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THE RELATIONSHIP BETWEEN ALGEBRAIC EQUATIONS AND (n, m)-FORMS, THEIR DEGREES AND RECURRENT FRACTIONS

Algebraic and recursion equations are widely used in different areas of mathematics, so various objects and methods of research that are associated with them are very important. In this article we investigate the relationship between (n, m)-forms with generalized Diophantine Pell's equation, algebraic equations of n degree and recurrent fractions. The properties of the $(n, m^n + 1)$-forms and their characteristic equation are considered. The author applied parafunctions of triangular matrices to the study of algebraic equations and corresponding recurrence equations. The form of adjacent roots of the annihilating polynomial of arbitrary (n, m)-forms over the field of rational numbers are explored.

The following question is very importan for some applied problems: Is a given form the largest by module among its adjacent roots? If it is so, then there is a periodic recurrence fraction of n-order that is equal to this (n, m)-form, and its mth rational shortening will be its rational approximation.The author has identified the class (nm)-forms with the largest module among their adjacent roots and showed how to find periodic recurrence fractions of n-order and rational approximations for them.

Key words and phrases: (n, m)-form, parapermanent, unit of field, Diophantine equation, recurrence fraction, rational approximation.

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1 PRELIMINARY CONCEPTS AND THEOREMS

1.1 Algebraic form of n order

Definition 1. *A real number*

$$x = s_0 + s_1 \sqrt[n]{m} + \ldots + s_{n-1} \sqrt[n]{m^{n-1}},\ n \in \mathbb{N},\ s_i, m \in \mathbb{Q}, \tag{1}$$

or corresponding n-dimensional vector

$$x = (s_0, s_1, \ldots, s_{n-1}) \tag{2}$$

is called an algebraic (n, m)-form or briefly (n, m)-form.

It is known that the set of (n, m)-forms with the usual operations of addition and multiplication is a field.

We check the isomorphism of (n, m)-forms with some class matrices. For each (n, m)-form (1) we put in correspondence the circular n order matrix

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$$X = \begin{pmatrix} s_0 & s_{n-1}\sqrt[n]{m^{n-1}} & s_{n-2}\sqrt[n]{m^{n-2}} & \cdots & s_2\sqrt[n]{m^2} & s_1\sqrt[n]{m} \\ s_1\sqrt[n]{m} & s_0 & s_{n-1}\sqrt[n]{m^{n-1}} & \cdots & s_3\sqrt[n]{m^3} & s_2\sqrt[n]{m^2} \\ s_2\sqrt[n]{m^2} & s_1\sqrt[n]{m} & s_0 & \cdots & s_4\sqrt[n]{m^4} & s_3\sqrt[n]{m^3} \\ \vdots & \dots & \dots & \dots & \dots & \vdots \\ s_{n-2}\sqrt[n]{m^{n-2}} & s_{n-3}\sqrt[n]{m^{n-3}} & s_{n-4}\sqrt[n]{m^{n-4}} & \cdots & s_0 & s_{n-1}\sqrt[n]{m^{n-1}} \\ s_{n-1}\sqrt[n]{m^{n-1}} & s_{n-2}\sqrt[n]{m^{n-2}} & s_{n-3}\sqrt[n]{m^{n-3}} & \cdots & s_1\sqrt[n]{m} & s_0 \end{pmatrix} \tag{3}$$

and for each (n,m)-form (2) we put in correspondence the matrix

$$X = \begin{pmatrix} s_0 & ms_{n-1} & ms_{n-2} & \cdots & ms_2 & ms_1 \\ s_1 & s_0 & ms_{n-1} & \cdots & ms_3 & ms_2 \\ s_2 & s_1 & s_0 & \cdots & ms_4 & ms_3 \\ \vdots & \dots & \dots & \dots & \dots & \vdots \\ s_{n-2} & s_{n-3} & s_{n-4} & \cdots & s_0 & ms_{n-1} \\ s_{n-1} & s_{n-2} & s_{n-3} & \cdots & s_1 & s_0 \end{pmatrix}. \tag{4}$$

Both matrices (3) and (4) are uniquely defined by their first columns.

The product of (n,m)-forms

$$\begin{aligned} x' &= s'_0 + s'_1\sqrt[n]{m} + \ldots + s'_{n-1}\sqrt[n]{m^{n-1}}, \\ x'' &= s''_0 + s''_1\sqrt[n]{m} + \ldots + s''_{n-1}\sqrt[n]{m^{n-1}} \end{aligned} \tag{5}$$

is the following (n,m)-form

$$x = s_0 + s_1\sqrt[n]{m} + \ldots + s_{n-1}\sqrt[n]{m^{n-1}},$$

where

$$s_i = \sum_{j=0}^{i} s'_j s''_{i-j} + m\sum_{j=i+1}^{n-1} s'_j s''_{i-j},\ i = 0,1,\ldots,n-1. \tag{6}$$

Thus, we have proved the following theorem.

Theorem 1. *The semigroups of* (n,m)*-forms* (1) *and* (2) *are isomorphic to the semigroups of matrices* (3) *and* (4)*, respectively.*

From the above it follows that k degree of (n,m)-form (5) is responsible k degree of matrix

$$X' = \begin{pmatrix} s'_0 & s'_{n-1}\sqrt[n]{m^{n-1}} & s'_{n-2}\sqrt[n]{m^{n-2}} & \cdots & s'_2\sqrt[n]{m^2} & s'_1\sqrt[n]{m} \\ s'_1\sqrt[n]{m} & s'_0 & s'_{n-1}\sqrt[n]{m^{n-1}} & \cdots & s'_3\sqrt[n]{m^3} & s'_2\sqrt[n]{m^2} \\ s'_2\sqrt[n]{m^2} & s'_1\sqrt[n]{m} & s'_0 & \cdots & s'_4\sqrt[n]{m^4} & s'_3\sqrt[n]{m^3} \\ \vdots & \dots & \dots & \dots & \dots & \vdots \\ s'_{n-2}\sqrt[n]{m^{n-2}} & s'_{n-3}\sqrt[n]{m^{n-3}} & s'_{n-4}\sqrt[n]{m^{n-4}} & \cdots & s'_0 & s'_{n-1}\sqrt[n]{m^{n-1}} \\ s'_{n-1}\sqrt[n]{m^{n-1}} & s'_{n-2}\sqrt[n]{m^{n-2}} & s'_{n-3}\sqrt[n]{m^{n-3}} & \cdots & s'_1\sqrt[n]{m} & s'_0 \end{pmatrix}$$

or matrix

$$X' = \begin{pmatrix} s'_0 & ms_{n-1} & ms'_{n-2} & \cdots & ms'_2 & ms'_1 \\ s'_1 & s'_0 & ms'_{n-1} & \cdots & ms'_3 & ms'_2 \\ s'_2 & s'_1 & s'_0 & \cdots & ms'_4 & ms'_3 \\ \vdots & \dots & \dots & \dots & \dots & \vdots \\ s'_{n-2} & s'_{n-3} & s'_{n-4} & \cdots & s'_0 & ms'_{n-1} \\ s'_{n-1} & s'_{n-2} & s'_{n-3} & \cdots & s'_1 & s'_0 \end{pmatrix}.$$

It is also obvious that if the last two matrices multiply by on the matrices columns

$$X'' = \begin{pmatrix} s_0'' \\ s_1'' \sqrt[n]{m} \\ \vdots \\ s_{n-2}'' \sqrt[n]{m^{n-2}} \\ s_{n-1}'' \sqrt[n]{m^{n-1}} \end{pmatrix}, \; X'' = \begin{pmatrix} s_0'' \\ s_1'' \\ \vdots \\ s_{n-2}'' \\ s_{n-1}'' \end{pmatrix},$$

then we get the matrices columns

$$X = \begin{pmatrix} s_0 \\ s_1 \sqrt[n]{m} \\ \vdots \\ s_{n-2} \sqrt[n]{m^{n-2}} \\ s_{n-1} \sqrt[n]{m^{n-1}} \end{pmatrix}, \; X = \begin{pmatrix} s_0 \\ s_1 \\ \vdots \\ s_{n-2} \\ s_{n-1} \end{pmatrix},$$

where s_i are defined by (6).

For any (n, m)-form

$$x = s_0 + s_1 \sqrt[n]{m} + \ldots + s_{n-1} \sqrt[n]{m^{n-1}}$$

there exists a unique (n, m)-form

$$\overline{x} = \overline{s_0} + \overline{s_1} \sqrt[n]{m} + \ldots + \overline{s_{n-1}} \sqrt[n]{m^{n-1}},$$

such that product $x\overline{x}$ is a real number. The (n, m)-form $\overline{x}$ is called conjugated to (n, m)-form x, and their product is called a norm of x and denoted by $|(n, m)|$.

Let X and $\overline{X}$ are matrices that corresponds to (n, m)-form

$$x = s_0 + s_1 \sqrt[n]{m} + \ldots + s_{n-1} \sqrt[n]{m^{n-1}}$$

and conjugate (n, m)-form $\overline{x}$. Then

$$X \cdot \overline{X} = |(n, m)| \cdot E,$$

where E be the identity matrix. The norm of (n, m)-form x is equal to $\det X$, and matrix that corresponds to conjugated (n, m)-form $\overline{x}$ is inverse to the matrix X multiplied on the determinant of X.

Therefore, the equation

$$\begin{vmatrix} s_0 & ms_{n-1} & ms_{n-2} & \cdots & ms_2 & ms_1 \\ s_1 & s_0 & ms_{n-1} & \cdots & ms_3 & ms_2 \\ s_2 & s_1 & s_0 & \cdots & ms_4 & ms_3 \\ \vdots & \ldots & \ldots & \ldots & \ldots & \vdots \\ s_{n-2} & s_{n-3} & s_{n-4} & \cdots & s_0 & ms_{n-1} \\ s_{n-1} & s_{n-2} & s_{n-3} & \cdots & s_1 & s_0 \end{vmatrix} = \pm 1$$

is an n-dimensional generalization Pell's equation

$$\begin{vmatrix} s_0 & ms_1 \\ s_1 & s_0 \end{vmatrix} = s_0^2 - ms_1^2 = \pm 1.$$

Using the polynomial formula it is easy to prove the equality

$$(s_0 + s_1\sqrt[n]{m} + \ldots + s_{n-1}\sqrt[n]{m^{n-1}})^k$$
$$= \sum_{\substack{\lambda_0+\lambda_1+\ldots+\lambda_{n-1}=k \\ \lambda_1+2\lambda_2+\ldots+(n-1)\lambda_{n-1}=ns+i}} \frac{k!}{\lambda_0!\lambda_1!\cdot\ldots\cdot\lambda_{n-1}!} s_0^{\lambda_0}s_1^{\lambda_1}\cdot\ldots\cdot s_{n-1}^{\lambda_{n-1}}m^s m^{\frac{i}{n}}.$$

However, the above formula is inconvenient for the elevation of (n,m)-forms to the k degree, because it is associated with orderly partition number of n on integer nonnegative summands.

1.2 Parafunctions of triangular matrices (tables)

Let K be some field of numbers.

Definition 2 ([2]). *Triangular table*

$$A = \begin{pmatrix} a_{11} & & & \\ a_{21} & a_{22} & & \\ \vdots & \vdots & \ddots & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}_n \tag{7}$$

of numbers in K is called a ***triangular matrix.***

To every element a_{ij} of the triangular matrix (7) we put in correspondence the $(i-j+1)$ elements a_{ik}, $k \in \{j,...,i\}$ which are called *derived elements* of triangular matrix, generated by a *key element* a_{ij}. A key element of a triangular matrix is also a derived element. The product of all derived elements generated by a key element of a_{ij} is denoted by $\{a_{ij}\}$ and is called a *factorial* product of this key element, i.e.,

$$\{a_{ij}\} = \prod_{k=j}^{i} a_{ik}.$$

Definition 3 ([2]). *The paradeterminant and parapermanent of the triangular matrix (7) are the numbers*

$$\operatorname{ddet}(A) = \sum_{r=1}^{n} \sum_{p_1+\ldots+p_r=n} (-1)^{n-r} \prod_{s=1}^{r} \{a_{p_1+\ldots+p_s,p_1+\ldots+p_{s-1}+1}\},$$

$$\text{pper(A)} = \sum_{r=1}^{n} \sum_{p_1+\ldots+p_r=n} \prod_{s=1}^{r} \{a_{p_1+\ldots+p_s,p_1+\ldots+p_{s-1}+1}\},$$

where the summation of over the set of natural solutions of the equality $p_1+\ldots+p_r=n$.

Paradeterminants and parapermanents of triangular matrices can be used in Algebra, Number Theory and Combinatorics (see [2] for more details and examples).

1.3 One-periodic recurrence fractions

Let us consider the algebraic equations of nth order

$$x^n = a_1 x^{n-1} + a_2 x^{n-2} + \ldots + a_n, \tag{8}$$

where $a_n \neq 0$, and the expression

$$\left[\begin{array}{c|ccccccc}
a_1 & & & & & & & \\
\frac{a_2}{a_1} & a_1 & & & & & & \\
\vdots & \cdots & \ddots & & & & & \\
\frac{a_{n-1}}{a_{n-2}} & \frac{a_{n-2}}{a_{n-3}} & \cdots & a_1 & & & & \\
\frac{a_n}{a_{n-1}} & \frac{a_{n-1}}{a_{n-2}} & \cdots & \frac{a_2}{a_1} & a_1 & & & \\
0 & \frac{a_n}{a_{n-1}} & \cdots & \frac{a_3}{a_2} & \frac{a_2}{a_1} & a_1 & & \\
\vdots & \cdots & \cdots & \cdots & \cdots & \cdots & \ddots & \\
0 & 0 & \cdots & \frac{a_n}{a_{n-1}} & \frac{a_{n-1}}{a_{n-2}} & \frac{a_{n-2}}{a_{n-3}} & \cdots & a_1
\end{array}\right]_m \tag{9}$$

which is closely related to (8). The expression (9) looks like as a symbol fraction, the numerator of which is a parapermanent P_m of order m formed by the removal columns from the expression pipe and the denominator of which is a parapermanent Q_m of order $m-1$ without first column of parapermanent of numerator.

If in the expression (9) we direct to the limit as $m \to \infty$, we obtain an one-periodic recurrent fraction of order n

$$\left[\begin{array}{c|ccccccc}
a_1 & & & & & & & \\
\frac{a_2}{a_1} & a_1 & & & & & & \\
\vdots & \cdots & \ddots & & & & & \\
\frac{a_{n-1}}{a_{n-2}} & \frac{a_{n-2}}{a_{n-3}} & \cdots & a_1 & & & & \\
\frac{a_n}{a_{n-1}} & \frac{a_{n-1}}{a_{n-2}} & \cdots & \frac{a_2}{a_1} & a_1 & & & \\
0 & \frac{a_n}{a_{n-1}} & \cdots & \frac{a_3}{a_2} & \frac{a_2}{a_1} & a_1 & & \\
\vdots & \cdots & \cdots & \cdots & \cdots & \cdots & \ddots & \\
0 & 0 & \cdots & \frac{a_n}{a_{n-1}} & \frac{a_{n-1}}{a_{n-2}} & \frac{a_{n-2}}{a_{n-3}} & \cdots & a_1
\end{array}\right]_\infty . \tag{10}$$

The expression (9) is called the mth approximant of (10).

Theorem 2 ([3])**.** *Let* (8) *be an algebraic equation from pairwise different roots. If for the m-rational shortening of one-periodic recurrent fraction of nth order* (10) *a finite non-zero real limit exists as $m \to \infty$, i.e.,*

$$\lim_{m\to\infty} \frac{P_m}{Q_m} = x \neq 0,$$

then a recurrent fraction of order n is an image of the real root of algebraic equations (8) *with the largest module.*

More information about recurrent fractions can be found in [3].

2 RELATIONSHIP (n,m)-FORM WITH ALGEBRAIC EQUATIONS

Let us find the integer coefficients of equation

$$x^n = a_{n1}x^{n-1} + a_{n2}x^{n-2} + \ldots + a_{n,n-1}x^1 + a_{n,n} \tag{11}$$

the root of which is the (n,m)-form

$$x = s_0 + s_1\sqrt[n]{m} + \ldots + s_{n-1}\sqrt[n]{m^{n-1}},$$

where $s_i \in \mathbb{Q}$, $m \in \mathbb{N}$.

The main minor of rth order of matrix

$$X = \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \ldots & \ldots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \tag{12}$$

is denoted by

$$X\begin{pmatrix} i_1 & i_2 & \ldots & i_r \\ i_1 & i_2 & \ldots & i_r \end{pmatrix} = \begin{vmatrix} a_{i_1,i_1} & a_{i_1,i_2} & \cdots & a_{i_1,i_r} \\ a_{i_2,i_1} & a_{i_2,i_2} & \cdots & a_{i_2,i_r} \\ \vdots & \ldots & \ldots & \vdots \\ a_{i_r,i_1} & a_{i_r,i_2} & \cdots & a_{i_r,i_r} \end{vmatrix},$$

where $i_1 < i_2 < \ldots < i_r$. The characteristic equation of matrix (12) is

$$\det(X - xE) = 0$$

or

$$x^n = \alpha_{n1}x^{n-1} + \alpha_{n2}x^{n-2} + \ldots + \alpha_{n,n-1}x^1 + \alpha_{nn},$$

where

$$\alpha_{nj} = (-1)^{j-1} \sum_{1 \le i_1 < i_2 < \ldots < i_j \le n} X\begin{pmatrix} i_1 & i_2 & \ldots & i_j \\ i_1 & i_2 & \ldots & i_j \end{pmatrix}. \tag{13}$$

According to theorem Hamilton-Cayley, each square matrix satisfies the characteristic equation, so

$$X^n = \alpha_{n1}X^{n-1} + \alpha_{n2}X^{n-2} + \ldots + \alpha_{n,n-1}X^1 + \alpha_{nn}, \tag{14}$$

with coefficients (13), where X is matrix (12).

If matrix X in (14) is given by (4), then the coefficients a_{nj} of equation (11), for which a (n,m)-form (2) is the root, can be found using the equalities (13). Thus, we prove

Theorem 3. *If the (n,m)-form*

$$x = s_0 + s_1\sqrt[n]{m} + \ldots + s_{n-1}\sqrt[n]{m^{n-1}}$$

is a root of equation

$$x^n = a_{n1}x^{n-1} + a_{n2}x^{n-2} + \ldots + a_{n,n-1}x^1 + a_{nn},$$

then the coefficients of this equation are equal

$$a_{nj} = (-1)^{j-1} \sum_{1 \le i_1 < i_2 < \ldots < i_j \le n} X \begin{pmatrix} i_1 & i_2 & \ldots & i_j \\ i_1 & i_2 & \ldots & i_j \end{pmatrix},$$

where

$$X \begin{pmatrix} i_1 & i_2 & \ldots & i_j \\ i_1 & i_2 & \ldots & i_j \end{pmatrix}$$

are major minors of matrix

$$X = \begin{pmatrix} s_0 & ms_{n-1} & ms_{n-2} & \cdots & ms_2 & ms_1 \\ s_1 & s_0 & ms_{n-1} & \cdots & ms_3 & ms_2 \\ s_2 & s_1 & s_0 & \cdots & ms_4 & ms_3 \\ \vdots & \ldots & \ldots & \ldots & \ldots & \vdots \\ s_{n-2} & s_{n-3} & s_{n-4} & \cdots & s_0 & ms_{n-1} \\ s_{n-1} & s_{n-2} & s_{n-3} & \cdots & s_1 & s_0 \end{pmatrix}.$$

Theorem 4. *The $(n, m^n + 1)$-form*

$$m^{n-1} + m^{n-2}\sqrt[n]{m^n+1} + \ldots + m\sqrt[n]{(m^n+1)^{n-2}} + \sqrt[n]{(m^n+1)^{n-1}}$$

is the root of an algebraic equation

$$x^n = \binom{n}{1} m^{n-1}x^{n-1} + \binom{n}{2} m^{n-2}x^{n-2} + \ldots + \binom{n}{n-1} mx + \binom{n}{n}.$$

Proof. Since all the major minors the same order of matrix

$$\begin{pmatrix} m^{n-1} & m^n+1 & m(m^n+1) & \ldots & m^{n-3}(m^n+1) & m^{n-2}(m^n+1) \\ m^{n-2} & m^{n-1} & m^n+1 & \ldots & m^{n-4}(m^n+1) & m^{n-3}(m^n+1) \\ m^{n-3} & m^{n-2} & m^{n-1} & \ldots & m^{n-5}(m^n+1) & m^{n-4}(m^n+1) \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ m & m^2 & m^3 & \ldots & m^{n-1} & m^n+1 \\ 1 & m & m^2 & \ldots & m^{n-2} & m^{n-1} \end{pmatrix}$$

are equal, we find one of them. Let us find the major minor of matrix

$$\begin{pmatrix} m^{n-1} & m^n+1 & m(m^n+1) & \cdots & m^{s-2}(m^n+1) \\ m^{n-2} & m^{n-1} & m^n+1 & \cdots & m^{s-3}(m^n+1) \\ m^{n-3} & m^{n-2} & m^{n-1} & \cdots & m^{s-4}(m^n+1) \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ m^{n-s} & m^{n-s+1} & m^{n-s+2} & \ldots & m^{n-1} \end{pmatrix}.$$

We multiply the first column on $-m^r$, $r = 1, 2, \ldots, s-1$ and add it to the $(r+1)$ column; then we get the determinant of matrix

$$\begin{pmatrix} m^{n-1} & 1 & m & \cdots & m^{s-2} \\ m^{n-2} & 0 & 1 & \cdots & m^{s-3} \\ m^{n-3} & 0 & 0 & \cdots & m^{s-4} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ m^{n-s} & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

Decomposing above determinant by elements of the first column, we get $(-1)^{s+1}m^{n-s}$.

Thus, according to the Theorem 3, coefficients a_{ns} are equal to

$$(-1)^{s-1}(-1)^{s+1}m^{n-s}\binom{n}{s} = m^{n-s}\binom{n}{s}.$$

$\square$

3 Some calculations related to an algebraic equations of n degree

Theorem 5. *If*

$$x^n = a_{n1}x^{n-1} + a_{n2}x^{n-2} + \ldots + a_{n,n-1}x + a_{nn}$$

and

$$x^m = A_{m1}x^{n-1} + A_{m2}x^{n-2} + \ldots + A_{m,n-1}x + A_{mn},\ n \leq m,$$

then for all $i = 1, 2, \ldots, n$

$$A_{mi} = \begin{bmatrix} a_{ni} & & & & & & & & & \\ \frac{a_{n,i+1}}{a_{n1}} & a_{n1} & & & & & & & & \\ \frac{a_{n,i+2}}{a_{n2}} & \frac{a_{n2}}{a_{n1}} & a_{n1} & & & & & & & \\ \vdots & \ldots & \ldots & \ddots & & & & & & \\ \frac{a_{nn}}{a_{n,n-i}} & \frac{a_{n,n-i}}{a_{n,n-i-1}} & \frac{a_{n,n-i-1}}{a_{n,n-i-2}} & \ldots & a_{n1} & & & & & \\ 0 & \frac{a_{nn}}{a_{n,n-i}} & \frac{a_{n,n-i}}{a_{n,n-i-1}} & \ldots & \frac{a_{n2}}{a_{n1}} & a_{n1} & & & & \\ \vdots & \ldots & \ldots & \ldots & \ldots & \ldots & \ddots & & & \\ 0 & \frac{a_{nn}}{a_{n,n-1}} & \frac{a_{n,n-1}}{a_{n,n-2}} & \ldots & \frac{a_{n,i+1}}{a_{ni}} & \frac{a_{ni}}{a_{n,i-1}} & \ldots & a_{n1} & & \\ \vdots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ddots & \\ 0 & 0 & 0 & \ldots & \frac{a_{nn}}{a_{n,n-1}} & \frac{a_{n,n-1}}{a_{n,n-2}} & \ldots & \frac{a_{n3}}{a_{n2}} & \frac{a_{n2}}{a_{n1}} & a_{n1} \end{bmatrix}_{m-n+1}. \tag{15}$$

Proof. Obviously, equality (15) is true at $m = n$. Let us show that the induction step is performed. We have

$$x^{m+1} = A_{m+1,1}x^{n-1} + \ldots + A_{m+1,i}x^{n-i} + \ldots + A_{m+1,n-1}x + A_{m+1,n}.$$

On the other side,

$$\begin{aligned} x^{m+1} &= A_{m1}x^n + A_{m2}x^{n-1} + \ldots + A_{m,n-1}x^2 + A_{mn}x \\ &= A_{m1}(a_{n1}x^{n-1} + a_{n2}x^{n-2} + \ldots + a_{n,n-1}x + a_{nn}) + A_{m2}x^{n-1} + \ldots + A_{m,n-1}x^2 + A_{mn}x \\ &= (a_{n1}A_{m1} + A_{m2})x^{n-1} + \ldots + (a_{ni}A_{m1} + A_{m,i+1})x^{n-i} \\ &\qquad + \ldots + (a_{n,n-1}A_{m1} + A_{mn})x + a_{nn}A_{m1}. \end{aligned}$$

Thus

$$A_{m+1,i} = a_{ni}A_{m1} + A_{m,i+1}.$$

It is easy to see that decomposing the parapermanent $A_{m+1,i}$ by elements of the first column, we get $a_{ni}A_{m1} + A_{m,i+1}$. $\square$

Corollary 1. *If*
$$x^n = a_{n1}x^{n-1} + a_{n2}x^{n-2} + \ldots + a_{n,n-1}x + a_{nn}$$
and
$$x^m = A_{m1}x^{n-1} + A_{m2}x^{n-2} + \ldots + A_{m,n-1}x + A_{mn},\ n \leq m,$$
then coefficients A_{mi} can be found from the recurrence equations
$$A_{mi} = a_{ni}A_{m-1,1} + a_{n,i+1}A_{m-2,1} + \ldots + a_{nn}A_{m-n+i-1,1},\ i = 1, 2, \ldots, n,$$
where
$$A_{n1} = a_{n1}, A_{n-1,1} = 1, A_{n-2,1} = \ldots = A_{0,1} = 0.$$

Proof. The proof it follows from the decomposition of parapermanent (15) by the elements of the first column. □

Example 1. *If*
$$x^3 = a_{31}x^2 + a_{32}x^1 + a_{33}$$
and
$$x^m = A_{m1}x^2 + A_{m2}x^1 + A_{m3},\ m \geq 3,$$
then coefficients $A_{mi}, i = 1, 2, 3$ can be found from the recurrence equations
$$\begin{aligned} A_{m1} &= a_{31}A_{m-1,1} + a_{3,2}A_{m-2,1} + a_{33}A_{m-3,1}, \\ A_{m2} &= a_{32}A_{m-1,1} + a_{3,3}A_{m-2,1}, \\ A_{m3} &= a_{33}A_{m-1,1},\ m \geq 4, \end{aligned}$$
where $A_{31} = a_{31}, A_{21} = 1, A_{11} = A_{01} = 0$.

For comparison, let us consider a similar algorithm of Delone and Fadeev ([1, p. 73]). Let
$$\omega^3 = S\omega^2 + Q\omega + N$$
and
$$\omega^m = U_m\omega^2 + V_m\omega + W_m,$$
then the coefficients U_m, V_m, W_m can be found from relations
$$\begin{aligned} U_m &= \sum_{\alpha+2\beta+3\gamma=m-2} \frac{(\alpha+\beta+\gamma)!}{\alpha!\beta!\gamma!} S^\alpha Q^\beta N^\gamma, \\ V_m &= U_{m+1} - U_m S, \\ W_m &= U_{m+2} - U_{m+1}S - U_m Q. \end{aligned}$$

Note that similar algorithms with $n > 3$ were not considered.

Theorem 6. *If (n, k)-form looks like*
$$x = s_0 + s_1\sqrt[n]{k} + \cdots + s_{n-1}\sqrt[n]{k^{n-1}},$$
then others of adjacent roots of diriment polynomial over the field of rational numbers of this form are as follows
$$x_i = s_0 + s_1\varepsilon^i\sqrt[n]{k} + \cdots + s_{n-1}\varepsilon^{(n-1)i}\sqrt[n]{k^{n-1}},$$
where ε is the primitive root of degree n of 1 and $i = 1, \ldots, n-1$.

Proof. To unify notation we also denoted (n,k)-form x by x_n. We will show that for every k, $S_m = \sum_{i=1}^{n} x_i^m$ does not depend on radicals and belongs the field of rational numbers.

Let us consider, first, $\sum_{i=1}^{n} \varepsilon^{ip} = \varepsilon^p \sum_{i=0}^{n-1} \varepsilon^{ip}$. Since ε is the primitive root of degree n of unit, then

$$\sum_{i=0}^{n-1} \varepsilon^{ip} = \sum_{i=1}^{n} \varepsilon^{ip}, \quad \text{i.e.,} \quad \sum_{i=1}^{n} \varepsilon^{ip} = 0,$$

if p is not a multiple of n.

In formula x_i^m each summand will looks like

$$s_{p_1}\varepsilon^{ip_1}\sqrt[n]{k^{p_1}} \cdot \ldots \cdot s_{p_m}\varepsilon^{ip_m}\sqrt[n]{k^{p_m}} = (s_{p_1}\ldots s_{p_m})\varepsilon^{i(p_1+\cdots+p_m)}\sqrt[n]{k^{p_1+\cdots+p_m}},$$

where $p_1,\ldots,p_m \in \{0,1,2,\ldots,n-1\}$. Then in $S_m = \sum_{i=1}^{n} x_i^m$ we can regroup the terms in groups of sets with the same $p_1,\ldots,p_m$; each a such group has representation:

$$\sum_{i=1}^{n}(s_{p_1}\ldots s_{p_m})\varepsilon^{i(p_1+\cdots+p_m)}\sqrt[n]{k^{p_1+\cdots+p_m}} = (s_{p_1}\ldots s_{p_m})\left(\sum_{i=1}^{n}\varepsilon^{i(p_1+\cdots+p_m)}\right)\sqrt[n]{k^{p_1+\cdots+p_m}}.$$

Hence, if $p_1+\cdots+p_m$ is not a multiple of n, then this group of summands is equal to zero; if $p_1+\cdots+p_m$ is a multiple of n, then this group of summands is a rational number.

According to Newton's formulas, σ_m (elementary symmetric expressions of $x_1,\ldots,x_n$), $m = 1,\ldots,n$ are expressed through the S_q, $q \le m$, i σ_r, $r < m$. Since $\sigma_1 = S_1$ and all S_m is a rational number, then all σ_m, where $m = 1,\ldots,n$, also a rational number. According to the Viete formulas $(x-x_1)\ldots(x-x_n) \in \mathbb{Q}[x]$, that proves the theorem. □

For some applications it less important to known the view of adjacent roots of (n,k)-form then the answer to the question: Is this form the largest by module? This question is quite difficult and requires a special investigation, but we have an obvious consequence.

Corollary 2. *If $s_0,s_1,\ldots,s_{n-1}$, $n \in \mathbb{N}$ are nonnegative rational numbers, then (n,k)-form $x = s_0 + s_1\sqrt[n]{k} + \cdots + s_{n-1}\sqrt[n]{k^{n-1}}$ is the largest by module among its adjacent roots diriment polynomial over the field of rational numbers.*

Thus, using Theorem 3 for each (n,m)-form (1) we can write an algebraic equation of order n. According to Corollary 2 and Theorem 2, by rational approximations of recurrent fractions we can build a mth rational shortening (9) to the (n,m)-form (1).

Theorem 7. *If (n,m)-form (1) with nonnegative coefficients $s_i, i = 0,1,\ldots,n-1$ is the root of an algebraic equation (8), then recurrent fraction (10) is it's image, and it's mth approximant (9) is it's rational approximation.*

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Алгебраїчні та рекурентні рівняння мають широке застосування не тільки в алгебрі але й в інших розділах математики, чим викликають неабияке зацікавлення до різного роду об'єктів та методів дослідження пов'язаних із ними. В цій статті досліджено зв'язок (n, m)-форм з узагальненими рівняннями Пеля, алгебраїчними рівняннями n-ого степеня і рекурентними дробами. Розглянуто властивості $(n, m^n + 1)$-форми і її характеристичного рівняння. Застосовано парафункції трикутних матриць до алгебраїчних рівнянь n-ого степеня та відповідних їм рекурентних рівнянь. Досліджено вигляд суміжних коренів анулюючого полінома довільної (n, m)-форми над полем раціональних чисел.

Для деяких прикладних задач велике значення має відповідь на питання: чи є дана (n, m)-форма найбільша за модулем серед своїх суміжних коренів? Тоді в цьому випадку існуватиме одноперіодичний рекурентний дріб n-ого порядку, який дорівнюватиме даній (n, m)-формі, а його m-те раціональне вкорочення буде її раціональним наближенням. Автор виділив клас (n, m)-форм, які є найбільшими за модулем серед своїх суміжних коренів, та показав як для них знайти одноперіодичні рекурентні дроби n-ого порядку й раціональні наближення.

Ключові слова і фрази: (n, m)-форма, параперманент, узагальнене рівняння Пеля, рекурентний дріб, раціональне наближення.