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NOOR M.A., NOOR K.I., IFTIKHAR S.

SOME INEQUALITIES FOR STRONGLY (p,h)-HARMONIC CONVEX FUNCTIONS

In this paper, we show that harmonic convex functions f is strongly (p,h)-harmonic convex functions if and only if it can be decomposed as $g(x) = f(x) - c(\frac{1}{x^p})^2$, where $g(x)$ is (p,h)-harmonic convex function. We obtain some new estimates class of strongly (p,h)-harmonic convex functions involving hypergeometric and beta functions. As applications of our results, several important special cases are discussed. We also introduce a new class of harmonic convex functions, which is called strongly (p,h)-harmonic log-convex functions. Some new Hermite-Hadamard type inequalities for strongly (p,h)-harmonic log-convex functions are obtained. These results can be viewed as important refinement and significant improvements of the new and previous known results. The ideas and techniques of this paper may stimulate further research.

Key words and phrases: p-harmonic convex functions, h-convex functions, strongly convex functions, Hermite-Hadamard type inequalities.

Department of Mathematics, COMSATS University Islamabad, 45550, Islamabad, Pakistan
E-mail: noormaslam@gmail.com (Noor M.A.), khalidan@gmail.com (Noor K.I.),
sabah.iftikhar22@gmail.com (Iftikhar S.)

1 INTRODUCTION

Inequalities have played an important role in the developments of various fields of pure and applied sciences. Convexity theory an inequalities theory are closely related with each other. It is well known that a function is a convex function if and only if it satisfies the integral inequality which is known as the Hermite-Hadamard inequality. Hermite-Hadmard type inequalities are used to obtain the error bounds for energy functions in the material sciences. For applications and other aspects of these inequalities and their generalized invariant forms, see [3–5, 7, 9, 10, 23, 26, 27].

In recent years, convex functions have been extended and generalized in various directions using novel and innovative techniques. Varosanec [25] introduced and studied a new class of convex functions involving an arbitrary non-negative function $h(.)$, which is known as h-convex function. With an appropriate and suitable choice of arbitrary function $h(\cdot)$, one can obtain a wide class of convex functions. This idea has been used to introduce various classes of convex functions in other fields. Polyak [24] introduced the concept of strongly convex functions, which include the convex functions as special cases. Strongly convex functions played a crucial role in optimization and variational inequalities problem. Motivated and inspired by its applications, Angulo et al. [2] introduced the notion of strongly h-convex functions and have shown that this class unifies other known and new classes of strongly convex functions. The class of strongly beta-convex functions has introduced and investigated by Noor et al. [19]. They obtained some integral inequalities involving hypergeometric and beta functions. The

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harmonic convex functions were introduced and studied by Anderson et al. [1] and Iscan [6]. Noor et al. [15] have introduced a class of strongly harmonic convex functions and established some Hermite-Hadamard type integral inequalities. Noor et al. [11] also introduced the concept of p-harmonic means, which includes the harmonic means, arithmetic means and geometric mean as special cases. Using this concept, they introduced and investigated the properties of p-harmonic convex sets and the p-harmonic convex functions. It have been shown that the p-harmonic convex functions include the harmonic convex functions and convex functions as special cases. For recent developments and generalizations, see [12–14, 16, 17, 20, 21].

Inspired and motivated by the ongoing research, Noor et al. [22] have introduced a concept of strongly (p,h)-harmonic convex functions with respect to an arbitrary non-negative function $h(\cdot)$ and obtained the integral inequalities. This class is more general and contains several known and new classes of convex functions as special cases. In this paper, study those conditions under which a function $f(.)$ is a strongly (p,h)-harmonic convex function, if it can be decomposed as $g(x) = f(x) - (\frac{1}{x^p})^2$, where $g(.)$ is (p,h)-harmonic convex functions. Some new estimates for the integral $\int_a^b (x^p - a^p)^\alpha (b^p - x^p)^\beta f(x)\mathrm{d}x$ in terms of hypergeometric and beta functions are obtained. Some special cases are discussed as applications of these new estimates. In addition, we introduce and study the strongly (p,h)-harmonic log-convex functions, which is quite general and unifying one. Hermite-Hadamard type integral inequalities are obtained. We would like to emphasize that the ideas and technqiues of this paper may stimulate further research in this dynamic field.

2 PRELIMINARIES

In this section, we introduce some new classes of harmonic convex functions. Throughout the paper, we will take $p \in \mathbb{R}$ and $I = [a,b] \subset (0,\infty)$ be an interval, unless otherwise specified.

Definition 1 ([11]). *A set I is said to be a p-harmonic convex set, if*

$$\left[\frac{x^p y^p}{tx^p + (1-t)y^p}\right]^{\frac{1}{p}} \in I, \qquad \forall x, y \in I,\ t \in [0,1].$$

We would like to point out that if $p = 1$, then p-harmonic convex set becomes harmonic convex set. If $p = -1$, then p-harmonic convex set becomes convex set and if $p = 0$, then p-harmonic convex set becomes geometrically convex set. This shows that the concept of p-harmonic convex set is quite general and unifying one.

Definition 2 ([11]). *Let I be a p-harmonic convex set. A function $f : I \to \mathbb{R}$ is said to p-harmonic convex, if*

$$f\left(\left[\frac{x^p y^p}{tx^p + (1-t)y^p}\right]^{\frac{1}{p}}\right) \le (1-t)f(x) + tf(y), \qquad \forall x, y \in I, t \in [0,1].$$

Noor et al. [11] have obtained the Hermite-Hadamard inequality for p-harmonic convex functions, which may be regarded as a refinement of the concept of convexity. In particular, it has been shown that f is a p-harmonic convex function, if and only if,

$$\begin{aligned} f\left(\left[\frac{2a^pb^p}{a^p+b^p}\right]^{\frac{1}{p}}\right) &\leq \frac{1}{2}\left[f\left(\left[\frac{4a^pb^p}{3a^p+b^p}\right]^{\frac{1}{p}}\right)+f\left(\left[\frac{4a^pb^p}{a^p+3b^p}\right]^{\frac{1}{p}}\right)\right] \\ &\leq \frac{pa^pb^p}{b^p-a^p}\int_a^b \frac{f(x)}{x^{1+p}}\mathrm{d}x \\ &\leq \frac{1}{2}\left[f\left(\left[\frac{2a^pb^p}{a^p+b^p}\right]^{\frac{1}{p}}\right)+\frac{f(a)+f(b)}{2}\right] \leq \frac{1}{2}[f(a)+f(b)]. \end{aligned} \tag{1}$$

The inequality (1) holds in reversed direction, if f is a p-harmonic concave function.

Definition 3 ([22]). *Let $h : J = [0,1] \to \mathbb{R}$ be an arbitrary nonnegative function. A function $f : I \to \mathbb{R}$ is said to be strongly (p,h)-harmonic convex function with respect to an arbitrary non-negative function h with modulus $c > 0$, if*

$$f\left(\left[\frac{x^py^p}{tx^p+(1-t)y^p}\right]^{\frac{1}{p}}\right) \leq h(1-t)f(x)+h(t)f(y)-ct(1-t)\left(\frac{x^p-y^p}{x^py^p}\right)^2. \tag{2}$$

The function f is said to be strongly (p,h)-harmonic concave function, if and only if, $-f$ is strongly (p,h)-harmonic convex function. For $t=\frac{1}{2}$ in (2), we have

$$f\left(\frac{2x^py^p}{x^p+y^p}\right)^{\frac{1}{p}} \leq h\left(\frac{1}{2}\right)[f(x)+f(y)]-\frac{c}{4}\left(\frac{x^p-y^p}{x^py^p}\right)^2, \quad x,y\in I. \tag{3}$$

The function f is called Jensen strongly (p,h)-harmonic convex function.

For $h(t)=h(t)h(1-t)$, in Definition 3, we obtain a new class of p-harmonic convex functions, called relative strongly p-harmonic tgs-convex functions.

Definition 4. *Let $h : J = [0,1] \to \mathbb{R}$ be an arbitrary nonnegative function. A function $f : I \to \mathbb{R}$ is said to be relative strongly p-harmonic tgs-convex with respect to an arbitrary non-negative function h with modulus $c > 0$, if*

$$f\left(\left[\frac{x^py^p}{tx^p+(1-t)y^p}\right]^{\frac{1}{p}}\right) \leq h(t)h(1-t)[f(x)+f(y)]-ct(1-t)\left(\frac{x^p-y^p}{x^py^p}\right)^2.$$

Remark 1.
- *(i) If $p=1$ in Definition 3, then it reduces to strongly harmonic h-convex functions introduced by Noor et al.* [18].
- *(ii) If $p=-1$ in Definition 3, then it reduces to strongly h-convex functions* [2].
- *(iii) If $p=0$ in Definition 3, then it reduces to strongly geometrically h-convex functions.*

Definition 5. *Let $h : J = [0,1] \to \mathbb{R}$ be an arbitrary nonnegative function. A function $f : I \to \mathbb{R}$ is said to be strongly geometrically h-convex function with respect to an arbitrary non-negative function h with modulus $c > 0$, if*

$$f\left(x^{1-t}y^t\right) \leq h(1-t)f(x)+h(t)f(y)-ct(1-t)\left(\ln x-\ln y\right)^2.$$

Now we discuss some special cases of strongly (p,h)-harmonic convex functions, which appears to be new ones.

I. If $h(t)=t$ in Definition 3, then it reduces to:

Definition 6. *A function $f : I \to \mathbb{R}$ is said to be strongly p-harmonic convex with modulus $c > 0$, if*

$$f\left(\left[\frac{x^p y^p}{tx^p + (1-t)y^p}\right]^{\frac{1}{p}}\right) \le (1-t)f(x) + tf(y) - ct(1-t)\left(\frac{x^p - y^p}{x^p y^p}\right)^2, \quad \forall x, y \in I, t \in (0,1).$$

II. If $h(t) = t^s$ in Definition 3, then it reduces to:

Definition 7. *A function $f : I \to \mathbb{R}$ is said to be strongly p-harmonic s-convex function in second sense with modulus $c > 0$, where $s \in [-1, 1]$, if*

$$f\left(\left[\frac{x^p y^p}{tx^p + (1-t)y^p}\right]^{\frac{1}{p}}\right) \le (1-t)^s f(x) + t^s f(y) - ct(1-t)\left(\frac{x^p - y^p}{x^p y^p}\right)^2, \quad \forall x, y \in I, t \in (0,1).$$

III. If $h(t) = t^s(1-t)^s$ in Definition 3, then it reduces to:

Definition 8. *A function $f : I \to \mathbb{R}$ is said to be generalized strongly p-harmonic s-convex with modulus $c > 0$, if*

$$f\left(\left[\frac{x^p y^p}{tx^p + (1-t)y^p}\right]^{\frac{1}{p}}\right) \le t^s(1-t)^s[f(x) + f(y) - c\left(\frac{x^p - y^p}{x^p y^p}\right)^2], \quad \forall x, y \in I, t \in (0,1).$$

IV. If $h(t) = t^{\mathfrak{p}}(1-t)^{\mathfrak{q}}$ in Definition 3, then it reduces to the definition of strongly p-harmonic beta-convex functions.

Definition 9. *A function $f : I \to \mathbb{R}$ is said to be strongly p-harmonic beta-convex with modulus $c > 0$, where $\mathfrak{p}, \mathfrak{q} > -1$, if*

$$f\left(\left[\frac{x^p y^p}{tx^p + (1-t)y^p}\right]^{\frac{1}{p}}\right) \le (1-t)^{\mathfrak{p}} t^{\mathfrak{q}} f(x) + t^{\mathfrak{p}}(1-t)^{\mathfrak{q}} f(y) - ct(1-t)\left(\frac{x^p - y^p}{x^p y^p}\right)^2$$
$$\forall x, y \in I, \ t \in (0,1).$$

If $p = 1, -1, 0$, then Definition 9 reduces to the definition of strongly harmonic beta-convex, strongly beta-convex functions and strongly geometrically beta-convex functions, respectively.

Since strongly (p,h)-harmonic convexity is a strengthening of the notion of (p,h)-harmonic convexity, some properties of strongly (p,h)-harmonic convex functions are just stronger version of known properties of (p,h)-harmonic convex functions. Using the technique of Nikodem [8] and Noor et al. [15], we prove the following result which shows the relationships between strongly (p,h)-harmonic convex (strongly (p,h)-harmonic mid-convex) and (p,h)-harmonic convex ((p,h)-harmonic mid-convex) functions.

Lemma 1. *i). Let a function $f : I \subset (0,\infty) \to \mathbb{R}$ be strongly (p,h)-harmonic convex function with modulus c. If $h(t) \le t$, then the function $g(x) = f(x) - c(\frac{1}{x^p})^2$ is (p,h)-harmonic convex.*
ii). Let a function $f : I \subset (0,\infty) \to \mathbb{R}$ be strongly (p,h)-harmonic mid convex with modulus c. If $h(t) \le t$, then the function $g(x) = f(x) - c(\frac{1}{x^p})^2$ is (p,h)-harmonic mid convex function.

Proof. i) Assume that f is strongly (p,h)-harmonic convex with modulus c. Using properties of the inner product, we have

$$
\begin{aligned}
g\left(\left[\frac{x^p y^p}{tx^p+(1-t)y^p}\right]^{\frac{1}{p}}\right) &= f\left(\left[\frac{x^p y^p}{tx^p+(1-t)y^p}\right]^{\frac{1}{p}}\right) - c\left(\frac{tx^p+(1-t)y^p}{x^p y^p}\right)^2 \\
&\leq h(1-t)f(x) + h(t)f(y) - ct(1-t)\left(\frac{x^p-y^p}{x^py^p}\right)^2 - c\left(\frac{tx^p+(1-t)y^p}{x^py^p}\right)^2 \\
&= h(1-t)f(x)+h(t)f(y) - c\Bigg(t(1-t)\left[\left(\frac{1}{x^p}\right)^2 - \frac{2}{x^py^p} + \left(\frac{1}{y^p}\right)^2\right] \\
&\qquad + (1-t)^2\left(\frac{1}{x^p}\right)^2 + \frac{2t(1-t)}{x^py^p} + t^2\left(\frac{1}{y^p}\right)^2\Bigg) \\
&= h(1-t)f(x)+h(t)f(y) - c(1-t)\left(\frac{1}{x^p}\right)^2 - ct\left(\frac{1}{y^p}\right)^2 \\
&\leq h(1-t)f(x)+h(t)f(y) - ch(1-t)\left(\frac{1}{x^p}\right)^2 - ch(t)\left(\frac{1}{y^p}\right)^2 \\
&= h(1-t)g(x)+h(t)g(y),
\end{aligned}
$$

which gives that g is (p,h)-harmonic convex function.
ii) Let f be strongly (p,h)-harmonic mid convex with modulus c. Then

$$
\begin{aligned}
g\left(\left[\frac{2x^py^p}{x^p+y^p}\right]^{\frac{1}{p}}\right) &= f\left(\left[\frac{2x^py^p}{x^p+y^p}\right]^{\frac{1}{p}}\right) - c\left(\frac{x^p+y^p}{2x^py^p}\right)^2 \\
&\leq h\left(\frac{1}{2}\right)[f(x)+f(y)] - \frac{c}{4}\left(\frac{x^p-y^p}{x^py^p}\right)^2 - \frac{c}{4}\left(\frac{x^p+y^p}{x^py^p}\right)^2 \\
&= h\left(\frac{1}{2}\right)[f(x)+f(y)] - \frac{c}{4}\left[2\left(\frac{1}{x^p}\right)^2 + 2\left(\frac{1}{y^p}\right)^2\right] \\
&\leq h\left(\frac{1}{2}\right)[f(x)+f(y)] - ch\left(\frac{1}{2}\right)\left[\left(\frac{1}{x^p}\right)^2 + \left(\frac{1}{y^p}\right)^2\right] \\
&= h\left(\frac{1}{2}\right)[g(x)+g(y)],
\end{aligned}
$$

which gives that g is (p,h)-harmonic mid convex function. □

Remark 2. *Under the condition $h(t) \geq t$, the converse of the Lemma 1 holds.*

The Euler Beta function is a special function defined by

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\mathrm{d}t = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}, \quad \forall x,y>0,$$

where $\Gamma(\cdot) = \int_0^\infty e^{-t}t^{x-1}\mathrm{d}t$ is a gamma function. The integral form of hypergeometric function is defined as:

$$ {}_2F_1[a,b;c;z] = \frac{1}{\beta(b,c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}\mathrm{d}t, $$

where $|z| < 1$, $c > b > 0$.

3 INTEGRAL INEQUALITIES

Some new and interesting estimates of the integral via strongly (p,h)-harmonic convex functions are obtained. These estimates can be viewed as refined bounds of the quadrature formula of Guass-Jacobi type. The quadrature formula of Guass-Jacobi type has the form

$$ \int_a^b (x-a)^\alpha (b-x)^\beta f(x)\mathrm{d}x = \sum_{k=0}^{m} B_{m,k} f(\gamma_k) + R_m[f], $$

for some $B_{m,k}$, γ_k and the remainder term $R_m[f]$.

Lemma 2. *If $f : I \to \mathbb{R}$ is a function such that $f \in L[a,b]$, then the following equality holds for some fixed $\alpha, \beta > 0$.*

$$ \int_a^b (x^p - a^p)^\alpha (b^p - x^p)^\beta f(x)\mathrm{d}x = a^{p(\alpha+1)} b^{p(\beta+1)} (b^p - a^p)^{\alpha+\beta+1} \int_0^1 \frac{t^\alpha (1-t)^\beta}{A_t^{p\alpha+p\beta+p+1}} f\left(\frac{ab}{A_t}\right)\mathrm{d}t, $$

where $A_t = [ta^p + (1-t)b^p]^{\frac{1}{p}}$ and $L[a,b]$ is the space of Lebesque integrable functions on $[a,b]$.

Theorem 1. *If $f : I \to \mathbb{R}$ is a function such that $f \in L[a,b]$ and $|f|$ is strongly (p,h)-harmonic convex function, $\alpha, \beta > 0$, then*

$$ \left| \int_a^b (x^p - a^p)^\alpha (b^p - x^p)^\beta f(x)\mathrm{d}x \right| \\ \leq a^{p(\alpha+1)} b^{p(\beta+1)} (b^p - a^p)^{\alpha+\beta+1} \left(\omega_1 |f(a)| + \omega_2 |f(b)| - c\left(\frac{a^p - b^p}{a^p b^p}\right)^2 \omega_3 \right), $$

where

$$ \omega_1 = \int_0^1 \frac{t^\alpha (1-t)^\beta h(1-t)}{A_t^{\alpha+\beta+2}}\mathrm{d}t, \tag{4} $$

$$ \omega_2 = \int_0^1 \frac{t^\alpha (1-t)^\beta h(t)}{A_t^{p\alpha+p\beta+p+1}}\mathrm{d}t, \tag{5} $$

$$ \begin{aligned} \omega_3 &= \int_0^1 \frac{t^{\alpha+1}(1-t)^{\beta+1}}{A_t^{p\alpha+p\beta+p+1}}\mathrm{d}t \\ &= \frac{B(\alpha+2,\beta+2)}{b^{p\alpha+p\beta+p+1}} {}_2F_1\left[\alpha+\beta+1+1/p, \alpha+2; \alpha+\beta+4, 1-\frac{a^p}{b^p}\right]. \end{aligned} \tag{6} $$

Proof. Using Lemma 2 and strongly (p,h)-harmonic convexity of $|f|$, we have

$$\begin{aligned}
&\left|\int_a^b (x^p-a^p)^\alpha (b^p-x^p)^\beta f(x)\mathrm{d}x\right| \\
&= a^{p(\alpha+1)}b^{p(\beta+1)}(b^p-a^p)^{\alpha+\beta+1}\int_0^1 \frac{t^\alpha(1-t)^\beta}{A_t^{p\alpha+p\beta+p+1}}\left|f\left(\frac{ab}{A_t}\right)\right|\mathrm{d}t \\
&\le a^{p(\alpha+1)}b^{p(\beta+1)}(b^p-a^p)^{\alpha+\beta+1}\int_0^1 \frac{t^\alpha(1-t)^\beta}{A_t^{p\alpha+p\beta+p+1}}\bigg\{h(1-t)|f(a)| \\
&\quad + h(t)|f(b)| - ct(1-t)\left(\frac{a^p-b^p}{a^pb^p}\right)^2\bigg\}\mathrm{d}t \\
&= a^{p(\alpha+1)}b^{p(\beta+1)}(b^p-a^p)^{\alpha+\beta+1}\bigg(|f(a)|\int_0^1 \frac{t^\alpha(1-t)^\beta h(1-t)}{A_t^{p\alpha+p\beta+p+1}}\mathrm{d}t \\
&\quad + |f(b)|\int_0^1 \frac{t^\alpha(1-t)^\beta h(t)}{A_t^{p\alpha+p\beta+p+1}}\mathrm{d}t - c\left(\frac{a^p-b^p}{a^pb^p}\right)^2\int_0^1 \frac{t^{\alpha+1}(1-t)^{\beta+1}}{A_t^{p\alpha+p\beta+p+1}}\mathrm{d}t\bigg) \\
&= a^{p(\alpha+1)}b^{p(\beta+1)}(b^p-a^p)^{\alpha+\beta+1}\left(\omega_1|f(a)|+\omega_2|f(b)|-c\left(\frac{a^p-b^p}{a^pb^p}\right)^2\omega_3\right).
\end{aligned}$$

This completes the proof. □

Corollary 1. *Under the conditions of Theorem* 1 *with* $p=1$ *and* $h(t)=t^{\mathfrak{p}}(1-t)^{\mathfrak{q}}$*, we have*

$$\begin{aligned}
&\left|\int_a^b (x-a)^\alpha (b-x)^\beta f(x)\mathrm{d}x\right| \\
&\le a^{\alpha+1}b^{\beta+1}(b-a)^{\alpha+\beta+1}\left(|f(a)|\omega_1^*+|f(b)|\omega_2^*-c\left(\frac{a-b}{ab}\right)^2\omega_3^*\right),
\end{aligned}$$

where

$$\begin{aligned}
\omega_1^* &= \int_0^1 \frac{t^{\alpha+\mathfrak{q}}(1-t)^{\beta+\mathfrak{p}}}{A_t^{\alpha+\beta+2}}\mathrm{d}t \\
&= \frac{B(\alpha+\mathfrak{q}+1,\beta+\mathfrak{p}+1)}{b^{\alpha+\beta+2}}\ {}_2F_1\left[\alpha+\beta+2,\alpha+\mathfrak{q}+1;\alpha+\beta+\mathfrak{p}+\mathfrak{q}+2,1-\frac{a}{b}\right],
\end{aligned} \tag{7}$$

$$\begin{aligned}
\omega_2^* &= \int_0^1 \frac{t^{\alpha+\mathfrak{p}}(1-t)^{\beta+q}}{A_t^{\alpha+\beta+2}}\mathrm{d}t \\
&= \frac{B(\alpha+\mathfrak{p}+1,\beta+\mathfrak{q}+1)}{b^{\alpha+\beta+2}}\ {}_2F_1\left[\alpha+\beta+2,\alpha+\mathfrak{p}+1;\alpha+\beta+\mathfrak{p}+\mathfrak{q}+2,1-\frac{a}{b}\right],
\end{aligned} \tag{8}$$

$$\begin{aligned}
\omega_3^* &= \int_0^1 \frac{t^{\alpha+1}(1-t)^{\beta+1}}{A_t^{\alpha+\beta+2}}\mathrm{d}t \\
&= \frac{B(\alpha+2,\beta+2)}{b^{\alpha+\beta+2}}\ {}_2F_1\left[\alpha+\beta+2,\alpha+2;\alpha+\beta+4,1-\frac{a}{b}\right].
\end{aligned} \tag{9}$$

Theorem 2. *If* $f: I\to\mathbb{R}$ *is a function such that* $f\in L[a,b]$ *and* $|f|^q$ *is strongly* (p,h)*-harmonic convex function,* $\alpha,\beta>0$*,* $q\ge 1$*, then*

$$\begin{aligned}
&\left|\int_a^b (x^p-a^p)^\alpha (b^p-x^p)^\beta f(x)\mathrm{d}x\right| \\
&\le a^{p(\alpha+1)}b^{p(\beta+1)}(b^p-a^p)^{\alpha+\beta+1}(\omega_4)^{1-\frac{1}{q}}\left(|f(a)|^q\omega_1+|f(b)|^q\omega_2-c\left(\frac{a^p-b^p}{a^pb^p}\right)^2\omega_3\right)^{\frac{1}{q}},
\end{aligned}$$

where ω_1, ω_2 *and* ω_3 *are given by* (4), (5) *and* (6) *respectively and*

$$\begin{aligned}\omega_4 &= \int_0^1 \frac{t^{\alpha}(1-t)^{\beta}}{A_t^{p\alpha+p\beta+p+1}}\mathrm{d}t \\ &= \frac{B(\alpha+1,\beta+1)}{b^{p\alpha+p\beta+p+1}}\,{}_2F_1\Big[\alpha+\beta+1+1/p,\alpha+1;\alpha+\beta+2,1-\frac{a^p}{b^p}\Big].\end{aligned}$$

Proof. Using Lemma 2, strongly (p,h)-harmonic convexity of $|f|^{\mathsf{q}}$ and power mean inequality, we have

$$\begin{aligned}&\Big|\int_a^b (x^p-a^p)^{\alpha}(b^p-x^p)^{\beta}f(x)\mathrm{d}x\Big| \\ &= a^{p(\alpha+1)}b^{p(\beta+1)}(b^p-a^p)^{\alpha+\beta+1}\int_0^1 \frac{t^{\alpha}(1-t)^{\beta}}{A_t^{p\alpha+p\beta+p+1}}\Big|f\Big(\frac{ab}{A_t}\Big)\Big|\mathrm{d}t \\ &\le a^{p(\alpha+1)}b^{p(\beta+1)}(b^p-a^p)^{\alpha+\beta+1}\Big(\int_0^1 \frac{t^{\alpha}(1-t)^{\beta}}{A_t^{p\alpha+p\beta+p+1}}\mathrm{d}t\Big)^{1-\frac{1}{\mathsf{q}}}\Big(\int_0^1 \frac{t^{\alpha}(1-t)^{\beta}}{A_t^{p\alpha+p\beta+p+1}}\Big|f\Big(\frac{ab}{A_t}\Big)\Big|^{\mathsf{q}}\mathrm{d}t\Big)^{\frac{1}{\mathsf{q}}} \\ &\le a^{p(\alpha+1)}b^{p(\beta+1)}(b^p-a^p)^{\alpha+\beta+1}\Big(\int_0^1 \frac{t^{\alpha}(1-t)^{\beta}}{A_t^{p\alpha+p\beta+p+1}}\mathrm{d}t\Big)^{1-\frac{1}{\mathsf{q}}} \\ &\quad \Big(\int_0^1 \frac{t^{\alpha}(1-t)^{\beta}}{A_t^{p\alpha+p\beta+p+1}}\Big\{h(1-t)|f(a)|^{\mathsf{q}}+h(t)|f(b)|^{\mathsf{q}}-ct(1-t)\Big(\frac{a^p-b^p}{a^pb^p}\Big)^2\Big\}\mathrm{d}t\Big)^{\frac{1}{\mathsf{q}}} \\ &= a^{p(\alpha+1)}b^{p(\beta+1)}(b^p-a^p)^{\alpha+\beta+1}\Big(\int_0^1 \frac{t^{\alpha}(1-t)^{\beta}}{A_t^{p\alpha+p\beta+p+1}}\mathrm{d}t\Big)^{1-\frac{1}{\mathsf{q}}} \\ &\quad \Big(|f(a)|^{\mathsf{q}}\int_0^1 \frac{t^{\alpha}(1-t)^{\beta}h(1-t)}{A_t^{p\alpha+p\beta+1+p}}\mathrm{d}t+|f(b)|^{\mathsf{q}}\int_0^1 \frac{t^{\alpha}(1-t)^{\beta}h(t)}{A_t^{p\alpha+p\beta+p+1}}\mathrm{d}t \\ &\quad -c\Big(\frac{a^p-b^p}{a^pb^p}\Big)^2\int_0^1 \frac{t^{\alpha+1}(1-t)^{\beta+1}}{A_t^{p\alpha+p\beta+p+1}}\mathrm{d}t\Big)^{\frac{1}{\mathsf{q}}} \\ &= a^{p(\alpha+1)}b^{p(\beta+1)}(b^p-a^p)^{\alpha+\beta+1}(\omega_4)^{1-\frac{1}{\mathsf{q}}}\Big(|f(a)|^{\mathsf{q}}\omega_1+|f(b)|^{\mathsf{q}}\omega_2-c\Big(\frac{a^p-b^p}{a^pb^p}\Big)^2\omega_3\Big)^{\frac{1}{\mathsf{q}}},\end{aligned}$$

which is the required result. □

Corollary 2. *Under the conditions of Theorem* 2 *with* $p=1$ *and* $h(t)=t^{\mathsf{p}}(1-t)^{\mathsf{q}}$*, we have*

$$\begin{aligned}&\Big|\int_a^b (x-a)^{\alpha}(b-x)^{\beta}f(x)\mathrm{d}x\Big| \\ &\qquad \le a^{\alpha+1}b^{\beta+1}(b-a)^{\alpha+\beta+1}(\omega_4^*)^{1-\frac{1}{q}}\Big(|f(a)|^q\omega_1^*+|f(b)|^q\omega_2^*-c\Big(\frac{a-b}{ab}\Big)^2\omega_3^*\Big)^{\frac{1}{q}},\end{aligned}$$

where ω_1^*, ω_2^* *and* ω_3^* *are given by* (7), (8) *and* (9) *respectively, and*

$$\begin{aligned}\omega_4^* &= \int_0^1 \frac{t^{\alpha}(1-t)^{\beta}}{A_t^{\alpha+\beta+2}}\mathrm{d}t \\ &= \frac{B(\alpha+1,\beta+1)}{b^{\alpha+\beta+2}}\,{}_2F_1\Big[\alpha+\beta+2,\alpha+1;\alpha+\beta+2,1-\frac{a}{b}\Big].\end{aligned}$$

Theorem 3. *If $f : I \to \mathbb{R}$ is a function such that $f \in L[a,b]$ and $|f|^q$ is strongly (p,h)-harmonic convex function, $\alpha, \beta > 0$, then*

$$\left| \int_a^b (x^p - a^p)^\alpha (b^p - x^p)^\beta f(x) \mathrm{d}x \right|$$
$$\leq a^{p(\alpha+1)} b^{p(\beta+1)} (b^p - a^p)^{\alpha+\beta+1} (\omega_5)^{\frac{1}{r}} \left([|f(a)|^q + |f(b)|^q] \int_0^1 h(t) \mathrm{d}t - \frac{c}{6} \left(\frac{a^p - b^p}{a^p b^p} \right)^2 \right)^{\frac{1}{q}},$$

where $r, q > 1$, $\frac{1}{r} + \frac{1}{q} = 1$ and

$$\omega_5 = \int_0^1 \frac{t^{\alpha r}(1-t)^{\beta r}}{A_t^{(p\alpha+p\beta+p+1)r}} \mathrm{d}t$$
$$= \frac{B(\alpha r+1, \beta r+1)}{b^{(p\alpha+p\beta+p+1)r}} \, {}_2F_1[(\alpha+\beta+1+1/p)r, \alpha r+1; (\alpha+\beta)r+2, 1-\frac{a^p}{b^p}].$$

Proof. Using Lemma 2, strongly (p,h)-harmonic convexity of $|f|^q$ and the Hölder's integral inequality, we have

$$\left| \int_a^b (x^p - a^p)^\alpha (b^p - x^p)^\beta f(x) \mathrm{d}x \right|$$
$$= a^{p(\alpha+1)} b^{p(\beta+1)} (b^p - a^p)^{\alpha+\beta+1} \int_0^1 \frac{t^\alpha (1-t)^\beta}{A_t^{p\alpha+p\beta+p+1}} \left| f\left(\frac{ab}{A_t}\right) \right| \mathrm{d}t$$
$$\leq a^{p(\alpha+1)} b^{p(\beta+1)} (b^p - a^p)^{\alpha+\beta+1} \left(\int_0^1 \frac{t^{\alpha r}(1-t)^{\beta r}}{A_t^{(p\alpha+p\beta+p+1)r}} \mathrm{d}t \right)^{\frac{1}{r}} \left(\int_0^1 \left| f\left(\frac{ab}{A_t}\right) \right|^q \mathrm{d}t \right)^{\frac{1}{q}}$$
$$\leq a^{p(\alpha+1)} b^{p(\beta+1)} (b^p - a^p)^{\alpha+\beta+1} \left(\int_0^1 \frac{t^{\alpha r}(1-t)^{\beta r}}{A_t^{(p\alpha+p\beta+p+1)r}} \mathrm{d}t \right)^{\frac{1}{r}}$$
$$\left(\int_0^1 \left\{ h(1-t)|f(a)|^q + h(t)|f(b)|^q - ct(1-t) \left(\frac{a^p - b^p}{a^p b^p} \right)^2 \right\} \mathrm{d}t \right)^{\frac{1}{q}}$$
$$= a^{p(\alpha+1)} b^{p(\beta+1)} (b^p - a^p)^{\alpha+\beta+1} (\omega_5)^{\frac{1}{r}} \left([|f(a)|^q + |f(b)|^q] \int_0^1 h(t) \mathrm{d}t - \frac{c}{6} \left(\frac{a^p - b^p}{a^p b^p} \right)^2 \right)^{\frac{1}{q}}.$$

This completes the proof. □

Corollary 3. *Under the conditions of Theorem 3 with $p = 1$ and $h(t) = t^p(1-t)^q$, we have*

$$\left| \int_a^b (x-a)^\alpha (b-x)^\beta f(x) \mathrm{d}x \right|$$
$$\leq a^{\alpha+1} b^{\beta+1} (b-a)^{\alpha+\beta+1} (\omega_5)^{\frac{1}{r}} \left([|f(a)|^q + |f(b)|^q] \int_0^1 h(t) \mathrm{d}t - \frac{c}{6} \left(\frac{a-b}{ab} \right)^2 \right)^{\frac{1}{q}},$$

where

$$\omega_5^* = \int_0^1 \frac{t^{\alpha r}(1-t)^{\beta r}}{A_t^{(\alpha+\beta+2)r}} \mathrm{d}t$$
$$= \frac{B(\alpha r+1, \beta r+1)}{b^{(\alpha+\beta+2)r}} \, {}_2F_1\left[(\alpha+\beta+2)r, \alpha r+1; (\alpha+\beta)r+2, 1-\frac{a}{b} \right].$$

Theorem 4. *If $f : I \to \mathbb{R}$ is a function such that $f \in L[a,b]$ and $|f|^q$ is strongly (p,h)-harmonic convex function, $\alpha, \beta > 0$, then*

$$\left| \int_a^b (x^p - a^p)^\alpha (b^p - x^p)^\beta f(x) \mathrm{d}x \right| \leq a^{p(\alpha+1)} b^{p(\beta+1)} (b^p - a^p)^{\alpha+\beta+1} B^{\frac{1}{r}}(\alpha r + 1, \beta r + 1)$$
$$\times \left(|f(a)|^q \omega_6 + |f(b)|^q \omega_7 - c \left(\frac{a^p - b^p}{a^p b^p} \right)^2 \omega_8 \right)^{\frac{1}{q}},$$

where $r, q > 1$, $\frac{1}{r} + \frac{1}{q} = 1$ and

$$\omega_6 = \int_0^1 \frac{h(1-t)}{A_t^{(p\alpha+p\beta+p+1)q}} \mathrm{d}t,$$
$$\omega_7 = \int_0^1 \frac{h(t)}{A_t^{(p\alpha+p\beta+p+1)q}} \mathrm{d}t,$$
$$\omega_8 = \int_0^1 \frac{t(1-t)}{A_t^{(p\alpha+p\beta+p+1)q}} \mathrm{d}t = \frac{1}{6b^{(p\alpha+p\beta+p+1)q}} \, {}_2F_1\left[(\alpha+\beta+1+1/p)q, 2; 4, 1 - \frac{a^p}{b^p} \right].$$

Proof. Using Lemma 2, strongly (p,h)-harmonic convexity of $|f|^q$ and the Hölder's integral inequality, we have

$$\left| \int_a^b (x^p - a^p)^\alpha (b^p - x^p)^\beta f(x) \mathrm{d}x \right| = a^{p(\alpha+1)} b^{p(\beta+1)} (b^p - a^p)^{\alpha+\beta+1} \int_0^1 \frac{t^\alpha (1-t)^\beta}{A_t^{p\alpha+p\beta+p+1}} \left| f\left(\frac{ab}{A_t} \right) \right| \mathrm{d}t$$
$$\leq a^{p(\alpha+1)} b^{p(\beta+1)} (b^p - a^p)^{\alpha+\beta+1} \left(\int_0^1 t^{\alpha r} (1-t)^{\beta r} \mathrm{d}t \right)^{\frac{1}{r}} \left(\int_0^1 \frac{1}{A_t^{(p\alpha+p\beta+p+1)q}} \left| f\left(\frac{ab}{A_t} \right) \right|^q \mathrm{d}t \right)^{\frac{1}{q}}$$
$$\leq a^{p(\alpha+1)} b^{p(\beta+1)} (b^p - a^p)^{\alpha+\beta+1} \left(\int_0^1 t^{\alpha r} (1-t)^{\beta r} \mathrm{d}t \right)^{\frac{1}{r}}$$
$$\left(\int_0^1 \frac{1}{A_t^{(p\alpha+p\beta+p+1)q}} \left\{ h(1-t)|f(a)|^q + h(t)|f(b)|^q - ct(1-t) \left(\frac{a^p - b^p}{a^p b^p} \right)^2 \right\} \mathrm{d}t \right)^{\frac{1}{q}}$$
$$= a^{p(\alpha+1)} b^{p(\beta+1)} (b^p - a^p)^{\alpha+\beta+1} \left(\int_0^1 t^{\alpha r} (1-t)^{\beta r} \mathrm{d}t \right)^{\frac{1}{r}}$$
$$\left(|f(a)|^q \int_0^1 \frac{h(1-t)}{A_t^{(p\alpha+p\beta+p+1)q}} \mathrm{d}t + |f(b)|^q \int_0^1 \frac{h(t)}{A_t^{(p\alpha+p\beta+p+1)q}} \mathrm{d}t \right.$$
$$\left. - c \left(\frac{a^p - b^p}{a^p b^p} \right)^2 \int_0^1 \frac{t(1-t)}{A_t^{(p\alpha+p\beta+p+1)q}} \right)^{\frac{1}{q}}$$
$$= a^{p(\alpha+1)} b^{p(\beta+1)} (b^p - a^p)^{\alpha+\beta+1} B^{\frac{1}{r}}(\alpha r + 1, \beta r + 1) \left(|f(a)|^q \omega_6 + |f(b)|^q \omega_7 - c \left(\frac{a^p - b^p}{a^p b^p} \right)^2 \omega_8 \right)^{\frac{1}{q}}.$$

This completes the proof. □

Corollary 4. *Under the conditions of Theorem 4 with $p = 1$ and $h(t) = t^{\mathfrak{p}}(1-t)^{\mathfrak{q}}$, we have*

$$\left| \int_a^b (x-a)^\alpha (b-x)^\beta f(x) \mathrm{d}x \right|$$
$$\leq a^{\alpha+1} b^{\beta+1} (b-a)^{\alpha+\beta+1} B^{\frac{1}{r}}(\alpha r + 1, \beta r + 1) \left(|f(a)|^q \omega_6^* + |f(b)|^q \omega_7^* - c \left(\frac{a-b}{ab} \right)^2 \omega_8^* \right)^{\frac{1}{q}},$$

where

$$\omega_6^* = \int_0^1 \frac{t^{\mathfrak{q}}(1-t)^{\mathfrak{p}}}{A_t^{(\alpha+\beta+2)q}}\mathrm{d}t = \frac{B(\mathfrak{p}+1,\mathfrak{q}+1)}{b^{(\alpha+\beta+2)q}}\, {}_2F_1\Big[(\alpha+\beta+2)q, \mathfrak{q}+1; \mathfrak{p}+\mathfrak{q}+2, 1-\frac{a}{b}\Big],$$

$$\omega_7^* = \int_0^1 \frac{t^{\mathfrak{p}}(1-t)^{\mathfrak{q}}}{A_t^{(\alpha+\beta+2)q}}\mathrm{d}t = \frac{B(\mathfrak{p}+1,\mathfrak{q}+1)}{b^{(\alpha+\beta+2)q}}\, {}_2F_1\Big[(\alpha+\beta+2)q, \mathfrak{p}+1; \mathfrak{p}+\mathfrak{q}+2, 1-\frac{a}{b}\Big],$$

$$\omega_8^* = \int_0^1 \frac{t(1-t)}{A_t^{(\alpha+\beta+2)q}}\mathrm{d}t = \frac{1}{6b^{(\alpha+\beta+2)q}}\, {}_2F_1\Big[(\alpha+\beta+2)q, 2; 4, 1-\frac{a}{b}\Big].$$

Theorem 5. *If $f : I \to \mathbb{R}$ is a function such that $f \in L[a,b]$ and $|f|^{\mathfrak{q}}$ is strongly (p,h)-harmonic convex function, $\alpha, \beta > 0$, then*

$$\left|\int_a^b (x^p-a^p)^\alpha(b^p-x^p)^\beta f(x)\mathrm{d}x\right| \le a^{p(\alpha+1)}b^{p(\beta+1)}(b^p-a^p)^{\alpha+\beta+1}(\omega_9)^{\frac{1}{r}}$$
$$\left(|f(a)|^{q}\omega_{10} + |f(b)|^{q}\omega_{11} - c\left(\frac{a^p-b^p}{a^pb^p}\right)^2 B(\alpha q+2, \beta q+2)\right)^{\frac{1}{q}},$$

where $r, q > 1$, $\frac{1}{r}+\frac{1}{q} = 1$ and

$$\omega_9 = \int_0^1 \frac{1}{A_t^{(p\alpha+p\beta+p+1)r}}\mathrm{d}t = \frac{{}_2F_1[(\alpha+\beta+1+1/p)r, 1; 2, 1-\frac{a^p}{b^p}]}{b^{(p\alpha+p\beta+p+1)r}},$$

$$\omega_{10} = \int_0^1 t^{\alpha\mathfrak{q}}(1-t)^{\beta\mathfrak{q}}h(1-t)\mathrm{d}t, \qquad \omega_{11} = \int_0^1 t^{\alpha\mathfrak{q}}(1-t)^{\beta\mathfrak{q}}h(t)\mathrm{d}t.$$

Proof. Using Lemma 2, strongly (p,h)-harmonic convexity of $|f|^{\mathfrak{q}}$ and the Hölder's integral inequality, we have

$$\left|\int_a^b (x^p-a^p)^\alpha(b^p-x^p)^\beta f(x)\mathrm{d}x\right| = a^{p(\alpha+1)}b^{p(\beta+1)}(b^p-a^p)^{\alpha+\beta+1}\int_0^1 \frac{t^\alpha(1-t)^\beta}{A_t^{p\alpha+p\beta+p+1}}\left|f\left(\frac{ab}{A_t}\right)\right|\mathrm{d}t$$

$$\le a^{p(\alpha+1)}b^{p(\beta+1)}(b^p-a^p)^{\alpha+\beta+1}\left(\int_0^1 \frac{1}{A_t^{(p\alpha+p\beta+p+1)\mathrm{r}}}\mathrm{d}t\right)^{\frac{1}{\mathrm{r}}}\left(\int_0^1 t^{\alpha\mathfrak{q}}(1-t)^{\beta\mathfrak{q}}\left|f\left(\frac{ab}{A_t}\right)\right|^{\mathfrak{q}}\mathrm{d}t\right)^{\frac{1}{\mathfrak{q}}}$$

$$\le a^{p(\alpha+1)}b^{p(\beta+1)}(b^p-a^p)^{\alpha+\beta+1}\left(\int_0^1 \frac{1}{A_t^{(p\alpha+p\beta+p+1)\mathrm{r}}}\mathrm{d}t\right)^{\frac{1}{\mathrm{r}}}$$
$$\left(\int_0^1 t^{\alpha\mathfrak{q}}(1-t)^{\beta\mathfrak{q}}\left\{h(1-t)|f(a)|^{\mathfrak{q}} + h(t)|f(b)|^{\mathfrak{q}} - ct(1-t)\left(\frac{a^p-b^p}{a^pb^p}\right)^2\right\}\mathrm{d}t\right)^{\frac{1}{\mathfrak{q}}}$$

$$= a^{p(\alpha+1)}b^{p(\beta+1)}(b^p-a^p)^{\alpha+\beta+1}\left(\int_0^1 \frac{1}{A_t^{(p\alpha+p\beta+p+1)\mathrm{r}}}\mathrm{d}t\right)^{\frac{1}{\mathrm{r}}}$$
$$\Bigg(|f(a)|^{\mathfrak{q}}\int_0^1 t^{\alpha\mathfrak{q}}(1-t)^{\beta\mathfrak{q}}h(1-t)\mathrm{d}t + |f(b)|^{\mathfrak{q}}\int_0^1 t^{\alpha\mathfrak{q}}(1-t)^{\beta\mathfrak{q}}h(t)\mathrm{d}t$$
$$- c\left(\frac{a^p-b^p}{a^pb^p}\right)^2\int_0^1 t^{\alpha\mathfrak{q}+1}(1-t)^{\beta\mathfrak{q}+1}\mathrm{d}t\Bigg)^{\frac{1}{\mathfrak{q}}}$$

$$= a^{p(\alpha+1)}b^{p(\beta+1)}(b^p-a^p)^{\alpha+\beta+1}(\omega_9)^{\frac{1}{\mathrm{r}}}$$
$$\left(|f(a)|^{\mathfrak{q}}\omega_{10} + |f(b)|^{\mathfrak{q}}\omega_{11} - c\left(\frac{a^p-b^p}{a^pb^p}\right)^2 B(\alpha\mathfrak{q}+2, \beta\mathfrak{q}+2)\right)^{\frac{1}{\mathfrak{q}}}.$$

This completes the proof. □

Corollary 5. *Under the conditions of Theorem* 5 *with* $p = 1$ *and* $h(t) = t^{\mathfrak{p}}(1-t)^{\mathfrak{q}}$, *we have*

$$\left| \int_a^b (x-a)^{\alpha}(b-x)^{\beta} f(x)\mathrm{d}x \right|$$
$$\leq a^{\alpha+1}b^{\beta+1}(b-a)^{\alpha+\beta+1}(\omega_9)^{\frac{1}{r}}\bigg(|f(a)|^q B(\alpha q+\mathfrak{q}+1,\beta q+\mathfrak{p}+1)$$
$$+ |f(b)|^q B(\alpha q+\mathfrak{p}+1,\beta q+\mathfrak{q}+1) - c\left(\frac{a-b}{ab}\right)^2 B(\alpha q+2,\beta q+2)\bigg)^{\frac{1}{q}},$$

where

$$\omega_9^* = \int_0^1 \frac{1}{A_t^{(\alpha+\beta+2)r}}\mathrm{d}t = \frac{{}_2F_1[(\alpha+\beta+2)r,1;2,1-\frac{a}{b}]}{b^{(\alpha+\beta+2)r}}.$$

Remark 3. *For* $p = -1$ *and* $h(t) = t^{\mathfrak{p}}(1-t)^{\mathfrak{q}}$, *our results reduces to the previously known results obtained by Noor et al.* [19] *for strongly beta-convex functions.*

4 STRONGLY p-HARMONIC log-CONVEX FUNCTIONS

In this section, we define the class of strongly (p,h)-harmonic log-convex functions and obtain the integral inequalities.

Definition 10. *Let* $h : J = [0,1] \to \mathbb{R}$ *a nonnegative function. A function* $f : I \to (0,\infty)$ *is said to be strongly* (p,h)*-harmonic* log*-convex function with modulus* $c > 0$, *if*

$$f\left(\left[\frac{x^p y^p}{tx^p+(1-t)y^p}\right]^{\frac{1}{p}}\right) \leq \big(f(x)\big)^{h(1-t)}\big(f(y)\big)^{h(t)} - ct(1-t)\left(\frac{x^p-y^p}{x^p y^p}\right)^2.$$

For $t = \frac{1}{2}$, we have

$$f\left(\left[\frac{2x^p y^p}{x^p+y^p}\right]^{\frac{1}{p}}\right) \leq \big(f(x)f(y)\big)^{h(\frac{1}{2})} - \frac{c}{4}\left(\frac{x^p-y^p}{x^p y^p}\right)^2, \qquad \forall x,y \in I.$$

The function f is called Jensen type strongly (p,h)-harmonic log-convex function.

Now we discuss some special cases of Definition 10.

I. If $p = 1$, then Definition 10 reduces to:

Definition 11. *Let* $h : J = [0,1] \to \mathbb{R}$ *a nonnegative function. A function* $f : I \to (0,\infty)$ *is said to be strongly* h*-harmonic* log*-convex function with modulus* $c > 0$, *if*

$$f\left(\frac{xy}{tx+(1-t)y}\right) \leq \big(f(x)\big)^{h(1-t)}\big(f(y)\big)^{h(t)} - ct(1-t)\left(\frac{x-y}{xy}\right)^2.$$

II. If $p = -1$, then Definition 10 reduces to:

Definition 12. *Let* $h : J = [0,1] \to \mathbb{R}$ *a nonnegative function. A function* $f : I \to (0,\infty)$ *is said to be strongly* h-log*-convex function with modulus* $c > 0$, *if*

$$f((1-t)x+ty) \leq \big(f(x)\big)^{h(1-t)}\big(f(y)\big)^{h(t)} - ct(1-t)\big(x-y\big)^2.$$

III. If $p=0$, then Definition 10 reduces to:

Definition 13. *Let $h: J=[0,1]\to\mathbb{R}$ a nonnegative function. A function $f: I\to(0,\infty)$ is said to be strongly h-geometrically* log*-convex function with modulus $c>0$, if*

$$f\left(x^{1-t}y^{t}\right)\le\left(f(x)\right)^{h(1-t)}\left(f(y)\right)^{h(t)}-ct(1-t)\left(\ln x-\ln y\right)^{2}.$$

We now consider the following definitions of special means, which are used in our coming results. For arbitrary $a,b(a\neq b)\in(0,\infty)$, we have

1) the arithmetic mean

$$A(a,b)=\frac{a+b}{2};$$

2) the generalized logarithmic mean

$$L_{\rho}(a,b)=\begin{cases}\left[\frac{b^{\rho+1}-a^{\rho+1}}{(\rho+1)(b-a)}\right]^{\frac{1}{\rho}}, & \rho\neq-1,0,\\ \frac{a-b}{\log a-\log b}, & a\neq b,\ \rho=-1,\\ \frac{1}{e}\left(\frac{b^{b}}{a^{a}}\right)^{\frac{1}{b-a}}, & \rho=0.\end{cases}$$

Under the assumptions of Definition 10 with $h(t)=t$, we obtain the Hermite-Hadamard inequalities for strongly p-harmonic log-convex functions.

Theorem 6. *Let $f: I\to(0,\infty)$ be strongly p-harmonic* log*-convex function with modulus $c>0$. If $f\in L[a,b]$, then*

$$\frac{a^{p}b^{p}}{b^{p}-a^{p}}\int_{a}^{b}\frac{f(x)}{x^{1+p}}\mathrm{d}x\le\frac{f(a)+f(b)}{2}-\frac{c}{6}\left(\frac{a^{p}-b^{p}}{a^{p}b^{p}}\right)^{2}.$$

Proof. Let f be strongly p-harmonic log-convex function with modulus $c>0$. Then

$$\begin{aligned}\frac{a^{p}b^{p}}{b^{p}-a^{p}}\int_{a}^{b}\frac{f(x)}{x^{1+p}}\mathrm{d}x &= \int_{0}^{1}f\left(\left[\frac{a^{p}b^{p}}{ta^{p}+(1-t)b^{p}}\right]^{\frac{1}{p}}\right)\mathrm{d}t\\ &\le \int_{0}^{1}\left[\left(f(a)\right)^{1-t}\left(f(b)\right)^{t}-ct(1-t)\left(\frac{a^{p}-b^{p}}{a^{p}b^{p}}\right)^{2}\right]\mathrm{d}t\\ &= f(a)\int_{0}^{1}\left(\frac{f(b)}{f(a)}\right)^{t}\mathrm{d}t-\int_{0}^{1}ct(1-t)\left(\frac{a^{p}-b^{p}}{a^{p}b^{p}}\right)^{2}\mathrm{d}t\\ &= \frac{f(b)-f(a)}{\ln f(b)-\ln f(a)}-\frac{c}{6}\left(\frac{a^{p}-b^{p}}{a^{p}b^{p}}\right)^{2}\\ &= L(f(a),f(b))-\frac{c}{6}\left(\frac{a^{p}-b^{p}}{a^{p}b^{p}}\right)^{2}\\ &\le \frac{f(a)+f(b)}{2}-\frac{c}{6}\left(\frac{a^{p}-b^{p}}{a^{p}b^{p}}\right)^{2}.\end{aligned}$$

□

Theorem 7. *Let $f, g : I \to (0,\infty)$ be strongly p-harmonic* log*-convex functions with modulus $c > 0$. If $f, g \in L[a,b]$, then*

$$\begin{aligned}
&\frac{a^pb^p}{b^p-a^p}\int_a^b \frac{f(x)g\big([\frac{a^pb^px^p}{(a^p+b^p)x^p-a^pb^p}]^{\frac{1}{p}}\big)}{x^{1+p}}\mathrm{d}x\\
&\leq \alpha\frac{f(a)+f(b)}{2}\Big[L_{(\frac{1}{\alpha}-1)}(f(b),f(a))\Big]^{\frac{\alpha}{1-\alpha}}+\beta\frac{g(a)+g(b)}{2}\Big[L_{(\frac{1}{\beta}-1)}(g(b),g(a))\Big]^{\frac{\beta}{1-\beta}}\\
&\quad-\frac{2c\big(\frac{a^p-b^p}{a^pb^p}\big)^2}{[\ln(f(b))-\ln(f(a))]^2}\big[A(f(a),f(b))-L(f(a),f(b))\big]\\
&\quad-\frac{2c\big(\frac{a^p-b^p}{a^pb^p}\big)^2}{[\ln(g(b))-\ln(g(a))]^2}\big[A(g(a),g(b))-L(g(a),g(b))\big]+\frac{c^2\big(\frac{a^p-b^p}{a^pb^p}\big)^4}{30}.
\end{aligned}$$

Proof. Let f, g be strongly p-harmonic log-convex functions with modulus $c > 0$. Then

$$\begin{aligned}
&\frac{a^pb^p}{b^p-a^p}\int_a^b \frac{f(x)g\big([\frac{a^pb^px^p}{(a^p+b^p)x^p-a^pb^p}]^{\frac{1}{p}}\big)}{x^{1+p}}\mathrm{d}x=\int_0^1 f\bigg(\Big[\frac{a^pb^p}{ta^p+(1-t)b^p}\Big]^{\frac{1}{p}}\bigg)g\bigg(\Big[\frac{a^pb^p}{(1-t)a^p+tb^p}\Big]^{\frac{1}{p}}\bigg)\mathrm{d}t\\
&\leq\int_0^1\Big[\big(f(a)\big)^{1-t}\big(f(b)\big)^t-ct(1-t)\big(\frac{a^p-b^p}{a^pb^p}\big)^2\Big]\Big[\big(g(a)\big)^t\big(g(b)\big)^{1-t}-ct(1-t)\big(\frac{a^p-b^p}{a^pb^p}\big)^2\Big]\mathrm{d}t\\
&=\int_0^1\big(f(a)\big)^{1-t}\big(f(b)\big)^t\big(g(a)\big)^t\big(g(b)\big)^{1-t}\mathrm{d}t-c\big(\frac{a^p-b^p}{a^pb^p}\big)^2f(a)\int_0^1 t(1-t)\bigg(\frac{f(b)}{f(a)}\bigg)^t\mathrm{d}t\\
&\quad-c\big(\frac{a^p-b^p}{a^pb^p}\big)^2g(b)\int_0^1 t(1-t)\bigg(\frac{g(a)}{g(b)}\bigg)^t\mathrm{d}t+c^2\big(\frac{a^p-b^p}{a^pb^p}\big)^4\int_0^1 t^2(1-t)^2\mathrm{d}t\\
&\leq\int_0^1\alpha\big[\big(f(a)\big)^{1-t}\big(f(b)\big)^t\big]^{\frac{1}{\alpha}}+\beta\big[\big(g(a)\big)^t\big(g(b)\big)^{1-t}\big]^{\frac{1}{\beta}}\mathrm{d}t-\frac{c\big(\frac{a^p-b^p}{a^pb^p}\big)^2f(a)}{\ln[\frac{f(b)}{f(a)}]}\int_0^1(2t-1)\bigg(\frac{f(b)}{f(a)}\bigg)^t\mathrm{d}t\\
&\quad-\frac{c\big(\frac{a^p-b^p}{a^pb^p}\big)^2g(b)}{\ln[\frac{g(a)}{g(b)}]}\int_0^1(2t-1)\bigg(\frac{g(a)}{g(b)}\bigg)^t\mathrm{d}t+\frac{c^2\big(\frac{a^p-b^p}{a^pb^p}\big)^4}{30}\\
&=\alpha^2\big(f(a)\big)^{\frac{1}{\alpha}}\int_0^{\frac{1}{\alpha}}\bigg(\frac{f(b)}{f(a)}\bigg)^w\mathrm{d}w+\beta^2\big(g(b)\big)^{\frac{1}{\beta}}\int_0^{\frac{1}{\beta}}\bigg(\frac{g(a)}{g(b)}\bigg)^w\mathrm{d}w\\
&\quad-\frac{c\big(\frac{a^p-b^p}{a^pb^p}\big)^2f(a)}{\ln[\frac{f(b)}{f(a)}]}\Bigg[\frac{f(a)+f(b)}{f(a)\ln[\frac{f(b)}{f(a)}]}-2\int_0^1\frac{\big(\frac{f(b)}{f(a)}\big)^t}{\ln[\frac{f(b)}{f(a)}]}\mathrm{d}t\Bigg]\\
&\quad-\frac{c\big(\frac{a^p-b^p}{a^pb^p}\big)^2g(b)}{\ln[\frac{g(a)}{g(b)}]}\Bigg[\frac{g(a)+g(b)}{g(b)\ln[\frac{g(a)}{g(b)}]}-2\int_0^1\frac{\big(\frac{g(a)}{g(b)}\big)^t}{\ln[\frac{g(a)}{g(b)}]}\mathrm{d}t\Bigg]+\frac{c^2\big(\frac{a^p-b^p}{a^pb^p}\big)^4}{30}\\
&=\alpha^2\frac{\big(f(b)\big)^{\frac{1}{\alpha}}-\big(f(a)\big)^{\frac{1}{\alpha}}}{\ln(f(b))-\ln(f(a))}+\beta^2\frac{\big(g(b)\big)^{\frac{1}{\beta}}-\big(g(a)\big)^{\frac{1}{\beta}}}{\ln(g(b))-\ln(g(a))}\\
&\quad-c\big(\frac{a^p-b^p}{a^pb^p}\big)^2\bigg[\frac{f(a)+f(b)}{[\ln(f(b))-\ln(f(a))]^2}-\frac{2f(b)-2f(a)}{[\ln(f(b))-\ln(f(a))]^3}\\
&\quad+\frac{g(a)+g(b)}{[\ln(g(a))-\ln(g(b))]^2}-\frac{2g(a)-2g(b)}{[\ln(g(a))-\ln(g(b))]^3}\bigg]+\frac{c^2\big(\frac{a^p-b^p}{a^pb^p}\big)^4}{30}
\end{aligned}$$

$$
\begin{aligned}
&= \alpha^2 \frac{(f(b))^{\frac{1}{\alpha}} - (f(a))^{\frac{1}{\alpha}}}{f(b)-f(a)} L(f(b),f(a)) + \beta^2 \frac{(g(b))^{\frac{1}{\beta}} - (g(a))^{\frac{1}{\beta}}}{g(b)-g(a)} L(g(b),g(a)) \\
&\quad - \frac{2c\big(\frac{a^p-b^p}{a^pb^p}\big)^2}{[\ln(f(b)) - \ln(f(a))]^2} \big[A(f(a),f(b)) - L(f(a),f(b))\big] \\
&\quad - \frac{2c\big(\frac{a^p-b^p}{a^pb^p}\big)^2}{[\ln(g(b)) - \ln(g(a))]^2} \big[A(g(a),g(b)) - L(g(a),g(b))\big] + \frac{c^2\big(\frac{a^p-b^p}{a^pb^p}\big)^4}{30} \\
&= \alpha \Big[L_{(\frac{1}{\alpha}-1)}(f(b),f(a))\Big]^{\frac{\alpha}{1-\alpha}} L(f(b),f(a)) + \beta \Big[L_{(\frac{1}{\beta}-1)}(g(a),g(b))\Big]^{\frac{\beta}{1-\beta}} L(g(b),g(a)) \\
&\quad - \frac{2c\big(\frac{a^p-b^p}{a^pb^p}\big)^2}{[\ln(f(b)) - \ln(f(a))]^2} \big[A(f(a),f(b)) - L(f(a),f(b))\big] \\
&\quad - \frac{2c\big(\frac{a^p-b^p}{a^pb^p}\big)^2}{[\ln(g(b)) - \ln(g(a))]^2} \big[A(g(a),g(b)) - L(g(a),g(b))\big] + \frac{c^2\big(\frac{a^p-b^p}{a^pb^p}\big)^4}{30} \\
&\le \alpha \frac{f(a)+f(b)}{2} \Big[L_{(\frac{1}{\alpha}-1)}(f(b),f(a))\Big]^{\frac{\alpha}{1-\alpha}} + \beta \frac{g(a)+g(b)}{2} \Big[L_{(\frac{1}{\beta}-1)}(g(b),g(a))\Big]^{\frac{\beta}{1-\beta}} \\
&\quad - \frac{2c\big(\frac{a^p-b^p}{a^pb^p}\big)^2}{[\ln(f(b)) - \ln(f(a))]^2} \big[A(f(a),f(b)) - L(f(a),f(b))\big] \\
&\quad - \frac{2c\big(\frac{a^p-b^p}{a^pb^p}\big)^2}{[\ln(g(b)) - \ln(g(a))]^2} \big[A(g(a),g(b)) - L(g(a),g(b))\big] + \frac{c^2\big(\frac{a^p-b^p}{a^pb^p}\big)^4}{30},
\end{aligned}
$$

which is the required result. □

Theorem 8. *Let $f : I \to (0,\infty)$ be strongly p-harmonic* log-*convex function with modulus $c > 0$. If $f \in L[a,b]$, then*

$$
\begin{aligned}
&\frac{a^pb^p}{b^p-a^p} \int_a^b \frac{f(x) f\big([\frac{a^pb^px^p}{(a^p+b^p)x^p - a^pb^p}]^{\frac{1}{p}}\big)}{x^{1+p}} \mathrm{d}x \\
&\qquad \le f(a)f(b) - \frac{4c\big(\frac{a^p-b^p}{a^pb^p}\big)^2}{[\ln(f(b)) - \ln(f(a))]^2} \big[A(f(a),f(b)) - L(f(a),f(b))\big] + \frac{c^2\big(\frac{a^p-b^p}{a^pb^p}\big)^4}{30}.
\end{aligned}
$$

Proof. Let f be strongly harmonic log-convex functions with modulus $c > 0$. Then

$$
\begin{aligned}
&\frac{a^pb^p}{b^p-a^p} \int_a^b \frac{f(x) f\big([\frac{a^pb^px^p}{(a^p+b^p)x^p - a^pb^p}]^{\frac{1}{p}}\big)}{x^{1+p}} \mathrm{d}x = \int_0^1 f\bigg(\Big[\frac{a^pb^p}{ta^p + (1-t)b^p}\Big]^{\frac{1}{p}}\bigg) f\bigg(\Big[\frac{a^pb^p}{(1-t)a^p + tb^p}\Big]^{\frac{1}{p}}\bigg) \mathrm{d}t \\
&\le \int_0^1 \Big[\big(f(a)\big)^{1-t}\big(f(b)\big)^t - ct(1-t)\big(\frac{a^p-b^p}{a^pb^p}\big)^2\Big]\Big[\big(f(a)\big)^t\big(f(b)\big)^{1-t} - ct(1-t)\big(\frac{a^p-b^p}{a^pb^p}\big)^2\Big] \mathrm{d}t \\
&= \int_0^1 f(a)f(b)\mathrm{d}t - c\big(\frac{a^p-b^p}{a^pb^p}\big)^2 f(a) \int_0^1 t(1-t)\bigg(\frac{f(b)}{f(a)}\bigg)^t \mathrm{d}t \\
&\qquad - c\big(\frac{a^p-b^p}{a^pb^p}\big)^2 f(b) \int_0^1 t(1-t)\bigg(\frac{f(a)}{f(b)}\bigg)^t \mathrm{d}t +^2 \big(\frac{a^p-b^p}{a^pb^p}\big)^4 \int_0^1 t^2(1-t)^2 \mathrm{d}t \\
&= f(a)f(b) - \frac{c\big(\frac{a^p-b^p}{a^pb^p}\big)^2 f(a)}{\ln[\frac{f(b)}{f(a)}]} \int_0^1 (2t-1)\bigg(\frac{f(b)}{f(a)}\bigg)^t \mathrm{d}t
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{c\left(\frac{a^p-b^p}{a^pb^p}\right)^2 f(b)}{\ln[\frac{f(a)}{f(b)}]} \int_0^1 (2t-1)\left(\frac{f(a)}{f(b)}\right)^t \mathrm{d}t + \frac{c^2\left(\frac{a^p-b^p}{a^pb^p}\right)^4}{30} \\
&= f(a)f(b) - \frac{c\left(\frac{a^p-b^p}{a^pb^p}\right)^2 f(a)}{\ln[\frac{f(b)}{f(a)}]}\left[\frac{f(a)+f(b)}{f(a)\ln[\frac{f(b)}{f(a)}]} - 2\int_0^1 \frac{\left(\frac{f(b)}{f(a)}\right)^t}{\ln[\frac{f(b)}{f(a)}]}\mathrm{d}t\right] \\
&- \frac{c\left(\frac{a^p-b^p}{a^pb^p}\right)^2 f(b)}{\ln[\frac{f(a)}{f(b)}]}\left[\frac{f(a)+f(b)}{f(b)\ln[\frac{f(a)}{f(b)}]} - 2\int_0^1 \frac{\left(\frac{f(a)}{f(b)}\right)^t}{\ln[\frac{f(a)}{f(b)}]}\mathrm{d}t\right] + \frac{c^2\left(\frac{a^p-b^p}{a^pb^p}\right)^4}{30} \\
&= f(a)f(b) - c\left(\frac{a^p-b^p}{a^pb^p}\right)^2\left[\frac{f(a)+f(b)}{[\ln(f(b))-\ln(f(a))]^2} - \frac{2f(b)-2f(a)}{[\ln(f(b))-\ln(f(a))]^3}\right. \\
&\left. + \frac{f(a)+f(b)}{[\ln(f(a))-\ln(f(b))]^2} - \frac{2f(a)-2f(b)}{[\ln(f(a))-\ln(f(b))]^3}\right] + \frac{c^2\left(\frac{a^p-b^p}{a^pb^p}\right)^4}{30} \\
&= f(a)f(b) - \frac{4c\left(\frac{a^p-b^p}{a^pb^p}\right)^2}{[\ln(f(b))-\ln(f(a))]^2}\left[A(f(a),f(b)) - L(f(a),f(b))\right] + \frac{c^2\left(\frac{a^p-b^p}{a^pb^p}\right)^4}{30},
\end{aligned}
$$

which is the required result. □

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Нур М.А., Нур К.І., Іфтіхар С. *Деякі нерівності для сильно (p,h)-гармонійних опуклих функцій* // Карпатські матем. публ. — 2019. — Т.11, №1. — C. 119–135.

У даній статті ми показуємо, що гармонійні опуклі функції f є сильно (p,h)-гармонійно опуклими функціями тоді і тільки тоді коли їх можна подати у вигляді $g(x) = f(x) - c(\frac{1}{x^p})^2$, де $g(x)$ є (p,h)-гармонійно опуклою функцією. Отримано деякі нові оцінки класу сильно (p,h)-гармонійно опуклих функцій, включаючи гіпергеометричні та бета-функції. Як застосування наших результатів розглянуто кілька важливих особливих випадків. Також введено новий клас гармонійних опуклих функцій, які називаються сильно (p,h)-гармонійними log-опуклими функціями. Отримано деякі нові нерівності типу Ерміта-Адамара для сильно (p,h)-гармонійних log-опуклих функцій. Ці результати можна розглядати як важливе уточнення і суттєве покращення нових і попередніх відомих результатів. Ідеї та методики цієї роботи можуть бути підґрунтям для подальших досліджень.

Ключові слова і фрази: p-гармонійно опуклі функції, h-опуклі функції, сильно опуклі функції, нерівності типу Ерміта-Адамара.