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STOROZH O.G.

ON AN APPROACH TO THE CONSTRUCTION OF THE FRIEDRICHS AND NEUMANN-KREIN EXTENSIONS OF NONNEGATIVE LINEAR RELATIONS

Let L_0 be a closed linear nonnegative (probably, positively defined) relation ("multivalued operator") in a complex Hilbert space H. In terms of the so called boundary value spaces (boundary triples) and corresponding Weyl functions and Kochubei-Strauss characteristic ones, the Friedrichs (hard) and Neumann-Krein (soft) extensions of L_0 are constructed.

It should be noted that every nonnegative linear relation L_0 in a Hilbert space H has two extremal nonnegative selfadjoint extensions: the Friedrichs extension L_F and the Neumann-Krein extension L_K, satisfying the following property:

$$(\forall \varepsilon > 0)(L_F + \varepsilon 1)^{-1} \leq (\widetilde{L} + \varepsilon 1)^{-1} \leq (L_K + \varepsilon 1)^{-1}$$

in the set of all nonnegative selfadjoint subspace extensions $\widetilde{L}$ of L_0.

The boundary triple approach to the extension theory was initiated by F. S. Rofe-Beketov, M. L. and V. I. Gorbachuk, A. N. Kochubei, V. A. Mikhailets, V. O. Dercach, M. N. Malamud, Yu. M. Arlinskii and other mathematicians.

In addition, it is showed that the construction of the mentioned extensions may be realized in a more simple way under the assumption that initial relation is a positively defined one.

Key words and phrases: Hilbert space, relation, operator, extension, boundary value space.

Ivan Franko National University, 1 Universytetska str., 79000, Lviv, Ukraine
E-mail: storog@ukr.net

INTRODUCTION

Beginning with the work by R. Arens [2], the efforts of many authors were directed at the studying of linear relations (multivalued operators), in particular, at the investigations concerning the extension theory of the linear relations in Hilbert space (see, e.g., [4,5,8,9]). A number of problems arising in the mentioned theory have been solved in terms of the so called boundary value spaces (boundary triples) and corresponding Weyl functions (see Definitions 1, 2 and [3,6,7,10,11]).

Let $\oplus$ and $\ominus$ be the symbols of orthogonal sum and orthogonal complement, respectively. Explain that under (closed) linear relation in H, where H is a fixed complex Hilbert space equipped with the inner product $(\cdot|\cdot)$ and norm $\|\cdot\|$, we understand a (closed) linear manifold in $H^2 \stackrel{def}{=} H \oplus H$ and that in the theory of linear relations every linear operator is identified with its graph. Each such relation T has the inverse $T^{-1} \stackrel{def}{=} \{(y', y) \in H^2 \,|\, (y, y') \in T\}$ and the adjoint $T^* = H^2 \ominus JT \left(= J(H^2 \ominus T)\right)$, where $\forall\, h_1, h_2 \in H \quad J\,(h_1, h_2) \stackrel{def}{=} (-ih_2, ih_1)$. This

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circumstance (the inverse and adjoint existence) makes the theory of linear relations extremely useful in the study of various problems.

Remind that a linear relation S in H is said to be nonnegative (in symbols $S \geq 0$) if for all $(y, y') \in S \;\; (y'|y) \geq 0$, positively defined (in symbols $S \gg 0$) if, in addition,

$$\inf S \stackrel{def}{=} \inf \left\{ \left(u'|u\right) \mid \left(u, u'\right) \in S, \; \|u\| = 1 \right\} > 0,$$

and selfadjoint if $S = S^*$.

In this paper the role of initial object is played by a closed linear nonnegative relation L_0 in H. It is known [5] that there exist selfadjoint extensions (probably, subspace ones) L_F and L_K of L_0 satisfying the following property:

selfadjoint extension L_1 of L_0 is nonnegative iff for any $\varepsilon > 0$

$$\forall y \in H \quad \left((L_F + \varepsilon 1_H)^{-1} y|y \right) \leq \left((L_1 + \varepsilon 1_H)^{-1} y|y \right) \leq \left((L_K + \varepsilon 1_H)^{-1} y|y \right). \tag{1}$$

In the case when L_0 is a densely defined operator, this fact was proved by M. Krein [14].

The extensions L_F and L_K are called the Friedrichs and Neumann-Krein extensions of L_0, respectively. If L_0 is a positively defined, the first of the inequalities (1) holds under $\varepsilon = 0$, too.

The aim of this article is to construct the mentioned extensions in the terms of boundary value spaces and corresponding Weyl functions. We widely use the results exposed in [1,3,6, 7,16,19], but our approach is different from ones of these papers. In particular, we (as in our previous articles [17] and [18]) deal with Cayley transforms $U(\lambda)$ of Weyl functions (Strauss-Kochubei characteristic functions in the sence of [13] and [20]). But the papers are mentioned above devoted to the investigation of $U(\lambda)$ under $\mathrm{Im}\lambda \neq 0$, while we are interested to consider the behaviour of $U(\lambda)$ in the case when $\lambda \in \mathcal{R}$, first of all in the situations as $\lambda \to -0$ and $\lambda \to -\infty$.

1 NOTATIONS AND PRELIMINARY RESULTS

Through this paper we use the following notations:

$D(T),\; R(T),\; \ker T$ are, respectively, the domain, range, and kernel of a (linear) relation (in partial, operator) T;

$D(T) = \{y \in H | \, (\exists y' \in H) : \, (y, y') \in T\} ; \;\; R(T) = \{y' \in H | \, (\exists y \in H) : \, (y, y') \in T\} ;$

$\ker T = \{y \in H | \, (y, 0) \in T\} ;$

if $\lambda \in \mathbb{C}$ then $T - \lambda = \{(y, y' - \lambda y) \, | \, (y, y') \in T\}$, and so

$\ker(T - \lambda) = \{y \in H | \, (y, 0) \in T - \lambda\} \; (= \{y \in H | \, (y, \lambda y) \in T\});$

$\overset{\wedge}{\ker}(T - \lambda) = \{(y, \lambda y : y \in \ker(T - \lambda)\} ;$

$\rho(T) = \{\lambda \in \mathbb{C} | \ker(T - \lambda) = \{0\}, \; R(T - \lambda) = H\}$ (the resolvent set of T);

1_X is the identity in X.

If X, Y are Hilbert spaces then $(\cdot|\cdot)_X$ is the symbol of scalar product in X, $\mathcal{B}(X, Y)$ is the set of linear bounded operators $A : X \to Y$ such that $D(A) = X$; $\mathcal{B}(X) \stackrel{def}{=} \mathcal{B}(X, X)$.

If $A_i : X \to Y_i \, (i = 1, 2)$ are linear operators then the notation $A = A_1 \oplus A_2$ means that $Ax = \begin{pmatrix} A_1 x \\ A_2 x \end{pmatrix}$ for every $x \in X$. Let $s - \lim$ denotes the strong limit.

Under L_0 we understand the linear relation described in the Introduction, and $L \stackrel{def}{=} L_0^*$.

Definition 1. *Let $\mathcal{H}$ be a Hilbert space and $\Gamma_1, \Gamma_2 \in \mathcal{B}(L, \mathcal{H})$. The triple $(\mathcal{H}, \Gamma_1, \Gamma_2)$ is called the boundary value space (BVS) for the linear relation L_0 if*

$$R(\Gamma_1 \oplus \Gamma_2) = \mathcal{H} \oplus \mathcal{H}, \quad \ker(\Gamma_1 \oplus \Gamma_2) = L_0$$

and for any $\hat{y} = (y, y')$, $\hat{z} = (z, z') \in L$ we have

$$(y'|z) - (y|z') = (\Gamma_1 \hat{y}|\Gamma_2 \hat{z})_{\mathcal{H}} - (\Gamma_2 \hat{y}|\Gamma_1 \hat{z})_{\mathcal{H}}.$$

Through the paper we suppose that (the selfadjoint) relation $L_2 \overset{def}{=} \ker \Gamma_2$ is nonnegative, and so $\forall \lambda < \inf L_2$ the following operators are correctly defined:

$$L_\lambda = (L_2 - \lambda)^{-1} \in \mathcal{B}(H),\ \hat{L}_\lambda = \begin{pmatrix} L_\lambda \\ 1_H + \lambda L_\lambda \end{pmatrix} \in \mathcal{B}\left(H, H^2\right), \tilde{L}_\lambda = (L_\lambda,\ 1_H + \lambda L_\lambda) \in \mathcal{B}\left(H^2, H\right),$$

i.e. $\forall y \in H\ \ \hat{L}_\lambda y = \begin{pmatrix} L_\lambda y \\ y + \lambda L_\lambda y \end{pmatrix}$, $\forall \hat{y} = (y, y') \in H^2\ \tilde{L}_\lambda \hat{y} = L_\lambda y + (y' + \lambda L_\lambda y')$ (it is easy to see that $R(\hat{L}_\lambda) = L_2$ and $\hat{L}^*_\lambda = \tilde{L}_\lambda$). Put

$$Z_\lambda = \left(\Gamma_1 \hat{L}_\lambda\right)^*, \quad \hat{Z}_\lambda = \begin{pmatrix} Z_\lambda \\ \lambda Z_\lambda \end{pmatrix}.$$

Definition 2. *A $\mathcal{B}(\mathcal{H})$-valued function*

$$M(\lambda) = \Gamma_1 \hat{Z}_\lambda \ \ (\lambda < \inf L_2)$$

is called the Weyl function of the relation L_0 corresponding to its boundary value space $(\mathcal{H}, \Gamma_1, \Gamma_2)$.

Note that $M(\lambda) = M(\lambda)^*$.

Remark 1. *The notion of BVS had been introduced at first in [12] under the assumption that L_0 is a densely defined symmetric operator having equal defect numbers. In [16] this notion was extended onto the case of nondensely defined Hermitian operators. The conception of Weyl function corresponding to a given BVS was appeared in [6] and had found its development in many papers (see, for example, [7, 10, 11] and references therein). It is easy to see that Definition 2 is equivalent to suitable defintions from the mentioned articles. It becomes clear after analyzing the results of the monograph [15] (see also [17] and [18]).*

Theorem 1. *For arbitrary $\lambda, \mu \in (-\infty, \inf L_2)$ $M(\lambda) - M(\mu) = (\lambda - \mu) Z^*_\lambda Z_\mu \left(= (\lambda - \mu)\, Z^*_\mu Z_\lambda\right)$, in particular, $\mu < \lambda$ implies $M(\lambda) - M(\mu) \gg 0$. Hence for any $z < \inf L_2$ there exist*

$$s - \lim_{\lambda \to -0} (M(\lambda) - M(z))^{-1} \overset{def}{=} R_0\,(\geq 0)\,,$$
$$s - \lim_{\lambda \to -\infty} (M(\lambda) - M(z))^{-1} \overset{def}{=} R_{-\infty}\,(\leq 0)\,.$$

Theorem 2. *Let $L_A = \ker(A_1\Gamma_1 + A_2\Gamma_2)$, where $A_1, A_2 \in \mathcal{B}(\mathcal{H})$ and*

$$A_\lambda \overset{def}{=} A_1 M(\lambda) + A_2 \ \ (\lambda < \inf L_2)\,.$$

If $A_\lambda^{-1} \in \mathcal{B}(\mathcal{H})$, then $\lambda \in \rho(L_A)$ and

$$(L_A - \lambda)^{-1} = (L_2 - \lambda)^{-1} - Z_\lambda A_\lambda^{-1} A_1 Z^*_\lambda. \tag{2}$$

Theorem 3. *The linear relation L_1 is a selfadjoint extension of L_0 iff there exists a unitary operator $K \in \mathcal{B}(\mathcal{H})$ such that $L_1 = \ker\left[(K - 1_{\mathcal{H}})\,\Gamma_1 + i\,(K + 1_{\mathcal{H}})\,\Gamma_2\right]$.*

Put

$$L^{(\lambda)} = L_0 \dot{+} \overset{\wedge}{\ker}\,(L - \lambda) \quad (\lambda < \inf L_2)\,. \tag{3}$$

Theorem 4. $L^{(\lambda)} = \ker\left(\Gamma_1 - M(\lambda)\Gamma_2\right)$.

Theorem 5. *Suppose that $z < \inf L_2$, $\lambda < \inf L_2$ and $z \neq \lambda$. Then $L^{(\lambda)}$ is a selfadjoint relation and $z \in \rho(L^{(\lambda)})$. Moreover,*

$$(L_F - z)^{-1} = s - \lim_{\lambda \to -\infty} \left(L^{(\lambda)} - z\right)^{-1}, \quad (L_K - z)^{-1} = s - \lim_{\lambda \to -0} \left(L^{(\lambda)} - z\right)^{-1}.$$

Remark 2. *The results mentioned in Theorems 1–5 above are well known or are immediate consequences of such ones (see, e. g., [1,3,5,7,9,16]).*

2 Main results

Let λ and z be as above. Before formulating the main results let us introduce the following (defined on $\rho(L_2)$) operator-functions by setting

$$\begin{gathered} R(\lambda) = (M(\lambda) - M(z))^{-1}, \quad \Omega_{\pm}(\lambda) = (M(\lambda) \pm i)\,R(\lambda), \\ U(\lambda) = (M(\lambda) - i)\,(M(\lambda) + i)^{-1}. \end{gathered} \tag{4}$$

It is easily to check by calculation that

$$U(\lambda) = \Omega_{-}(\lambda)\Omega_{+}^{-1}(\lambda), \tag{5}$$

$$\Omega_{\pm}(\lambda) = 1_{\mathcal{H}} + (M(z) \pm i)\,R(\lambda), \tag{6}$$

$$\Omega_{\pm}^{-1}(\lambda) = 1_{\mathcal{H}} - (M(z) \pm i)\,(M(\lambda) \pm i)^{-1}. \tag{7}$$

Lemma 1.

$$L^{(\lambda)} = \left\{\hat{y} \in L \,|\, (U(\lambda) - 1_{\mathcal{H}})\,\Gamma_1\hat{y} + i\,(U(\lambda) + 1_{\mathcal{H}})\,\Gamma_2\hat{y} = 0\right\}. \tag{8}$$

Proof. It is clear that (4) yields

$$(U(\lambda) - 1_{\mathcal{H}})\,M(\lambda) = -i\,(U(\lambda) + 1_{\mathcal{H}})\,. \tag{9}$$

Let us denote (temporarily) the relation from the right side of (8) by $L^{[\lambda]}$. Taking into account (9) we obtain the following:

$$\hat{y} \in L^{(\lambda)} \Rightarrow \Gamma_1\hat{y} - M(\lambda)\Gamma_2\hat{y} = 0 \Rightarrow (U(\lambda) - 1_{\mathcal{H}})\,\Gamma_1\hat{y} + i\,(U(\lambda) + 1_{\mathcal{H}})\,\Gamma_2\hat{y} = 0 \Rightarrow y \in L^{[\lambda]}.$$

Thus $L^{(\lambda)} \subset L^{[\lambda]}$. But $L^{(\lambda)}$, $L^{[\lambda]}$ are selfadjoint relations (see Theorem 3), therefore $L^{(\lambda)} = L^{[\lambda]}$. □

Lemma 2. *Let B and R be selfadjoint operators from $\mathcal{B}(\mathcal{H})$ and*

$$\Omega_{\pm} \overset{def}{=} 1_{\mathcal{H}} + BR \pm iR.$$

Then $\Omega_{\pm}^{-1} \in \mathcal{B}(\mathcal{H})$.

Proof. One can readily check by calculations that

$$\begin{pmatrix} B-i & -\Omega_- \\ -(B+i) & \Omega_+ \end{pmatrix}\begin{pmatrix} \Omega_-^* & \Omega_+^* \\ B+i & B-i \end{pmatrix} = \begin{pmatrix} \Omega_-^* & \Omega_+^* \\ B+i & B-i \end{pmatrix}\begin{pmatrix} B-i & -\Omega_- \\ -(B+i) & \Omega_+ \end{pmatrix} = -2i1_{\mathcal{H}\oplus\mathcal{H}},$$

in particular

$$\Omega_-^*\Omega_- = \Omega_+^*\Omega_+, \tag{10}$$

$$\Omega_-^*(B-i) - \Omega_+^*(B+i) = -2i1_{\mathcal{H}}, \tag{11}$$

$$(B-i)\,\Omega_+^* = \Omega_-(B-i), \quad (B+i)\,\Omega_-^* = \Omega_+(B+i). \tag{12}$$

It follows from (10) that $\|\Omega_- h\| = \|\Omega_+ h\|$ for each $h \in \mathcal{H}$. This yields that there exists an isometry $K : R(\Omega_-) \to R(\Omega_+)$ such that $\Omega_+ = K\Omega_-$, consequently there exist $K_+, K_- \in \in \mathcal{B}(\mathcal{H})$, satisfying the equalities $\Omega_-^* = \Omega_+^* K_+,\ \Omega_+^* = \Omega_-^* K_-$. Thus $R(\Omega_-^*) = R(\Omega_+^*)$. Taking into account (11) we see that $R(\Omega_-^*) + R(\Omega_+^*) = \mathcal{H}$, therefore

$$R(\Omega_-^*) = R(\Omega_+^*) = \mathcal{H}. \tag{13}$$

The equalities (13) imply

$$\ker\Omega_+ = \ker\Omega_- = \{0\}. \tag{14}$$

In view of (12) and (14) we obtain $\ker\Omega_-^* = \ker\Omega_+^* = \{0\}$. To complete the proof it is sufficient to apply (13). □

Proposition 1. *There exist the unitary operators $U_{-\infty}$, $U_0 \in \mathcal{B}(\mathcal{H})$ defined as follows:*

$$U_{-\infty} = s - \lim_{\lambda\to-\infty} U(\lambda), \quad U_0 = s - \lim_{\lambda\to-0} U(\lambda). \tag{15}$$

Moreover,

$$U_{-\infty} = (1_{\mathcal{H}} + (M(z)-i)R_{-\infty}))\,(1_{\mathcal{H}} + (M(z)+i)R_{-\infty}))^{-1}, \tag{16}$$

$$U_0 = (1_{\mathcal{H}} + (M(z)-i)R_0))\,(1_{\mathcal{H}} + (M(z)+i)R_0))^{-1}, \tag{17}$$

where $R_{-\infty}$ and R_0 are as in the Theorem 1.

Proof. It follows from Theorem 1, from (6) and from Lemma 2, applied to the operators $1_{\mathcal{H}} + BR \pm iR$ with $B = M(z)$, $R = R_{-\infty}$, that $s - \lim\limits_{\lambda\to-\infty} \Omega_\pm(\lambda) = 1_{\mathcal{H}} + (M(z) \pm i)\,R_{-\infty}$ and the operators in the right side of the latter equality are invertible in $\mathcal{B}(\mathcal{H})$. Further, in view of (7) we obtain $\left\|\Omega_+^{-1}(\lambda)\right\| \le 1 + \|M(z)+i\| \cdot \left\|(M(\lambda)+i)^{-1}\right\|$.

On the other hand, using the elementary properties of the resolvent of a selfadjoint operator we conclude that for each $\lambda < \inf L_2$ $\left\|(M(\lambda)+i)^{-1}\right\| \le 1$. Thus the family

$$\left\{\Omega_+^{-1}(\lambda) \mid -\infty < \lambda < \inf L_2\right\}$$

is uniformly bounded in $\mathcal{B}(\mathcal{H})$, therefore

$$s - \lim_{\lambda\to-\infty} \Omega_+^{-1}(\lambda)\ \left(= s - \lim_{\lambda\to-\infty} \Omega_+(\lambda)\right)^{-1} = (1_{\mathcal{H}} + (M(z)+i)\,R_{-\infty})^{-1}.$$

Whence using (5) we conclude that there exists the first limit in (15) and the equality (16) holds. Similar arguments show that there exists the second limit in (15) and the equality (17) holds.

Finally, taking into account (15) and the invertibility in $\mathcal{B}(\mathcal{H})$ of the operators in right sides of (16)–(17), we conclude that the unitarity of $U(\lambda)$ under $\lambda < \inf L_2$ yields the unitarity of $U_{-\infty}$ and U_0. □

Theorem 6.

$$L_F = \{\hat{y} \in L \,|\, (U_{-\infty} - 1_{\mathcal{H}})\,\Gamma_1 \hat{y} + i\,(U_{-\infty} + 1_{\mathcal{H}})\,\Gamma_2 \hat{y} = 0\}\,, \tag{18}$$

$$L_K = \{\hat{y} \in L \,|\, (U_0 - 1_{\mathcal{H}})\,\Gamma_1 \hat{y} + i\,(U_0 + 1_{\mathcal{H}})\,\Gamma_2 \hat{y} = 0\}\,, \tag{19}$$

where $U_{-\infty}$ and U_0 are defined according to (15).

Proof. Applying (2) under $A_1 = 1_{\mathcal{H}},\ A_2 = -M(\lambda)$ and Theorem 4 we obtain

$$\left(L^{(\lambda)} - z\right)^{-1} = (L_2 - z)^{-1} - Z_z\,(M(z) - M(\lambda))^{-1}\,Z_\lambda^* \quad (\lambda, z < \inf L_2,\ z \neq \lambda)$$

(recall that $L^{(\lambda)}$ is defined by (3)). The latter equality together Theorem 1 and Theorem 5 implies

$$(L_F - z)^{-1} = (L_2 - z)^{-1} + Z_z R_{-\infty} Z_z^*, \quad (L_K - z)^{-1} = (L_2 - z)^{-1} + Z_z R_0 Z_z^*. \tag{20}$$

On the other hand, Theorem 3 shows that there exists an unitary operator $K \in \mathcal{B}(\mathcal{H})$ such that $L_1 = \ker\,[(K - 1_{\mathcal{H}})\,\Gamma_1 + i\,(K + 1_{\mathcal{H}})\,\Gamma_2]\,.$

Applying Theorem 2 under $A_1 = (K - 1_{\mathcal{H}})\,,\ A_2 = i\,(K + 1_{\mathcal{H}})$ we conclude that

$$(L_F - z)^{-1} = (L_2 - z)^{-1} - Z_z\,[(K - 1_{\mathcal{H}})\,M(z) + i\,(K + 1_{\mathcal{H}})]^{-1}\,(K - 1_{\mathcal{H}})\,Z_z^*. \tag{21}$$

Comparing (20) and (21) we see that

$$[(K - 1_{\mathcal{H}})\,M(z) + i\,(K + 1_{\mathcal{H}})]^{-1}\,(K - 1_{\mathcal{H}}) + R_{-\infty} = 0,$$

i. e. (multiplying this identity from left by the expression contained in square brackets)

$$K\,[1_{\mathcal{H}} + M(z)R_{-\infty} + iR_{-\infty}] = 1_{\mathcal{H}} + M(z)R_{-\infty} - iR_{-\infty}.$$

Whence using (16) we obtain $K = U_{-\infty}$. The relation (18) is proved. The proof of relation (19) is analogous. □

The construction of Friedrichs and Neumann-Krein extensions of L_0 may be realized in a more simple way in the case when L_2 (and hence L_0) is a positively defined relation. Before considering this case note that the Theorem 5 implies

$$L_0 \gg 0 \Rightarrow L_F^{-1} = s - \lim_{\lambda \to -\infty} \left(L^{(\lambda)}\right)^{-1}. \tag{22}$$

Further, put

$$B \overset{def}{=} s - \lim_{\lambda \to -\infty} (M(\lambda) - M(0))^{-1}. \tag{23}$$

It follows from the Theorem 1 that the limit in (23) exists. Moreover, $B \in \mathcal{B}(\mathcal{H})$ and $B \leq 0$.

Theorem 7. *Assume that $L_2 \gg 0$ and put*

$$\gamma_1 \hat{y} = \Gamma_1 \hat{y} - M(0)\Gamma_2 \hat{y}, \tag{24}$$

$$\gamma_2 \hat{y} = \Gamma_2 \hat{y} - B\gamma_1 \hat{y} \equiv -B\Gamma_1 \hat{y} + (1_{\mathcal{H}} + BM(0))\,\Gamma_2 \hat{y}, \tag{25}$$

where $\hat{y}$ runs through L and B is defined according to (23). *Then* *i)* $(\mathcal{H}, \gamma_1, \gamma_2)$ *is a BVS for* L_0*;*

ii) $L_F = \ker \gamma_2 \equiv \{\hat{y} \in L \mid \gamma_2 \hat{y} = 0\}$;

iii) $L_K = \ker \gamma_1 \equiv \{\hat{y} \in L \mid \gamma_1 \hat{y} = 0\}$.

Proof. Since $L_K = L_0 \dot{+} \overset{\wedge}{\ker} L$ (see [5] and [3, Prop. 3.2.1]) the statement iii) is an immediate consequence of (3) and Theorem 4 under $\lambda = 0$. Further, thinking as in the proof of Theorem 6 we obtain

$$\left(L^{(\lambda)}\right)^{-1} = L_2^{-1} + Z_0 \left(M(\lambda) - M(0)\right)^{-1} Z_0^* \quad (\lambda < 0),$$

$$\tilde{L}^{-1} = L_2^{-1} + Z_0 \left[-BM(0) + (1_{\mathcal{H}} + BM(0))\right]^{-1} BZ_0^* = L_2^{-1} + Z_0 B Z_0^*,$$

where $\tilde{L} = \ker \gamma_2$. So, item *ii)* follows from (22) and (23).

Furthermore, (24), (25) may be written in the following form:

$$\begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix} = \begin{pmatrix} 1_{\mathcal{H}} & -M(0) \\ -B & 1_{\mathcal{H}} + BM(0) \end{pmatrix} \begin{pmatrix} \Gamma_1 \\ \Gamma_2 \end{pmatrix}. \tag{26}$$

It is clear that the matrix operator in the right side of (26) is invertible in $\mathcal{B}(H \oplus H)$ and

$$\begin{pmatrix} \Gamma_1 \\ \Gamma_2 \end{pmatrix} = \begin{pmatrix} 1_{\mathcal{H}} + M(0)B & M(0) \\ B & 1_{\mathcal{H}} \end{pmatrix} \begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix}.$$

Moreover, the equality

$$\begin{pmatrix} 1_{\mathcal{H}} & -M(0) \\ -B & 1_{\mathcal{H}} + BM(0) \end{pmatrix} \begin{pmatrix} 0 & 1_{\mathcal{H}} \\ -1_{\mathcal{H}} & 0 \end{pmatrix} \begin{pmatrix} 1_{\mathcal{H}} & -B \\ -M(0) & 1_{\mathcal{H}} + BM(0) \end{pmatrix} = \begin{pmatrix} 0 & 1_{\mathcal{H}} \\ -1_{\mathcal{H}} & 0 \end{pmatrix}$$

implies that for any $\hat{y}, \hat{z} \in L$ $(\Gamma_1 \hat{y} | \Gamma_2 \hat{z})_{\mathcal{H}} - (\Gamma_2 \hat{y} | \Gamma_1 \hat{z})_{\mathcal{H}} = (\gamma_1 \hat{y} | \gamma_2 \hat{z})_{\mathcal{H}} - (\gamma_2 \hat{y} | \gamma_1 \hat{z})_{\mathcal{H}}$. Hence (see [15] for the details) $(\mathcal{H}, \gamma_1, \gamma_2)$ is a boundary value space for L_0.

□

REFERENCES

[1] Ando T., Nishio K. *Positive self-adjoint extensions of positive symmetric operators.* Tohoku Math. J. 1970, **22**, 65–75.

[2] Arens R. *Operational calculus of linear relations.* Pacif. J. Math. 1961, **11** (1), 9–23. doi:10.2140/pjm.1961.11.9

[3] Arlinskii Yu.M. *Positive spaces of boundary values and sectorial extensions of a nonnegative symmetric operator.* Ukrain. Mat. Zh. 1988, **40** (1), 8–14. (in Russian)

[4] Coddington E.A. *Self-adjoint subspace extensions of nondensely defined linear operators.* Bull. Amer. Math. Sos. 1973, **79** (4), 712–715.

[5] Coddington E.A., de Snoo H.S.V. *Positive self-adjoint extensions of positive symmetric subspaces.* Math. Z. 1978, **159** (3), 203–214.

[6] Derkach V.A., Malamud M.M. *On the Weyl function and Hermite operators with lacunae.* Docl. Akad. Nauk SSSR 1987, **293** (5), 1041–1046. (in Russian)

[7] Derkach V.A., Malamud M.M. Extension theory of symmetric operators and boundary value problems. Proceedings of the Institute of Mathematics of the National Academy of Sciences of Ukraine. Kyiv, 2017. (in Russian)

[8] Dijksma A., de Snoo H.S.V. *Self-adjoint subspace extensions of symmetric operators.* Pacific J. Math. 1974, **54** (1), 71–100. doi:10.2140/pjm.1974.54.71

[9] Gorbachuk V.I., Gorbachuk M.L. Boundary Value Problems for Differential-Operator Equations. Kluver Academic Publishers, Dordrecht-Boston-London, 1991. (in Russian)

[10] Hassi S., de Snoo H.S.V., Sterk A., Winkler H. *Finite-dimensional graph perturbations of selfadjoint Sturm-Liouville operators.* Tiberiu Constantinescu Memorial Volume. Theta Foundation, 2007, 205–226.

[11] Hassi S., de Snoo H.S.V., Szafraniec F.H. *Infinite-dimensional perturbations, maximally nondensely defined symmetric operators, and some matrix representations.* Indag. Math., 2012, **23**, 1087–1117.

[12] Kochubei A.N. *On extensions of symmetric operators and symmetric binary relations.* Math. Notes 1975, **17** (1), 25–28. (translation of Mat. Zametki 1975, **17** (1), 41–48. (in Russian))

[13] Kochubei A.N. *On characteristic functions of symmetric operators.* Izv. Akad. Nauk Arm. SSR. Ser. Math. 1980, **15** (2), 218–232. (in Russian)

[14] Krein M.G. *Theory of selfadjoint extensions of semibounded operators.* Mat. Sb. 1947, **20** (3), 431–495. (in Russian)

[15] Lyantse V.E., Storozh O.G. *Methods of the Theory of Unbounded Operators*, Naukova Dumka, Kyiv, 1983. (in Russian)

[16] Malamud M.M. *On one approach to extension theory of nondensely defined Hermitian operator.* Dokl. Akad. Nauk USSR 1990, **3**, 20–25. (in Russian)

[17] Storozh O.G. *Extremal extensions of nonnegative operator and accretive boundary problems.* Ukr. Math. J. 1990, **42** (6), 858–860. (in Russian)

[18] Storozh O.G. *The Extension Theory and Abstract Boundary Operators – 3.* In: Proc. of Sixth Summer School "Algebra, Topology and Analysis", Lviv-Kozyova, Ukraine, August 4–14, 2009, Lviv, 2009, 13 p.

[19] Strauss A.V. *On extensions of semibounded operator.* Dokl. Akad. Nauk USSR 1973, **231** (3), 543–546. (in Russian)

[20] Strauss A.V. *On extensions and characteristic function of symmetric operator.* Izv. Akad. Nauk SSSR. Ser. Math. 1968, **32** (1), 186–207. (in Russian)

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Сторож О.Г. *Про один підхід до побудови розширень Фрідріхса та Неймана-Крейна невід'ємного лінійного відношення* // Карпатські матем. публ. — 2018. — Т.10, №2. — С. 387–394.

Нехай L_0 — замкнене лінійне невід'ємне (можливо, додатно визначене) відношення ("багатозначний оператор") у комплексному гільбертовому просторі H. У термінах так званих просторів граничних значень (граничних трійок) і віповідних функцій Вейля та характеристичних функцій Кочубея-Штрауса побудовано розширення Фрідріхса (жорстке розширення) та Неймана-Крейна (м'яке розширення) відношення L_0.

Зазначимо, що кожне невід'ємне лінійне відношення L_0 у гільбертовому просторі H має два екстремальні невід'ємні самоспряжені розширення: розширення Фрідріхса L_F та розширення Неймана-Крейна L_K, які володіють такою властивістю:

$$(\forall \varepsilon > 0)(L_F + \varepsilon 1)^{-1} \leq (\widetilde{L} + \varepsilon 1)^{-1} \leq (L_K + \varepsilon 1)^{-1}$$

на множині всіх невід'ємних самоспряжених розширень-відношень $\widetilde{L}$ відношення L_0.

Розвивається підхід, заснований на понятті граничної трійки. Цей підхід був започаткований Ф. С. Рофе-Бекетовим, М. Л. Горбачуком та В. І. Горбачук, А. Н. Кочубеєм, В. А. Михайлецем, В. О. Деркачем, М. Н. Маламудом, Ю. М. Арлінським та іншими математиками.

Показано, що побудова згаданих розширень може бути реалізованою простішим шляхом у випадку, коли відношення L_0 є додатно визначеним.

Ключові слова і фрази: гільбертів простір, відношення, оператор, розширення, простір граничних значень.