



MAMALYHA KH.V., OSYPOCHUK M.M.

ON SINGLE-LAYER POTENTIALS FOR A CLASS OF PSEUDO-DIFFERENTIAL EQUATIONS RELATED TO LINEAR TRANSFORMATIONS OF A SYMMETRIC α -STABLE STOCHASTIC PROCESS

In this article an arbitrary invertible linear transformations of a symmetric α -stable stochastic process in d -dimensional Euclidean space \mathbb{R}^d are investigated. The result of such transformation is a Markov process in \mathbb{R}^d whose generator is the pseudo-differential operator defined by its symbol $-(Q\xi, \xi)^{\alpha/2}$, $\xi \in \mathbb{R}^d$ with some symmetric positive definite $d \times d$ -matrix Q and fixed exponent $\alpha \in (1, 2)$. The transition probability density of this process is the fundamental solution of some parabolic pseudo-differential equation. The notion of a single-layer potential for that equation is introduced and its properties are investigated. In particular, an operator is constructed whose role in our consideration is analogous to that the gradient in the classical theory. An analogy to the classical theorem on the jump of the co-normal derivative of the single-layer potential is proved. This result can be applied for solving some boundary-value problems for the parabolic pseudo-differential equations under consideration. For $\alpha = 2$, the process under consideration is a linear transformation of Brownian motion, and all the investigated properties of the single-layer potential are well known.

Key words and phrases: pseudo-differential equation, single-layer potential, α -stable stochastic process, jump theorem.

Vasyl Stefanyk Precarpathian National University, 57 Shevchenka str., 76018, Ivano-Frankivsk, Ukraine
E-mail: mamalygakhrystyna@gmail.com (Mamalyha Kh.V.), myosyp@gmail.com (Osypchuk M.M.)

INTRODUCTION

Let us consider a symmetric α -stable process $(x_0(t))_{t \geq 0}$ in the d -dimensional Euclidean space \mathbb{R}^d (we denote by (\cdot, \cdot) the inner product in this space), that is, a Markov process with its transition probability density given by the equality

$$g_0(t, x, y) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{i(\xi, x-y) - t|\xi|^\alpha} d\xi, \quad t > 0, x \in \mathbb{R}^d, y \in \mathbb{R}^d,$$

where the exponent $\alpha \in (1, 2)$ is fixed. The class of all symmetric α -stable processes can be obtained from the process $(x_0(t))_{t \geq 0}$ by multiplying it on some positive constants. More complex processes can be obtained in the following way.

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Let P be some invertible $d \times d$ -matrix and $x(t) = Px_0(t), t \geq 0$. This process is obviously Markov process and its transition probability density is given by the equality

$$g(t, x, y) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{i(\xi, x-y) - t(Q\xi, \xi)^{\alpha/2}} d\xi, \quad t > 0, x \in \mathbb{R}^d, y \in \mathbb{R}^d, \quad (1)$$

where $Q = PP^T$. It is clear that the process $(x(t))_{t \geq 0}$ is stochastically equivalent to the Markov process $(Lx_0(t))_{t \geq 0}$, where L is some lower triangular matrix which satisfies the equality $Q = LL^T$.

The function g is the fundamental solution of the pseudo-differential equation

$$\frac{\partial u(t, x)}{\partial t} = \mathbf{A}u(t, \cdot)(x), \quad t > 0, x \in \mathbb{R}^d, \quad (2)$$

where operator \mathbf{A} is a pseudo-differential operator whose symbol is given by the function $-(Q\xi, \xi)^{\alpha/2}_{\xi \in \mathbb{R}^d}$. The operator \mathbf{A} is the generator of Markov process $(x(t))_{t \geq 0}$.

For a given surface S , which separates \mathbb{R}^d into two open sets D_- and D_+ ($\mathbb{R}^d = D_- \cup S \cup D_+$) and a given continuous function $(\psi(t, x))_{t \geq 0, x \in S}$, we consider a function

$$v(t, x) = \int_0^t d\tau \int_S g(t - \tau, x, y)\psi(\tau, y)d\sigma_y, \quad t > 0, x \in \mathbb{R}^d,$$

where the inner integral is a surface one. The function v is called a single-layer potential on the surface S with the density ψ for equation (2).

In this article, we determine the existence conditions of the single-layer potential and investigate its properties. The case of $Q = c^{2/\alpha}I$ ($c > 0$ and I is a unit $d \times d$ -matrix) was considered in article [3]. We will use several methods from [3]. In the case of $\alpha = 2$, the theory of single-layer potentials is well-known (see, for example, [2]).

1 SOME AUXILIARY RESULTS

1.1 The function g

The function g defined above by formula (1) is continuous on the domain $t > 0, x \in \mathbb{R}^d, y \in \mathbb{R}^d$, and is uniformly continuous on each set of the type $(t, x, y) \in [\tau, +\infty) \times \mathbb{R}^d \times \mathbb{R}^d$ with $\tau > 0$. The following estimations of g and its derivatives are known (see [1, Ch.4]):

$$|\mathbf{D}^k g(t, \cdot, y)(x)| \leq N_k \frac{t}{(t^{1/\alpha} + |y - x|)^{d+\alpha+k}}, \quad t > 0, x \in \mathbb{R}^d, y \in \mathbb{R}^d; \quad (3)$$

$$|\mathbf{D}^\varkappa g(t, \cdot, y)(x)| \leq \tilde{N}_\varkappa \frac{1}{(t^{1/\alpha} + |y - x|)^{d+\varkappa}}, \quad t > 0, x \in \mathbb{R}^d, y \in \mathbb{R}^d.$$

Here \mathbf{D}^k means a differential operator of the order k ($k = 0, 1, 2, \dots$), \mathbf{D}^\varkappa means a pseudo-differential operator with a homogeneous symbol $(p_\varkappa(\xi))_{\xi \in \mathbb{R}^d}$ of the order \varkappa which has all derivatives of the orders $l < M$ with some $M \geq 2d + \varkappa + \alpha + 1$ and $|p_\varkappa^{(l)}(\xi)| \leq C_M |\xi|^{\varkappa-1}$ with some constant $C_M \geq 0$ for all $\xi \neq 0$, and N_k and \tilde{N}_\varkappa are some positive constants.

1.2 The operator \mathbf{A}

An action of the operator \mathbf{A} defined in Introduction on a smooth (with at least Lipschitz continuous gradient) and bounded together with its derivatives function $\varphi(x)_{x \in \mathbb{R}^d}$ is given by the expression

$$\mathbf{A}\varphi(x) = \frac{q_\alpha}{(\det Q)^{1/2}} \int_{\mathbb{R}^d} (\varphi(x+y) - \varphi(x) - (\nabla\varphi(x), y))(Q^{-1}y, y)^{-(d+\alpha)/2} dy, \quad (4)$$

where

$$q_\alpha = \frac{\alpha\Gamma((3-\alpha)/2)\Gamma((d+\alpha)/2)}{\pi^{(d+1)/2}\Gamma(2-\alpha)}.$$

The value of the constant q_α can be obtained by applying the operator \mathbf{A} to the function $\varphi_\xi(x) = e^{i(\xi, x)}$, $x \in \mathbb{R}^d$ with some fixed $\xi \in \mathbb{R}^d$.

1.3 An operator \mathbf{B}

Let us introduce the operator \mathbf{B} using its symbol $(i(Q\xi, \xi)^{\alpha/2-1}\xi)_{\xi \in \mathbb{R}^d}$. Some simple calculations lead us to the relation $\mathbf{A} = (\nabla, Q\mathbf{B})$. The action of the operator \mathbf{B} on a bounded Lipschitz continuous function $(\varphi(x))_{x \in \mathbb{R}^d}$ is defined by the following formula

$$\mathbf{B}\varphi(x) = \frac{q_\alpha}{\alpha(\det Q)^{1/2}} \int_{\mathbb{R}^d} (\varphi(x+y) - \varphi(x))(Q^{-1}y, y)^{-(d-\alpha)/2} Q^{-1}y dy,$$

where q_α has the above meaning.

Let ν be some fixed ort in \mathbb{R}^d . Consider the operator $\mathbf{B}_\nu = 2(Q\nu, \mathbf{B})$. We denote the result of its action on the function g with respect to the second argument by $g_\nu(t, x, y)$. Using representation (1) of the function g and the integration by parts, it is easy to obtain the relation

$$g_\nu(t, x, y) = \frac{2}{\alpha} \frac{(y-x, \nu)}{t} g(t, x, y), \quad t > 0, x \in \mathbb{R}^d, y \in \mathbb{R}^d. \quad (5)$$

1.4 A surface of the class $H^{1+\gamma}$

Let some surface S in \mathbb{R}^d (a manifold of dimension $d-1$) divide the set \mathbb{R}^d into two open sets: outer D_+ and inner D_- (i.e., $\mathbb{R}^d = D_- \cup S \cup D_+$). Suppose that this surface has a tangent hyperplane at each point $x \in S$. We will denote $\nu(x)$ the unit vector of the outer normal to the surface S at the point $x \in S$. Choose the point $x \in S$ and consider a local orthogonal coordinate system with the origin at this point, such that $\nu(x)$ is the ort of its last axis. Assume the surface S is such that for some $\delta > 0$ each part $S_\delta(x) = S \cap B_\delta(x)$, $x \in S$, of the surface S (here $B_\delta(x)$ is a ball with the radius $\delta > 0$ and the center at the point x) can be described in the mentioned above local coordinate system by the equation $y_d = F_x(y_1, \dots, y_{d-1})$ with a single-valued function F_x .

The bounded closed surface S belongs to the class $H^{1+\gamma}$ if the function F_x has all partial derivatives $\frac{\partial F_x}{\partial y_k}$, $k = 1, 2, \dots, d-1$, satisfy Hölder's condition with a power $\gamma \in (0; 1)$ and the constant does not dependent on x .

Among the properties of the surface S which belongs to the class $H^{1+\gamma}$ we will use the following one (see [2, Ch.5]): there are some positive numbers δ_0 and r_0 , and a finite set of points x_1, x_2, \dots, x_m on the surface S , such that $S \setminus S_{r_0/2}(x) \subset \cup_{k \in I_x} S_{r_0/2}(x_k)$ for each $x \in S$, and $\min_{k \in I_x} \inf_{y \in S_{r_0/2}(x_k)} |y - x| \geq \delta_0$, where I_x is some subset of the indices $\{1, 2, \dots, m\}$.

2 A SINGLE-LAYER POTENTIAL

2.1 Existence conditions

Let S be a bounded closed surface of the class $H^{1+\gamma}$ with some $\gamma \in (0;1)$. Consider some continuous function $(\psi(t, x))_{t \geq 0, x \in S}$ and define the function $(v(t, x))_{t \geq 0, x \in \mathbb{R}^d}$ by the following equality

$$v(t, x) = \int_0^t d\tau \int_S g(t - \tau, x, y) \psi(\tau, y) d\sigma_y, \quad t > 0, x \in \mathbb{R}^d, y \in \mathbb{R}^d, \tag{6}$$

where an inner integral is a surface one. This function is called a single-layer potential on the surface S with the density ψ . The following statement contains the conditions under which a single-layer potential is well defined.

Lemma 1. *Let S be a bounded closed surface of the class $H^{1+\gamma}$ with some $\gamma \in (0;1)$ and $(\psi(t, x))_{t \geq 0, x \in S}$ be a continuous function, which satisfies the inequality $|\psi(t, x)| \leq C_T t^{-\beta}$ in each set of $(t; x) \in (0; T] \times S$ with some constants $\beta < 1$ and $C_T > 0$ (the last one can depend on $T > 0$). Then, the single-layer potential (6) is finite for all $t > 0$ and $x \in \mathbb{R}^d$.*

Proof. Estimation (3) with $k = 0$ and the fact that (see [3])

$$\int_S \frac{d\sigma_y}{(t^{1/\alpha} + |y - x|)^{d+\alpha}} \leq K t^{-1-1/\alpha}, t > 0, x \in \mathbb{R}^d$$

with some constant $K > 0$ imply to the inequality

$$|v(t, x)| \leq KN_0 C_T \int_0^t \frac{d\tau}{\tau^\beta (t - \tau)^{1/\alpha}} = KN_0 C_T B(1 - \beta, 1 - 1/\alpha) t^{1-\beta-1/\alpha}$$

for all $t \in (0; T]$, $x \in \mathbb{R}^d$ and each $T > 0$. □

2.2 Properties of the single-layer potential

Classically (when $\alpha = 2$), a single-layer potential satisfies the appropriate parabolic differential equation in the domain $(0; +\infty) \times (\mathbb{R}^d \setminus S)$ (see [2, Ch.5]). Let us prove an analogous statement in our case ($1 < \alpha < 2$).

Theorem 1. *Let S be a bounded closed surface of the class $H^{1+\gamma}$ with some $\gamma \in (0;1)$, and $(\psi(t, x))_{t \geq 0, x \in S}$ be a continuous function satisfying the inequality $|\psi(t, x)| \leq C_T t^{-\beta}$ in each set of $(t; x) \in (0; T] \times S$ with some constants $\beta < 1$ and $C_T > 0$ (the last one can depend on $T > 0$). Then the single-layer potential (3) satisfies the equation*

$$\frac{\partial v(t, x)}{\partial t} = \mathbf{A}v(t, \cdot)(x), \quad t > 0, x \in \mathbb{R}^d$$

in the domain $(t; x) \in (0; \infty) \times (\mathbb{R}^d \setminus S)$.

Proof. It has already been mentioned that the function g is the fundamental solution of equation (2), and therefore, for all $t > 0, x \in \mathbb{R}^d, y \in \mathbb{R}^d$ the equality $\frac{\partial g(t, x, y)}{\partial t} = \mathbf{A}g(t, \cdot, y)(x)$ holds true. So, we only have to prove that the operator \mathbf{A} with respect to the variable x can be moved under the integral symbol in the right-hand part of (6) and the equality

$$\lim_{\varepsilon \rightarrow 0^+} \int_S g(\varepsilon, x, y) \psi(t, y) d\sigma_y = 0,$$

holds true for $t > 0, x \in \mathbb{R}^d \setminus S$. The last is due to estimation (3) with $k = 0$ and the inequality

$$\left| \int_S g(\varepsilon, x, y) \psi(t, y) d\sigma_y \right| \leq N_0 \varepsilon \frac{|S|}{(\rho(x, S))^{d+\alpha}} C_T t^{-\beta},$$

where $|S|$ is the area of the surface S , and $\rho(x, S)$ is the distance from the point x to the surface S .

Next, we will take presentation (4) of the operator \mathbf{A} and prove the possibility to change the order of integrating in the integral

$$\int_0^t d\tau \int_S \psi(\tau, y) d\sigma_y \int_{\mathbb{R}^d} \frac{g(t - \tau, x + z, y) - g(t - \tau, x, y) - (\nabla g(t - \tau, \cdot, y)(x), z)}{(Q^{-1}z, z)^{(d+\alpha)/2}} dz.$$

Take into account that the following inequalities $\frac{1}{M}|z|^2 \leq (Q^{-1}z, z) \leq M|z|^2$ hold true with some constant $M > 0$ for all $x \in \mathbb{R}^d$. Divide the last integral into the sum of two integrals I_1 and I_2 taken from the same function: the first of them is by $(0; t) \in S \times B_\varepsilon$, and another is by $(0; t) \in S \times (\mathbb{R}^d \setminus B_\varepsilon)$, where B_ε is a ball of some small enough radius $\varepsilon > 0$ centered at the origin.

Since for $0 < \tau < t, x \in \mathbb{R}^d, y \in S, z \in B_\varepsilon$ the following equality

$$g(t - \tau, x + z, y) - g(t - \tau, x, y) - (\nabla g(t - \tau, \cdot, y)(x), z) = \frac{1}{2} \sum_{i,j=1}^d \frac{\partial^2 g(t - \tau, x + \theta z, y)}{\partial x_i \partial x_j} z_i z_j$$

is true, where $\theta = \theta(\tau; y) \in (0; 1)$, the absolute value of the integrand in I_1 is estimated by the expression

$$C_t \tau^{-\beta} \frac{1}{2} d^2 N_2 \frac{t - \tau}{((t - \tau)^{1/\alpha} + |y - x - \theta z|)^{d+\alpha+2}} |z|^2 M^{(d+\alpha)/2} |z|^{-d-\alpha},$$

where estimation (3) for $k = 2$ is used. Take a sufficiently small $\varepsilon > 0$ such that the inequality $\inf_{y \in S, z \in B_\varepsilon, \theta \in (0,1)} |y - x - \theta z| = \rho_0 > 0$ holds true. Therefore, I_1 is absolutely convergent.

The absolute value of the integrand in I_2 is estimated by the expression (estimation (3) is also taken into account)

$$\begin{aligned} & C_t \tau^{-\beta} M^{(d+\alpha)/2} |z|^{-d-\alpha} \left(\frac{N_0(t - \tau)}{((t - \tau)^{1/\alpha} + |y - x - z|)^{d+\alpha}} + \frac{N_0(t - \tau)}{((t - \tau)^{1/\alpha} + |y - x|)^{d+\alpha}} \right. \\ & \qquad \qquad \qquad \left. + \frac{N_1(t - \tau)}{((t - \tau)^{1/\alpha} + |y - x|)^{d+\alpha+1}} |z| \right) \\ & \leq C_t \tau^{-\beta} M^{(d+\alpha)/2} |z|^{-d-\alpha} \left(\frac{N_0(t - \tau)}{((t - \tau)^{1/\alpha} + |y - x - z|)^{d+\alpha}} + \frac{N_0(t - \tau)}{(\rho(x, S))^{d+\alpha}} + \frac{N_1(t - \tau)}{(\rho(x, S))^{d+\alpha+1}} |z| \right). \end{aligned}$$

Observe that the second and the third terms in this expression are integrable. Consider the integral of the first term and change the variable z into u using the equality $y - x - z = (t - \tau)^{1/\alpha} u$. We have got

$$N_0 C_t M^{(d+\alpha)/2} \int_0^t \frac{(t - \tau)^{1+d/\alpha}}{\tau^\beta} d\tau \int_S d\sigma_y \int_{D(\tau, y)} |y - x - (t - \tau)^{1/\alpha} u|^{-d-\alpha} (1 + |u|)^{-d-\alpha} du,$$

where $D(\tau, y) = \{u \in \mathbb{R}^d : |y - x - (t - \tau)^{1/\alpha} u| > \varepsilon\}$. Surely, this integral converges, and for completing the proof of the theorem, we have to use the Fubini's theorem. \square

2.3 The jump theorem

In a classical theory (with $\alpha = 2$) of a single-layer potential the jump theorem takes an essential place. It is the theorem on the jump of the co-normal derivative of a single-layer potential. This section is devoted to an analogue of that theorem in our situation ($1 < \alpha < 2$).

Lemma 2. *Let the surface S and the function $(\psi(t, x))_{t \geq 0, x \in S}$ satisfy the conditions of Theorem 1. Then for each $t > 0$ and $x \in S$ the following integral*

$$\int_0^t d\tau \int_S \mathbf{B}_{\nu(x)} g(t - \tau, \cdot, y)(x) \psi(\tau, y) d\sigma_y \tag{7}$$

is finite.

Proof. By equality (5), we can rewrite integral (7) in the following form

$$\frac{2}{\alpha} \int_0^t \frac{d\tau}{t - \tau} \int_S (y - x, \nu(x)) g(t - \tau, x, y) \psi(\tau, y) d\sigma_y.$$

Taking into account estimation (3) and the properties of the surface S (see Section 1.1), we can obtain for $\tau < t, x \in S$, the inequality

$$\begin{aligned} \left| \int_S (y - x, \nu(x)) g(t - \tau, x, y) \psi(\tau, y) d\sigma_y \right| &\leq C_t \tau^{-\beta} K((t - \tau)^{\gamma/2} + ((t - \tau)^{1/2} + \delta_0)^{-d-\alpha+1}(t - \tau)) \\ &\leq \text{const}_t \tau^{-\beta} ((t - \tau)^{\gamma/2} + (t - \tau)), \end{aligned}$$

where $K > 0$ is some constant and const_t is some positive constant, which probably depends on t . Hence the statement of the lemma is proved. \square

Remark 1. *Integral (7) is called a direct value of the action result of the operator $\mathbf{B}_{\nu(x)}$, $x \in S$ on single-layer potential (6) at the point $x \in S$. We will denote it by $\mathbf{B}_{\nu(x)}^{(dv)} v(t, \cdot)(x)$.*

The next statement is the jump theorem mentioned above.

Theorem 2. *Let S be a bounded closed surface of the class $H^{1+\gamma}$ with some $\gamma \in (0; 1)$ in \mathbb{R}^d , and $(\psi(t, x))_{t \geq 0, x \in S}$ be a continuous function satisfying the inequality $|\psi(t, x)| \leq C_T t^{-\beta}$, $0 < t \leq T, x \in S$ with some constants $\beta < 1$ and $C_T > 0$ (the last one can depend on T) for each $T > 0$. Then for each $t \geq 0, x \in S$ the following equality*

$$\lim_{y \rightarrow x \pm} \mathbf{B}_{\nu(x)} v(t, \cdot)(y) = \mp \psi(t, x) + \mathbf{B}_{\nu(x)}^{(dv)} v(t, \cdot)(x),$$

holds true, where $y \rightarrow x \pm$ means that y approaches x staying in some closed bounded cone $\mathcal{K} \subset \mathbb{R}^d$ with the vertex at the point x and $\mathcal{K} \subset D_{\pm} \cup \{x\}$.

Proof. Similar to the classic case it is sufficient to consider only the case of $y = x + \delta \nu(x)$ and $\delta \rightarrow 0 \pm$. Therefore, taking into account formula (5) we will obtain

$$\begin{aligned} \mathbf{B}_{\nu(x)} v(t, \cdot)(y) &= \frac{2}{\alpha} \int_0^t \frac{d\tau}{t - \tau} \int_S (z - x, \nu(x)) g(t - \tau, y, z) \psi(\tau, z) d\sigma_z \\ &\quad - \delta \frac{2}{\alpha} \int_0^t \frac{d\tau}{t - \tau} \int_S g(t - \tau, y, z) \psi(\tau, z) d\sigma_z \\ &= \mathbf{B}_{\nu(x)}^{(dv)} v(t, \cdot)(x) + \frac{2}{\alpha} \int_0^t \frac{d\tau}{t - \tau} \int_S (z - x, \nu(x)) (g(t - \tau, y, z) - g(t - \tau, x, z)) \psi(\tau, z) d\sigma_z \\ &\quad - \delta \frac{2}{\alpha} \int_0^t \frac{d\tau}{t - \tau} \int_S g(t - \tau, y, z) \psi(\tau, z) d\sigma_z. \end{aligned}$$

Denote the integrals on the right-hand side of this equality by I_1 and I_2 accordingly.

First, let us prove that $\lim_{\delta \rightarrow 0} I_1 = 0$. In order to get this proof, rewrite I_1 in the form of the sum of the following expressions

$$J_1^{(1)} = \frac{2}{\alpha} \int_0^{t-\rho} \frac{d\tau}{t-\tau} \int_S (z-x, \nu(x))(g(t-\tau, x+\delta\nu(x), z) - g(t-\tau, x, z))\psi(\tau, z)d\sigma_z,$$

$$J_2^{(1)} = \frac{2}{\alpha} \int_{t-\rho}^t \frac{d\tau}{t-\tau} \int_{S_{r_{0/2}}(x)} (z-x, \nu(x))(g(t-\tau, x+\delta\nu(x), z) - g(t-\tau, x, z))\psi(\tau, z)d\sigma_z,$$

$$J_3^{(1)} = \frac{2}{\alpha} \int_{t-\rho}^t \frac{d\tau}{t-\tau} \int_{S \setminus S_{r_{0/2}}(x)} (z-x, \nu(x))(g(t-\tau, x+\delta\nu(x), z) - g(t-\tau, x, z))\psi(\tau, z)d\sigma_z,$$

where $0 < \rho < t$ is some constant (t is fixed), which should be chosen. We estimate each of these expressions. Taking into account the properties of the surface S , we can obtain $|(z-x, \nu(x))| \leq |z-x|^{1+\gamma}$ for $z \in S_{r_{0/2}}(x)$. As a result, we have

$$|J_2^{(1)}| \leq C_t \frac{2}{\alpha} \int_{t-\rho}^t \frac{d\tau}{\tau^\beta} \int_{S_{r_{0/2}}(x)} \frac{|z-x|^{1+\gamma}}{((t-\tau)^{1/\alpha} + |z-x-\delta\nu(x)|)^{d+\alpha}} d\sigma_z$$

$$+ C_t \frac{2}{\alpha} \int_{t-\rho}^t \frac{d\tau}{\tau^\beta} \int_{S_{r_{0/2}}(x)} \frac{|z-x|^{1+\gamma}}{((t-\tau)^{1/\alpha} + |z-x|)^{d+\alpha}} d\sigma_z.$$

Let \tilde{z} be the orthogonal projection of the point $z \in S_{r_{0/2}}(x)$ on the tangent hyperplane to S at the point x . Hence, taking into account the inequalities $|z-x| \geq |\tilde{z}-x|$, $|z-x-\delta\nu(x)| \geq |\tilde{z}-x|$, $|z-x| \geq \text{const}|\tilde{z}-x|$, where const is some positive constant that does not depend on the point x (see [2, Ch.5]), we will obtain to the inequalities

$$|J_2^{(1)}| \leq \hat{C}_t \frac{2}{\alpha} \int_{t-\rho}^t \frac{d\tau}{\tau^\beta} \int_{\Delta_{K_{r_{0/2}}}(x)} \frac{|\tilde{z}|d\tilde{z}}{((t-\tau)^{1/\alpha} + |\tilde{z}|)^{d+\alpha}} d\sigma_z \leq \frac{\tilde{C}_t}{(t-\rho)^\beta} \rho^{\gamma/2},$$

where \hat{C}_t, \tilde{C}_t are positive constants that probably depend on t , and $\Delta_{r_{0/2}}(x) \subset \mathbb{R}^{d-1}$ is some bounded set. It means that $\Delta_{r_{0/2}}(x)$ is the orthogonal projection of $S_{r_{0/2}}(x)$ on the tangent hyperplane to S at the point $x \in S$ in the coordinate system of this hyperplane.

Next, we will estimate $J_3^{(1)}$:

$$|J_3^{(1)}| \leq C_t \int_{t-\rho}^t \frac{d\tau}{\tau^\beta} \int_{S \setminus S_{r_{0/2}}(x)} \frac{|z-x|}{((t-\tau)^{1/\alpha} + |z-x-\delta\nu(x)|)^{d+\alpha}} d\sigma_z$$

$$+ C_t \int_{t-\rho}^t \frac{d\tau}{\tau^\beta} \int_{S \setminus S_{r_{0/2}}(x)} \frac{|z-x|}{((t-\tau)^{1/\alpha} + |z-x|)^{d+\alpha}} d\sigma_z.$$

Taking into account that for $z \in S \setminus S_{r_{0/2}}(x)$ the inequalities $|z-x| \geq \delta_0$, $|z-x-\delta\nu(x)| \geq |z-x| - |\delta| \geq \delta_0 - |\delta|$ are true (choose δ to be the one that $|\delta| < \delta_0$), we will have

$$|J_3^{(1)}| \leq \hat{C}_t (\delta_0 - |\delta|)^{-d-\alpha} \int_{t-\rho}^t \frac{d\tau}{\tau^\beta}.$$

Thus, the sum $J_2^{(1)} + J_3^{(1)}$ can be made as small as we want by choosing $\rho > 0$.

Now, consider $J_1^{(1)}$. Since the function $g(t - \tau, x, z)$ is uniformly continuous in the sets of the type $(\tau, x, z) \in [0; t - \rho] \times K_1 \times K_2$, where K_1 and K_2 are some compacta in \mathbb{R}^d , and taking into account the integrability of the function ψ on $[0; t - \rho] \times S$ and the boundary of the function $(z - x, \nu(x))$ as the function of z on S , we will obtain that $\lim_{\delta \rightarrow 0} J_1^{(1)} = 0$. Therefore, $I_1 \rightarrow 0$ as $\delta \rightarrow 0$.

Now, consider the behavior of I_2 as $\delta \rightarrow 0$. Put I_2 in the form of the sum of the following expressions

$$\begin{aligned} J_1^{(2)} &= \frac{2}{\alpha} \delta \psi(t, x) \int_{t-\rho}^t \frac{d\tau}{t-\tau} \int_{S_\varepsilon(x)} g(t-\tau, x + \delta \nu(x), z) d\sigma_z, \\ J_2^{(2)} &= \frac{2}{\alpha} \delta \int_{t-\rho}^t \frac{d\tau}{t-\tau} \int_{S_\varepsilon(x)} g(t-\tau, x + \delta \nu(x), z) (\psi(\tau, z) - \psi(t, x)) d\sigma_z, \\ J_3^{(2)} &= \frac{2}{\alpha} \delta \int_0^{t-\rho} \frac{d\tau}{t-\tau} \int_{S_\varepsilon(x)} g(t-\tau, x + \delta \nu(x), z) \psi(\tau, z) d\sigma_z, \\ J_4^{(2)} &= \frac{2}{\alpha} \delta \int_0^t \frac{d\tau}{t-\tau} \int_{S \setminus S_\varepsilon(x)} g(t-\tau, x + \delta \nu(x), z) \psi(\tau, z) d\sigma_z, \end{aligned}$$

where $\rho > 0, \varepsilon > 0$ are rather small constants.

Let us estimate each of these terms. We start with the last one. Taking into account the properties of the surface S , there are numbers l_0 (natural) and $p_0 > 0$, such that we can find points $x_k \in S \setminus S_\varepsilon(x), k = 1, 2, \dots, l_0$ that $S \setminus S_\varepsilon(x) \subset \cup_{k=1}^{l_0} S_{r_0/2}(x_k)$ and $\inf_{|\xi| \leq |\delta|} \inf_{z \in S_{r_0/2}(x_k)} |z - x - \zeta \nu(x)| \geq p_0$. Then estimation (3) implies

$$|J_4^{(2)}| \leq \frac{2}{\alpha} C_t N_0 |S| l_0 |\delta| \int_0^t \tau^{-\beta} ((t-\tau)^{1/\alpha} + p_0)^{-d-\alpha} d\tau \rightarrow 0, \quad \delta \rightarrow 0.$$

Similarly, using inequality (3) we will get

$$\begin{aligned} |J_3^{(2)}| &\leq \frac{2}{\alpha} |\delta| C_t N_0 \int_0^{t-\rho} \frac{d\tau}{t^\beta} \int_{S_\varepsilon(x)} \frac{d\sigma_z}{((t-\tau)^{1/\alpha} + |z-x-\delta \nu(x)|)^{d+\alpha}} \\ &\leq \frac{2}{\alpha} |\delta| C_t N_0 |S| \int_0^{t-\rho} \frac{d\tau}{t^\beta (t-\tau)^{1+d/\alpha}} \rightarrow 0, \quad \delta \rightarrow 0. \end{aligned}$$

Here $|S|$ means the area of the surface S like mentioned above.

Now, prove the existence of a limit of $J_1^{(2)}$ as $\delta \rightarrow 0$. By the way, it means that by choosing $\rho > 0$ and $\varepsilon > 0$ the term $J_2^{(2)}$ can be made as small as you like. It is sufficiently to note that the function ψ is uniformly continuous on the set $[t - \rho; t] \times S_\varepsilon(x)$. Denote the tangent hyperplane to the surface S at the point $x \in S$ by Π_x , and consider

$$R = \frac{2}{\alpha} \delta \int_{t-\rho}^t \frac{d\tau}{t-\tau} \int_{\Pi_x} g(t-\tau, x + \delta \nu(x), z) d\sigma_z.$$

To prove that $\lim_{\delta \rightarrow 0} (J_1^{(2)} - \psi(t, x)R) = 0$, consider

$$\begin{aligned} &\delta \int_0^\rho \frac{d\tau}{\tau} \left(\int_{S_\varepsilon(x)} - \int_{\Pi_x} \right) g(\tau, x + \delta \nu(x), z) d\sigma_z \\ &= \delta \int_0^\rho \frac{d\tau}{\tau} \left(\int_{S_\varepsilon(x)} - \int_{\Pi_x} \right) g(\tau, x + \delta \nu(x), z) d\sigma_z \\ &\quad - \delta \int_0^\rho \frac{d\tau}{\tau} \int_{\Pi_x \setminus \Pi_\varepsilon(x)} g(\tau, x + \delta \nu(x), z) d\sigma_z \\ &= J' + J'', \end{aligned}$$

where $\Pi_\varepsilon(x)$ is the orthogonal projection $S_\varepsilon(x)$ on Π_x . Taking into account the properties of the surface S , it is easy to see that there is a constant $\theta > 0$ such that for all $z \in \Pi_x \setminus \Pi_\varepsilon(x)$ the inequality $|z| \geq \theta$ is true. Therefore the following estimation

$$|J''| \leq C|\delta|\rho \int_\theta^\infty \frac{r^{d-2}dr}{(\delta^2 + r^2)^{(d+\alpha)/2}} = C\rho \int_{\theta/|\delta|}^\infty \frac{r^{d-2}}{(1 + r^2)^{(d+\alpha)/2}} \frac{dr}{|\delta|^\alpha}$$

holds true with some constant $C > 0$. As a result, taking L'Hôpital's rule, we will obtain that $J'' \rightarrow 0$ as $\delta \rightarrow 0$.

In order to estimate J' let us transfer it to the local coordinate system with its origin at the point x and the vector $\nu(x)$ as the ort of its last axis. We have

$$S_\varepsilon(x) = \{u \in \mathbb{R}^d : u^d = F_x(u^{<d>}), u^{<d>} \in D_\varepsilon(x) \subset \mathbb{R}^{d-1}\},$$

$$\Pi_\varepsilon(x) = \{u \in \mathbb{R}^d : u^d = 0, u^{<d>} \in D_\varepsilon(x) \subset \mathbb{R}^{d-1}\},$$

where $D_\varepsilon(x)$ is some bounded closed set depended only on properties of the surface S , $u^{<d>}$ is the vector $(u_1, u_2, \dots, u_{d-1})$, and F_x is some single-valued function with Hölder continuous gradient (see Section 1.1) with power $\gamma \in (0; 1)$. Talking into account inequality (3), it is not difficult to state that

$$\begin{aligned} |J'| &\leq K|\delta| \int_0^\rho \frac{d\tau}{\tau} \int_{D_\varepsilon(x)} \frac{\tau|u|^\gamma(1 + |u|^\gamma)du}{(\tau^{1/\alpha} + k\sqrt{|u|^2 + \delta^2})^{d+\alpha}} \\ &\leq \hat{K}|\delta| \int_0^\rho d\tau \int_0^{\varepsilon_0} \frac{r^{d-2+\gamma}dr}{(\tau^{1/\alpha} + k\sqrt{r^2 + \delta^2})^{d+\alpha}}, \end{aligned}$$

where $K > 0, \hat{K} > 0, k > 0, \varepsilon_0 > 0$ are some constants. Changing the order of integration in the last integral and taking into account the equality $\int_0^\infty \frac{d\tau}{(\tau^{1/\alpha+a})^{d+\alpha}} = \alpha B(d, \alpha)a^{-d}$ that is correct for all $a > 0$, we will obtain the estimation

$$|J'| \leq \tilde{K}|\delta| \int_0^{\varepsilon_0} \frac{r^{d-2+\gamma}dr}{(\sqrt{r^2 + \delta^2})^d} \leq \tilde{K} \int_0^\infty \frac{r^{d-2+\gamma}dr}{(\sqrt{r^2 + 1})^d} |\delta|^\delta$$

with some constant $\tilde{K} > 0$. Therefore $J' \rightarrow 0$ as $\delta \rightarrow 0$ and $\lim_{\delta \rightarrow 0} (J_1^{(2)} - \psi(t, x)R) = 0$. Thus, we have to prove the existence of $\lim_{\delta \rightarrow 0\pm} R$ and to find it. In order to prove this we will take the equality proved in [3] ($\hat{\nu}$ is a fixed ort in \mathbb{R}^d)

$$\int_\Pi h_d(x + \lambda\hat{\nu})d\sigma_x = \frac{1}{\pi} \int_0^\infty e^{-cr^\alpha} \cos \lambda r dr, \tag{8}$$

where $\lambda \in \mathbb{R}^d, \Pi = \{x \in \mathbb{R}^d : (x, \hat{\nu}) = 0\}$ and $h_d(x) = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{i(x;\xi) - c|\xi|^\alpha} d\xi, x \in \mathbb{R}^d$. Therefore, after simple transformations related with the changing of the variables in the surface integral, we will obtain the equality

$$I = \int_{\Pi_x} g(t, x + \delta\nu(x), y)d\sigma_y = t^{-1/\alpha} \int_\Pi h_d(z - \delta t^{-1/\alpha}\hat{\nu})d\sigma_z,$$

where $\Pi = \{z \in \mathbb{R}^d : (z, \hat{\nu}) = 0\}, \hat{\nu}$ is some ort in \mathbb{R}^d . Equality (8) implies

$$I = t^{-1/\alpha} \frac{1}{\pi} \int_0^\infty e^{-cr^\alpha} \cos r\delta t^{-1/\alpha} dr = \frac{1}{\pi} \int_0^\infty e^{-ctr^\alpha} \cos r\delta dr.$$

Thus,

$$R = \frac{2}{\alpha\pi} \delta \int_{t-\rho}^t \frac{d\tau}{t-\tau} \int_0^\infty e^{-c(t-\tau)r^\alpha} \cos \delta r dr.$$

By changing the order of integration in this integral, we obtain the equality

$$R = \text{sign}\delta - \frac{2}{\pi} \int_0^\infty e^{-c\rho r^\alpha} \frac{\sin \delta r}{r} dr$$

and, therefore, we have that $\lim_{\delta \rightarrow 0^\pm} R = \pm 1$. Hence, $\lim_{\delta \rightarrow 0^\pm} J_1^{(2)} = \pm \psi(t, x)$, $t > 0$, $x \in S$ and the theorem has been proved. \square

2.4 The single-layer potential with a hyperplane as a carrier

Let S be a hyperplane defined by the equation $(x, \nu) = r$, where $\nu \in \mathbb{R}^d$ is some unit vector, and $r \in \mathbb{R}$ is a fixed real number. Let the function $(\psi(t, x))_{t \geq 0, x \in S}$ be continuous as mentioned above, and the inequality $|\psi(t, x)| \leq C_T t^{-\beta}$ be true for all $0 < t \leq T$, $x \in S$ and each $T > 0$. Here the constant $C_T > 0$ probably depends on T and $\beta < 1$. Taking into account inequality (3), we obtain the estimation $\int_S g(t, x, y) d\sigma_y \leq Kt^{-1/\alpha}$, $t > 0$, $x \in \mathbb{R}^d$ and for the fixed $x \in \mathbb{R}^d \setminus S$ we have $\int_S g(t, x, y) d\sigma_y \leq Mt$, $t > 0$, where $K > 0$ is some constant, and $M > 0$ is the constant depending on x . This, is analogous to the previous, one can state that the statements of Lemma 1 and Theorem 1 are true in this case (S is a hyperplane) as well.

Furthermore, formula (5) shows that $g_\nu(t, x, y) = 0$ for all $t > 0$, $x \in \mathbb{R}^d$, $y \in \mathbb{R}^d$. Therefore, the equality $\mathbf{B}_{\nu(x)}^{(d\nu)} v(t, \cdot)(x) = 0$, $t > 0$, $x \in S$ holds true. Thus, an analogue of Theorem 2 is valid.

Theorem 3. *Let S be a hyperplane with a unit normal vector $\nu \in \mathbb{R}^d$ and $(\psi(t, x))_{t \geq 0, x \in S}$ be a continuous function satisfying the following inequality $|\psi(t, x)| \leq C_T t^{-\beta}$, $0 < t \leq T$, $x \in S$ with some constants $\beta < 1$ and $C_T > 0$ (the last one can depend on T) for each $T > 0$. Then for all $t > 0$, $x \in S$ the following relations*

$$\lim_{y \rightarrow x^\pm} \mathbf{B}_\nu v(t, \cdot)(y) = \mp \psi(t, x)$$

hold true, where $y \rightarrow x+$, (or $y \rightarrow x-$) means that $y \rightarrow z$ in the way that $(y - x, \nu) > 0$ (or $(y - x, \nu) < 0$).

Proof. The proof of this theorem repeats the proof of Theorem 2 with some simplification. \square

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Мамалыга Х.В., Осипчук М.М. *Потенціали простого шару для одного класу псевдодиференціальних рівнянь пов'язаних з лінійними перетвореннями симетричного α -стійкого випадкового процесу // Карпатські матем. публ. — 2019. — Т.11, №2. — С. 350–360.*

Стаття присвячена дослідженню невідродженого лінійного перетворення симетричного α -стійкого випадкового процесу в евклідовому просторі \mathbb{R}^d . Результат цього перетворення є процесом Маркова в \mathbb{R}^d , чий твірний оператор задається символом $(-(Q\xi, \xi)^{\alpha/2})_{\xi \in \mathbb{R}^d}$ з деякою симетричною додатно визначеною $d \times d$ -матрицею Q та фіксованим $\alpha \in (1, 2)$. Щільність ймовірності переходу цього процесу є фундаментальним розв'язком деякого параболічного псевдодиференціального рівняння. Вводиться поняття потенціалу простого шару та досліджуються його властивості. Зокрема встановлено оператор, який відіграє роль градієнта в класичній теорії. Доведено аналог класичної теореми про стрибок конормальної похідної потенціалу простого шару. Ця властивість потенціалу простого шару може бути використана для побудови розв'язків деяких крайових задач для розглянутих параболічних псевдодиференціальних рівнянь. Якщо $\alpha = 2$, розглянутий процес є лінійним перетворенням процесу броунівського руху і всі досліджені властивості потенціалу простого шару добре відомі.

Ключові слова і фрази: псевдодиференціальне рівняння, потенціал простого шару, α -стійкий випадковий процес, теорема про стрибок.