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SPECTRAL APPROXIMATIONS OF STRONGLY DEGENERATE ELLIPTIC DIFFERENTIAL OPERATORS

We establish analytical estimates of spectral approximations errors for strongly degenerate elliptic differential operators in the Lebesgue space $L_q(\Omega)$ on a bounded domain Ω. Elliptic operators have coefficients with strong degeneration near boundary. Their spectrum consists of isolated eigenvalues of finite multiplicity and the linear span of the associated eigenvectors is dense in $L_q(\Omega)$. The received results are based on an appropriate generalization of Bernstein-Jackson inequalities with explicitly calculated constants for quasi-normalized Besov-type approximation spaces which are associated with the given elliptic operator. The approximation spaces are determined by the functional $E\,(t,u)$, which characterizes the shortest distance from an arbitrary function $u \in L_q(\Omega)$ to the closed linear span of spectral subspaces of the given operator, corresponding to the eigenvalues such that not larger than fixed $t > 0$. Such linear span of spectral subspaces coincides with the subspace of entire analytic functions of exponential type not larger than $t > 0$. The approximation functional $E\,(t,u)$ in our cases plays a similar role as the modulus of smoothness in the functions theory.

Key words and phrases: elliptic operators, spectral approximations.

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1 INTRODUCTION

We investigate the problem of best approximations in the Lebesgue space $L_q(\Omega)$ on a bounded domain $\Omega \subset \mathbb{R}^n$ by using spectral subspaces $\mathcal{R}(A)$ of a strongly degenerate elliptic differential operator A. Our aims is to prove the inverse and direct theorems that give precise estimates of approximation errors and which are connected with appropriate estimations by Bernstein-Jackson type inequalities.

For this purpose we use the best approximation functional $E\left(t,u;\mathcal{R}(A),L_q(\Omega)\right)$ which characterizes a shortest distance from an arbitrary function $u \in L_q(\Omega)$ to the closed linear span $\mathcal{R}^t(A)$ of all spectral subspaces $\mathcal{R}_{\lambda_j}(A)$ of the given operator A, corresponding to the eigenvalues λ_j such that $|\lambda_j| < t$ with a fixed $t > 0$.

This best approximation problem we solve by finding exact values of constants in the Bernstein-Jackson inequalities. Namely, we establish the Bernstein-Jackson inequalities with explicitly calculated constants, using the suitable generalization of Besov's space $\mathcal{B}^s_r(A,L_q(\Omega))$, determined by a given operator A and an appropriate functional $E\left(t,u;\mathcal{R}(A),L_q(\Omega)\right)$.

It is essentially to note that the approximation functional $E\left(t,u;\mathcal{R}(A),L_q(\Omega)\right)$ in these inequalities plays a similar role as the modulus of smoothness in the functions theory. Earlier applications of smoothness modulus to approximation problems can be found in [5–7].

In this paper we continue the research started in [3,4].

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2 STRONGLY DEGENERATE ELLIPTIC DIFFERENTIAL OPERATORS

We shall follow the treatment given in [8, Sec. 6.2.1]. Let $\Omega \subset \mathbb{R}^n$ be an open bounded set with the infinitely smooth boundary $\partial\Omega$. As usual, $C^\infty(\Omega)$ denotes the space of all infinitely differentiable complex-valued functions defined on Ω. Suppose that $\rho(x) \in C^\infty(\Omega)$ is a positive function such that:

(i) for all multi-indices $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$, $|\alpha| = \alpha_1 + \dots + \alpha_n$ there exist positive numbers c_α such that

$$|D^\alpha \rho(x)| \le c_\alpha \, \rho^{1+|\alpha|}(x) \text{ for all } x \in \Omega;$$

(ii) for any positive number K there exist numbers $\varepsilon_K > 0$ and $r_K > 0$ such that $\rho(x) > K$, if $d(x) \le \varepsilon_K$ or $|x| \ge r_K$, $x \in \Omega$ (here, $d(x)$ is the distance to the boundary $\partial\Omega$).

In what follows, $S_{\rho(x)}(\Omega)$ denotes the locally convex space

$$S_{\rho(x)}(\Omega) = \Big\{ u : u \in C^\infty(\Omega),\ \|u\|_{l,\alpha} = \sup_{x\in\Omega} \rho^l(x)|D^\alpha u(x)| < \infty \text{ for all } \alpha \text{ and } l \in \mathbb{N}_0 \Big\}.$$

Let $m \in \mathbb{N}$, $\mu, \tau \in \mathbb{R}$ and $\tau > \mu + 2m$. We put

$$\aleph_l = \frac{1}{2m} \left(\tau\,(2m - l) + \mu\, l \right), \quad l = 0, 1, \dots, 2m,$$

and consider the differential elliptic operator

$$Au = \sum_{l=0}^{m} \sum_{|\alpha|=2l} \rho^{\aleph_{2l}}(x)\, b_\alpha(x)\, D^\alpha u + \sum_{|\beta|<2m} a_\beta(x)\, D^\beta u, \tag{1}$$

where $b_\alpha(x) \in C^\infty(\Omega)$ ($|\alpha| = 2l$, $l = 0, 1, \dots, m$) are real functions, all derivatives of which (inclusively the functions themselves) are bounded in Ω. In sequel we assume that there exists a positive number C such that for all $\xi \in \mathbb{R}^n$ and all $x \in \Omega$

$$(-1)^m \sum_{|\alpha|=2m} b_\alpha(x)\, \xi^\alpha \ge C\,|\xi|^{2m}, \quad b_{(0,\dots,0)}(x) \ge C,$$
$$(-1)^l \sum_{|\alpha|=2l} b_\alpha(x)\, \xi^\alpha \ge 0, \quad l = 1, \dots, m-1.$$

Moreover, let $a_\beta(x) \in C^\infty(\Omega)$ ($0 \le |\beta| < 2m$) and there exists a positive number $\delta > 0$ such that $D^\gamma a_\beta(\xi) = O\left(\rho^{\aleph_{|\beta|} + |\gamma| - \delta}\right)$ for $0 \le |\beta| < 2m$ and for all multi-indices γ.

Let $1 < q < \infty$, $\tau \ge \mu + sq$, $s \in \mathbb{N}_0$ and $\tau, \mu \in \mathbb{R}$. Consider the weighted Sobolev space $W^s_q(\Omega; \rho^\mu; \rho^\tau)$ endowed with the norm (see [8, Thm 3.2.4/2])

$$\|u\|_{W^s_q(\Omega;\rho^\mu;\rho^\tau)} = \left[\int_\Omega \Big(\sum_{|\alpha|=s} \rho^\mu(x)|D^\alpha u(x)|^q + \rho^\tau(x)|u(x)|^q \Big) dx \right]^{\frac{1}{q}}.$$

Let $\tau > 0$, $1 < q < \infty$ and $\rho^{-a}(x) \in L_1(\Omega)$ for an appropriate number $a \ge 0$. Then A given by (1) with the domain $\mathfrak{D}(A) = W^{2m}_q(\Omega; \rho^{q\mu}; \rho^{q\tau})$ is the closed operator in $L_q(\Omega)$ (see [8, Thm 6.6.2]). The spectrum of A consists of isolated eigenvalues $\{\lambda_j \in \mathbb{C} \colon j \in \mathbb{N}\}$ of finite algebraic multiplicity and its eigenvectors belongs to $S_{\rho(x)}(\Omega)$, as well as, its linear span is dense in $S_{\rho(x)}(\Omega)$ and, as a consequence, it is dense in $L_q(\Omega)$.

Let $\mathcal{R}_{\lambda_j}(A) = \{u \in \mathfrak{D}^\infty(A) = \bigcap_{k\in\mathbb{N}} \mathfrak{D}^k(A) \colon (\lambda_j - A)^{r_j} u = 0\}$ be the spectral subspace, corresponding to the eigenvalue λ_j of multiplicity r_j. Denote by $\mathcal{R}^\nu(A)$ the complex linear span in $L_q(\Omega)$ of all spectral subspaces $\mathcal{R}_{\lambda_j}(A)$ such that $|\lambda_j| < \nu$. Following to [4], let $\mathcal{R}(A) := \bigcup_{\nu>0} \mathcal{R}^\nu(A)$ be endowed with the quasi-norm

$$|u|_{\mathcal{R}(A)} = \|u\|_{L_q(\Omega)} + \inf\{\nu > 0 \colon u \in \mathcal{R}^\nu(A)\}.$$

3 ANALYTICAL ESTIMATES OF SPECTRAL APPROXIMATIONS

Let us consider the subspace of all exponential type vectors $\mathcal{E}(A)$ of the elliptic operator A as the union $\bigcup_{\nu>0} \mathcal{E}^\nu(A)$ which is endowed with the quasi-norm

$$|u|_{\mathcal{E}(A)} = \|u\|_{L_q(\Omega)} + \inf\{\nu > 0 \colon u \in \mathcal{E}^\nu(A)\},$$

where for any $\nu > 0$ the subspace $\mathcal{E}^\nu(A) = \left\{u \in \mathcal{E}(A) \colon \|u\|_{\mathcal{E}^\nu(A)} < \infty\right\}$ is endowed with the norm $\|u\|_{\mathcal{E}^\nu(A)} = \sum_{k\in\mathbb{N}_0} \left\|(A/\nu)^k u\right\|_{L_q(\Omega)}$ (see [3,4]).

Let $0 < s < \infty$ and $0 < r \le \infty$ or $0 \le s < \infty$ and $r = \infty$. To investigate spectral approximation errors, we consider the appropriate Besov spaces

$$\mathcal{B}^s_r(A, L_q(\Omega)) = \left\{u \in L_q(\Omega) \colon |u|_{\mathcal{B}^s_r(A,L_q(\Omega))} < \infty\right\},$$

associated with the given operator A on the space $L_q(\Omega)$, which is endowed with the norm

$$|u|_{\mathcal{B}^s_r(A,L_q(\Omega))} = \begin{cases} \left(\int_0^\infty \left[t^s E(t, u; \mathcal{E}(A), L_q(\Omega))\right]^r \dfrac{dt}{t}\right)^{1/r}, & 0 < r < \infty, \\ \sup\limits_{t>0} t^s E(t, u; \mathcal{E}(A), L_q(\Omega)), & r = \infty, \end{cases}$$

where $E(t, u; \mathcal{E}(A), L_q(\Omega)) = \inf\left\{\|u - u^0\|_{L_q(\Omega)} \colon u^0 \in \mathcal{E}(A), |u^0|_{\mathcal{E}(A)} < t\right\}$ for all $u \in L_q(\Omega)$ and $t > 0$. Denote $E(t, u; \mathcal{R}(A), L_q(\Omega)) = \inf\left\{\|u - u^0\|_{L_q(\Omega)} \colon u^0 \in \mathcal{R}(A), |u^0|_{\mathcal{R}(A)} \le t\right\}$ for all $u \in L_q(\Omega)$.

Now, we consider the space $\mathcal{E}^\nu(D) = \{u \in C^\infty(\bar{\Omega}) \colon D^\alpha u \in L_q(\Omega), |\alpha| = k \in \mathbb{N}_0\}$ endowed with the norm $\|u\|_{\mathcal{E}^\nu(D)} = \sum_{k\ge 0} \sum_{|\alpha|=k} \nu^{-k} \|D^\alpha u\|_{L_q(\Omega)}$. On $\mathcal{E}(D) = \bigcup_{\nu>0} \mathcal{E}^\nu(D)$ we define the quasi-norm $|u|_{\mathcal{E}(D)} = \|u\|_{L_q(\Omega)} + \inf\{\nu > 0 \colon u \in \mathcal{E}^\nu(D)\}$.

In [3, Thm 9] it is proved that $\mathcal{E}(D)$ coincides with the space $\mathcal{M}_q(\Omega) = \bigcup_{\nu>0} \mathcal{M}^\nu_q(\Omega)$ endowed with the quasi-norm

$$|u|_{\mathcal{M}_q(\Omega)} = \inf_{v|_\Omega = u,\, v \in L_q(\mathbb{R}^n)} \left\{\|v\|_{L_q(\mathbb{R}^n)} + \sup_{\zeta \in \operatorname{supp} Fv} |\zeta|\right\},$$

where $\operatorname{supp} Fv$ denotes the support of the Fourier-image Fv of a function $v \in L_q(\mathbb{R}^n)$ and $\mathcal{M}^\nu_q(\Omega)$ means the space of entire analytic functions $v(z)$ of the complex variable $z \in \mathbb{C}^n$ of an exponential type $\nu > 0$ which restrictions to Ω belong to $L_q(\Omega)$.

Taking into account [1, Sec. 7.2] or [8, Sec. 2.5.4] and the mentioned above equality $\mathcal{E}(D) = \mathcal{M}_q(\Omega)$, the classic Besov space $B^s_{q,r}(\Omega)$ over Ω can be endowed with the norm

$$\|u\|_{B^s_{q,r}(\Omega)} = \begin{cases} \left(\int_0^\infty \left[t^s E(t,u;\mathcal{E}(D),L_q(\Omega))\right]^r \frac{dt}{t}\right)^{1/r}, & 0<r<\infty, \\ \sup_{t>0} t^s E(t,u;\mathcal{E}(D),L_q(\Omega)), & r=\infty. \end{cases}$$

In $B^s_{q,r}(\Omega)$ we consider the subspace which is associated with the function $\rho(x)$,

$$B^s_{q,r,\rho(x)}(\Omega) = \left\{u \in B^s_{q,r}(\Omega) \colon \sup_{x\in\Omega} \rho^l(x)|D^\alpha u(x)| < \infty \text{ for all } \alpha \text{ and } l \in \mathbb{N}_0\right\}.$$

Theorem 1. *The following Bernstein-Jackson inequalities hold,*

$$\|u\|_{B^s_{q,r}(\Omega)} \le c_{s,r}|u|^s_{\mathcal{R}(A)}\|u\|_{L_q(\Omega)}, \quad u \in \mathcal{R}(A), \tag{2}$$

$$t^s E(t,u;\mathcal{R}(A),L_q(\Omega)) \le C_{s,r}\|u\|_{B^s_{q,r}(\Omega)}, \quad u \in B^s_{q,r,\rho(x)}(\Omega) \tag{3}$$

with the constants $c_{s,r} = \left(rs^{-1}(s+1)^2\right)^{1/r}$ *and* $C_{s,r} = 2^{s+1}\left(r^{-1}s(s+1)^{-2}\right)^{1/r}$ *if* $r<\infty$, $c_{s,\infty} = C_{s,\infty} = 1$. *In addition, for each* $u \in B^s_{q,r,\rho(x)}(\Omega)$,

$$\inf\left\{\|u-u^0\|_{L_q(\Omega)} \colon u^0 \in \mathcal{R}^\nu(A)\right\} \le \nu^{-s} C_{s,r}\,\|u\|_{B^s_{q,r}(\Omega)}. \tag{4}$$

Proof. First, note that applying [2, Thm 2.2], we get the following equalities

$$\mathcal{E}(A) = \mathcal{R}(A), \quad |u|_{\mathcal{E}(A)} = |u|_{\mathcal{R}(A)} \quad \text{for all } u \in \mathcal{E}(A). \tag{5}$$

Now, we show that the following linear topological isomorphism holds,

$$\mathcal{B}^s_r(A, L_q(\Omega)) = B^s_{q,r,\rho(x)}(\Omega). \tag{6}$$

Using [8, Thm 6.5.2/1, Thm 3.2.4/3], we have

$$\mathfrak{D}^\infty(A) = \bigcap \mathfrak{D}^k(A) = \bigcap W^{2mk}_q(\Omega;\rho^{q\mu k};\rho^{q\tau k}) = S_{\rho(x)}(\Omega),$$

where the locally convex space $\mathfrak{D}^\infty(A)$ endowed with the semi-norms $\|A^k u\|_{L_q(\Omega)}$ for all $k \in \mathbb{N}_0$. Above, the equality also must be understood as linear topological isomorphism.

Let us prove the equality

$$\mathcal{E}(A) = \left\{u \in \mathcal{E}(D) \colon \sup_{x\in\Omega} \rho^l(x)|D^\alpha u(x)| < \infty \text{ for all } \alpha \text{ and } l \in \mathbb{N}_0\right\}. \tag{7}$$

Since $\|A^k u\|_{L_q(\Omega)} \le \nu^k \|u\|_{L_q(\Omega)} \le \nu^{2k}\left(\sum_{|\alpha|=k} \nu^{-k}\|D^\alpha u\|_{L_q(\Omega)} + \nu^{-k}\|u\|_{L_q(\Omega)}\right)$ for all $u \in \mathcal{E}^\nu(A)$, we get $\sum \nu^{-2k}\|A^k u\|_{L_q(\Omega)} \le \sum\left(\sum_{|\alpha|=k}\nu^{-k}\|D^\alpha u\|_{L_q(\Omega)} + \nu^{-k}\|u\|_{L_q(\Omega)}\right)$. Substituting $\sigma = \nu^2$ with $\nu > 1$, we have

$$\|u\|_{\mathcal{E}^\sigma(A)} \le \|u\|_{\mathcal{E}^\nu(D)} + \frac{\nu\|u\|_{L_q(\Omega)}}{\nu-1} \le \|u\|_{\mathcal{E}^\nu(D)} + \frac{\nu\|u\|_{\mathcal{E}^\nu(D)}}{\nu-1} = \frac{2\nu-1}{\nu-1}\|u\|_{\mathcal{E}^\nu(D)}.$$

It follows that $\left\{u \in \mathcal{E}^{\sqrt{\nu}}(D) \colon \sup_{x\in\Omega}\rho^l(x)|D^\alpha u(x)| < \infty \text{ for all } \alpha \text{ and } l \in \mathbb{N}_0\right\} \subset \mathcal{E}^\nu(A)$.

On the other hand, applying [8, Thm 6.5.2/1, Lemma 6.2.3] for any $k \in \mathbb{N}$, we obtain

$$\begin{aligned}\|A^k u\|_{L_q(\Omega)} &\geq c_k \|u\|_{W_q^{2mk}(\Omega;\rho^{q\mu k};\rho^{q\tau k})}\\ &= c_k \Big[\int_\Omega \Big(\sum_{|\alpha|=2mk} \rho^{q\mu k}(x) |D^\alpha u(x)|^q + \rho^{q\tau k}(x)|u(x)|^q \Big) dx \Big]^{\frac{1}{q}} \\ &\geq c_k c_\rho^k \Big[\int_\Omega \Big(\sum_{|\alpha|=2mk} |D^\alpha u(x)|^q + |u(x)|^q \Big) dx \Big]^{\frac{1}{q}} = c_k c_\rho^k \|u\|_{W_q^{2mk}(\Omega)},\end{aligned}$$

where $c_\rho > 0$ does not depend on k. Thus,

$$\begin{aligned}\|A^{k+1}u\|_{L_q(\Omega)} = \|A^k(Au)\|_{L_q(\Omega)} &\geq c_k c_\rho^k \|Au\|_{W_q^{2mk}(\Omega)}\\ &= c_k c_\rho^k \Big(\sum_{|\alpha|=2mk} \|D^\alpha Au\|^q_{L_q(\Omega)} + \|Au\|^q_{L_q(\Omega)} \Big)^{\frac{1}{q}}\\ &\geq c_k c_\rho^k \Big(\sum_{|\alpha|=2mk} \|AD^\alpha u\|^q_{L_q(\Omega)} + \|Au\|^q_{L_q(\Omega)} \Big)^{\frac{1}{q}}\\ &\geq c_k c_1 c_\rho^{k+1} \Big(\sum_{|\alpha|=2mk} \|D^\alpha u\|^q_{W_q^{2m}(\Omega)} + \|u\|^q_{W_q^{2m}(\Omega)} \Big)^{\frac{1}{q}} = c_{k+1} c_\rho^{k+1} \|u\|_{W_q^{2m(k+1)}(\Omega)},\end{aligned}$$

where $c_{k+1} = c_k c_1 = c_1^{k+1}$ by induction on k. Hence, for each $k \in \mathbb{N}$ and $u \in \mathfrak{D}^k(A)$, we have $\|A^k u\|_{L_q(\Omega)} \geq c_1^k c_\rho^k \|u\|_{W_q^{2mk}(\Omega)}$ for all $u \in \mathfrak{D}^k(A)$, where $c_1 > 0$ does not depend on k. This leads to the inequality $\sum \nu^{-k} \|A^k u\|_{L_q(\Omega)} \geq \sum ((c_1 c_\rho)^{-1}\nu)^{-k} \|u\|_{W_q^k(\Omega)}$ from which it follows that

$$\mathcal{E}^\nu(A) \subset \Big\{ u \in \mathcal{E}^{(c_1 c_\rho)^{-1}\nu}(D) \colon \sup_{x\in\Omega} \rho^l(x)|D^\alpha u(x)| < \infty \text{ for all } \alpha \text{ and } l \in \mathbb{N}_0 \Big\}.$$

Hence, equality (7) holds. Now applying [3, Thm 9], we obtain the required equality (6).

Using (5) and [4, Thm 2], as well as, taking into account (7), we obtain the required inequalities (2), (3), while (4) directly follows from (3) and [3, Thm 6]. □

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Встановлено аналітичні оцінки помилок спектральних апроксимацій сильно вироджених еліптичних диференціальних операторів в просторі Лебега $L_q(\Omega)$ над обмеженою областю Ω. Такі еліптичні оператори характеризуються сильним виродженням їх коефіцієнтів поблизу границі, їх спектр складається із ізольованих власних значень скінченної алгебраїчної кратності, а лінійна оболонка власних і приєднаних векторів щільна в просторі $L_q(\Omega)$. Отримані результати ґрунтуються на відповідному узагальненні нерівностей Бернштейна і Джексона з обчисленням точних констант для квазінормованих апроксимаційних просторів типу Бєсова, асоційованих з даним еліптичним оператором. Апроксимаційні простори визначаються за допомогою функціоналу $E(t,u)$, який характеризує найкоротшу відстань від заданої функції $u \in L_q(\Omega)$ до замкненої лінійної оболонки спектральних підпросторів заданого оператора, що відповідають власним значенням, які за абсолютною величиною не перевищують фіксоване число $t > 0$. При цьому вказана лінійна оболонка спектральних підпросторів співпадає з підпростором цілих аналітичних функцій експоненціального типу, що не перевищує $t > 0$. Апроксимаційний функціонал $E(t,u)$ в нашому випадку відіграє роль, подібну модулю гладкості в теорії функцій.

Ключові слова і фрази: еліптичні оператори, спектральні апроксимації.