (http://arxiv.org/abs/0811.0796).

6. Banakh T., Hryniv O. *Embedding topological semigroups into the hyperspaces over topological groups*, Acta Univ. Carolinae, Math. et Phys., **48**, 2 (2007), 3-18.

7. Bershadskii S.G. *Imbeddability of semigroups in a global supersemigroup over a group*, in: Semigroup varieties and semigroups of endomorphisms, Leningrad Gos. Ped. Inst., Leningrad, 1979, 47-49.

8. Bilyeu R.G., Lau A. *Representations into the hyperspace of a compact group*, Semigroup Forum **13** (1977), 267-270.

9. Engelking R. General Topology, PWN, Warsaw, 1977.

10. Gavrylkiv V. *The spaces of inclusion hyperspaces over noncompact spaces*, Mat. Stud., **28,** 1 (2007), 92-110.

11. Gavrylkiv V. *Right-topological semigroup operations on inclusion hyperspaces*, Mat. Stud., **29**, 1 (2008), 18-34.

12. Heyer H. *Probability Measures on Locally Compact Groups*, Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 94. Springer-Verlag, Berlin-New York, 1977.

13. Hindman N., Strauss D. Algebra in the Stone-Čech compactification, de Gruyter, Berlin, New York, 1998.

14. MacLane S. Categories for working mathematicican, Springer, 1971.

15. Parthasarathy K.R. Probability Measures on Metric Spaces, AMS Bookstore, 2005.

16. Protasov I. Combinatorics of Numbers, VNTL, Lviv, 1997.

17. Teleiko A., Zarichnyi M. Categorical Topology of Compact Hausdofff Spaces, VNTL, Lviv, 1999.

18. Trnkova V. *On a representation of commutative semigroups*, Semigroup Forum, **10,** 3 (1975), 203-214.

19. Zarichnyi M., Teleiko A. *Semigroups and monads*, in: Algebra and Topology, Lviv Univ. Press, 1996, 84-93 (in Ukrainian).

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Маючи неперервний монадичний функтор $T : \mathbf{Comp} \to \mathbf{Comp}$ в категорії компактів і дискретну топологічну напівгрупу X, ми продовжуємо напівгрупову операцію $\varphi : X \times X \to X$ до правотопологічної напівгрупової операції $\Phi : T\beta X \times T\beta X \to T\beta X$, топологічний центр Λ_Φ якої містить всюди щільну піднапівгрупу $T_f X$, яка складається з елементів $a \in T\beta X$ зі скінченним носієм в X.

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Пусть $T : \mathbf{Comp} \to \mathbf{Comp}$ – непрерывный монадический функтор в категории компактов и X – дискретная топологическая полугруппа. В работе построено продолжение полугрупповой операции $\varphi : X \times X \to X$ до правотопологической полугрупповой операции $\Phi : T\beta X \times T\beta X \to T\beta X$, топологический центр которой содержит всюду плотную подполугруппу $T_f X$, содержащую елементы $a \in T\beta X$ с конечным носителем в X.