ISSN 2075-9827 e-ISSN 2313-0210
Carpathian Math. Publ. 2018, **10** (2), 352–359
doi:10.15330/cmp.10.2.352-359

http://www.journals.pu.if.ua/index.php/cmp
Карпатськi матем. публ. 2018, Т.10, №2, С.352–359

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ON THE SIMILARITY OF MATRICES AB AND BA OVER A FIELD

Let A and B be n-by-n matrices over a field. The study of the relationship between the products of matrices AB and BA has a long history. It is well-known that AB and BA have equal characteristic polynomials (and, therefore, eigenvalues, traces, etc.). One beautiful result was obtained by H. Flanders in 1951. He determined the relationship between the elementary divisors of AB and BA, which can be treated as a criterion when two matrices C and D can be realized as $C = AB$ and $D = BA$. If one of the matrices (A or B) is invertible, then the matrices AB and BA are similar. If both A and B are singular then matrices AB and BA are not always similar. We give conditions under which matrices AB and BA are similar. The rank of matrices plays an important role in these investigations.

Key words and phrases: matrix, similarity, rank.

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1 INTRODUCTION

Let F be a field and let $M_{m,n}(\mathrm{F})$ denote the set of m-by-n matrices with entries from F. In what follows, $GL(n, \mathrm{F})$ the group of nonsingular matrices in $M_{n,n}(\mathrm{F})$, I_k is the identity $k \times k$ matrix, and $0_{m,n}$ is the zero $m \times n$ matrix.

Let $A, B \in M_{n,n}(\mathrm{F})$. It is well known that the characteristic polynomials of AB and BA are the same (see, for example, [6, 9, 10, 14]). If one of the matrices (A or B) is invertible, then the matrices AB and BA are similar. If both A and B are singular then matrices AB and BA are not always similar (see [6, Sec. 1.3]). It is clear that matrices AB and BA are similar if and only if the matrix polynomials $I_n\lambda - AB$ and $I_n\lambda - BA$ are equivalent. It is evident, if matrices A and B commute then AB and BA are similar.

Let $A \in M_{n,m}(\mathrm{F})$ and $B \in M_{m,n}(\mathrm{F})$. In paper [3], H. Flanders solved the problem of determining the relationship between the elementary divisors of AB and those of BA. Another proof of Flanders' theorem, with some generalizations, has been given in [11] (see also [1]). Robert C. Thompson [13] proposed a new proof of Flanders' theorem. It is obvious that some connection exists between the ranks of A and B and the intertwining of the elementary divisors of AB and BA. A constructive proof of Flanders' theorem was also given in [7]. Using the Weyr characteristic the relationship between the Jordan forms of the matrix products AB and BA for matrices A and B was given in [8]. Robert E. Hartwig [5] generalizes Flanders' result for matrices over a regular strongly-pi-regular ring. It will be observed that an extension of these results to rings would be valuable and interesting. The rank conditions under which matrices AB and BA are similar were proposed in [2, 3, 13].

Suppose that A and B are complex $n \times n$ matrices. The matrix AB is similar to BA if and only if $\operatorname{rank}(AB)^j = \operatorname{rank}(BA)^j$ for each $j = 1, 2, \ldots, n$ (see [6, Sec. 3]). If A is positive semidefinite matrix and B is normal matrix, in [4] it has been proved that AB and BA are

УДК 512.643
2010 *Mathematics Subject Classification:* 15A04, 15A21.

similar. The smallest nonnegative integer k such that $\operatorname{rank} A^{k+1} = \operatorname{rank} A^k$, is the index for A and denoted by $\operatorname{Ind}(A)$. In [8] was proved that matrices AB and BA are similar if and only if $\operatorname{Ind}(AB) = \operatorname{Ind}(BA) = k$ and $\operatorname{rank}(AB)^i = \operatorname{rank}(BA)^i$ for all $i = 1, 2, \dots, k-1$.

In this note we investigate the following widely known question: Let $A, B \in M_{n,n}(\mathrm{F})$. When are matrices AB and BA similar? We give conditions in terms of rank matrices, under which matrices AB and BA are similar. If matrices AB and BA are similar we give their canonical form with respect to similarity.

2 MAIN RESULTS

Let $A, B \in M_{n,n}(\mathrm{F})$ be singular matrices and let $\operatorname{rank} A = r$. We introduce the following notation for the matrices A and B. For A there exist matrices $U, V \in GL(n, \mathrm{F})$ such that

$$UAV = \begin{bmatrix} I_r & 0_{r,n-r} \\ 0_{n-r,r} & 0_{n-r,n-r} \end{bmatrix}.$$

Put $V^{-1}BU^{-1} = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$, where $B_{11} \in M_{r,r}(\mathrm{F})$. It is easy to make sure that

$$UABU^{-1} = C = \begin{bmatrix} B_{11} & B_{12} \\ 0_{n-r,r} & 0_{n-r,n-r} \end{bmatrix} \tag{1}$$

and

$$V^{-1}BAV = D = \begin{bmatrix} B_{11} & 0_{r,n-r} \\ B_{21} & 0_{n-r,n-r} \end{bmatrix}. \tag{2}$$

We will use these notations to give the characterization of similarity of matrices AB and BA. Thus, AB and BA are similar if and only if the polynomial matrices $I_n\lambda - C$ and $I_n\lambda - D$ are equivalent, i.e. the Smith normal forms of these polynomial matrices are coincide.

In view of the above, we give the following description of similarity of the matrices AB and BA.

Theorem 1. *Let $A, B \in M_{n,n}(\mathrm{F})$ be singular matrices. If*

(a) $\operatorname{rank} B_{11} = \operatorname{rank} \begin{bmatrix} B_{11} \\ B_{21} \end{bmatrix} = \operatorname{rank} \begin{bmatrix} B_{11} & B_{12} \end{bmatrix}$, *or*

(b) $B_{11} = 0_{r,r}$ *and* $\operatorname{rank} B_{21} = \operatorname{rank} B_{12}$, *or*

(c) the matrix $\begin{bmatrix} B_{11} & B_{12} \\ B_{21} & 0_{n-r,n-r} \end{bmatrix}$ *is symmetric,*

then matrices AB and BA are similar.

Proof. (a) Since $\operatorname{rank} B_{11} = \operatorname{rank} \begin{bmatrix} B_{11} \\ B_{21} \end{bmatrix} = \operatorname{rank} \begin{bmatrix} B_{11} & B_{12} \end{bmatrix}$, then the equations $XB_{11} = B_{21}$ and $B_{11}Y = B_{12}$ are solvable. Let matrices $X_1 \in M_{n-r,r}(\mathrm{F})$ and $Y_1 \in M_{r,n-r}(\mathrm{F})$ be the solutions to these equations respectively.

For matrix $T_1 = \begin{bmatrix} I_r & 0_{r,n-r} \\ -X_1 & I_{n-r} \end{bmatrix}$ we have

$$T_1 \begin{bmatrix} B_{11} & 0_{r,n-r} \\ B_{21} & 0_{n-r,n-r} \end{bmatrix} T_1^{-1} = \begin{bmatrix} B_{11} & 0_{r,n-r} \\ 0_{n-r,r} & 0_{n-r,n-r} \end{bmatrix}.$$

Similarly, for matrix $T_2 = \begin{bmatrix} I_r & -Y_1 \\ 0_{n-r,r} & I_{n-r} \end{bmatrix}$ we have

$$T_2^{-1} \begin{bmatrix} B_{11} & B_{12} \\ 0_{n-r,r} & 0_{n-r,n-r} \end{bmatrix} T_2 = \begin{bmatrix} B_{11} & 0_{r,n-r} \\ 0_{n-r,r} & 0_{n-r,n-r} \end{bmatrix}.$$

Hence, matrices $\begin{bmatrix} B_{11} & B_{12} \\ 0_{n-r,r} & 0_{n-r,n-r} \end{bmatrix}$ and $\begin{bmatrix} B_{11} & 0_{r,n-r} \\ B_{21} & 0_{n-r,n-r} \end{bmatrix}$ are similar. Thus, AB and BA are similar.

(b) Let $B_{11} = 0_{r,r}$ and $\operatorname{rank} B_{21} = \operatorname{rank} B_{12} = s$. For B_{12} there exist matrices $U_1 \in GL(r, \mathrm{F})$ and $V_1 \in GL(n-r, \mathrm{F})$ such that

$$U_1 B_{12} V_1 = \begin{bmatrix} 0_{s,n-r-s} & I_s \\ 0_{r-s,n-r-s} & 0_{r-s,s} \end{bmatrix}.$$

Thus, for the matrix $T_1 = \operatorname{diag}\left(U_1,\ V_1^{-1}\right)$ we have

$$T_1 \begin{bmatrix} 0_{r,r} & B_{12} \\ 0_{n-r,r} & 0_{n-r,n-r} \end{bmatrix} T_1^{-1} = \begin{bmatrix} 0_{s,n-s} & I_s \\ 0_{n-s,n-s} & 0_{n-s,s} \end{bmatrix}.$$

Similarly, for matrix B_{21} there exist $U_2 \in GL(n-r, \mathrm{F})$ and $V_2 \in GL(r, \mathrm{F})$ such that

$$U_2 B_{12} V_2 = \begin{bmatrix} 0_{n-r-s,s} & 0_{n-r-s,r-s} \\ I_s & 0_{s,r-s} \end{bmatrix}$$

and for the matrix $T_2 = \operatorname{diag}\left(V_2^{-1},\ U_2\right)$ we have

$$T_2 \begin{bmatrix} 0_{r,r} & B_{12} \\ 0_{n-r,r} & 0_{n-r,n-r} \end{bmatrix} T_2^{-1} = \begin{bmatrix} 0_{n-s,s} & 0_{n-s,n-s} \\ I_s & 0_{s,n-s} \end{bmatrix}.$$

It is obvious that matrices $\begin{bmatrix} 0_{r,r} & B_{12} \\ 0_{n-r,r} & 0_{n-r,n-r} \end{bmatrix}$ and $\begin{bmatrix} 0_{r,r} & 0_{r,n-r} \\ B_{21} & 0_{n-r,n-r} \end{bmatrix}$ are similar. Thus, AB and BA are similar.

(c) Matrix $\begin{bmatrix} B_{11} & B_{12} \\ 0_{n-r,r} & 0_{n-r,n-r} \end{bmatrix}$ and its transpose $\begin{bmatrix} B_{11} & B_{12} \\ 0_{n-r,r} & 0_{n-r,n-r} \end{bmatrix}^T$ are similar. Hence, we have

$$\begin{bmatrix} B_{11} & B_{12} \\ 0_{n-r,r} & 0_{n-r,n-r} \end{bmatrix}^T = \begin{bmatrix} B_{11}^T & 0_{r,n-r} \\ B_{12}^T & 0_{n-r,n-r} \end{bmatrix} = \begin{bmatrix} B_{11} & 0_{r,n-r} \\ B_{21} & 0_{n-r,n-r} \end{bmatrix}.$$

Thus, matrices AB and BA are similar. The proof of Theorem 1 is complete. □

From Theorem 1 we have the following statement.

Corollary 1. *Let $A, B \in M_{n,n}(\mathrm{F})$ be singular matrices. If $\det B_{11} \neq 0$ then matrices AB and BA are similar.*

Consider the following example.

Example. *Let* $\mathrm{F} = \mathbb{Q}$ *be the field of rational numbers and let*

$$A = \begin{bmatrix} 7 & -3 & -11 & 9 \\ 5 & -2 & -10 & 8 \\ -12 & 5 & 21 & -17 \\ 12 & -5 & -16 & 13 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} -23 & -18 & -2 & 16 \\ -55 & -43 & -5 & 38 \\ -65 & -52 & -4 & 48 \\ -80 & -64 & -5 & 59 \end{bmatrix}$$

be matrices over $\mathbb{Q}$. *For nonsingular matrices*

$$U = \begin{bmatrix} 0 & 1 & 1 & 1 \\ -1 & 0 & 1 & 1 \\ -1 & -1 & 0 & 1 \\ -1 & -1 & -1 & 0 \end{bmatrix} \quad \text{and} \quad V = \begin{bmatrix} 3 & 2 & 1 & 2 \\ 7 & 5 & 2 & 5 \\ 0 & 0 & 9 & 4 \\ 0 & 0 & 11 & 5 \end{bmatrix}$$

over $\mathbb{Q}$ *we have*

$$UAV = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} I_3 & 0_{3,1} \\ 0_{3,1} & 0 \end{bmatrix}$$

and

$$V^{-1}BU^{-1} = \left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 1 & 3 & 1 \\ \hline 0 & 1 & 2 & 2 \end{array}\right] = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}, \text{ where } B_{11} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 1 & 3 \end{bmatrix}.$$

Thus, $\operatorname{rank} B_{11} = \operatorname{rank} \begin{bmatrix} B_{11} \\ B_{21} \end{bmatrix} = \operatorname{rank} \begin{bmatrix} B_{11} & B_{12} \end{bmatrix} = 2$. *By statement (a) of Theorem 1 matrices* AB *and* BA *are similar to the matrix* B_{11}.

Lemma 1. *Let* $A, B \in M_{n,n}(\mathrm{F})$ *be singular matrices. If* $\operatorname{rank} AB = \operatorname{rank} BA = 1$, *then* AB *and* BA *are similar.*

To prove the Lemma we need the following proposition (see also Chapter 2 in [6] and Theorem 1 in [12]).

Proposition 1. *Let* $C \in M_{n,n}(\mathrm{F})$ *be a matrix of rank one and* $\operatorname{tr} C = c$. *The matrix* C *is similar to one of the matrices*

$$D_1 = \operatorname{diag}(c, 0, \dots, 0) \text{ if } c \neq 0$$

or

$$D_2 = \operatorname{diag}\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, 0, \dots, 0 \right) \text{ if } c = 0.$$

Proof. The proof of the Proposition is algorithmic. The matrix C we write in the form $C = \overline{p} \cdot \overline{q}$, where $\overline{p} \in M_{n,1}(\mathrm{F})$ and $\overline{q} \in M_{1,n}(\mathrm{F})$. For the vector $\overline{p}$ there exists a matrix $P \in GL(n, \mathrm{F})$ such that $P \cdot \overline{p} = \begin{bmatrix} 1 & 0 & \dots & 0 \end{bmatrix}^T$. Then C is similar to a matrix of the form

$$PCP^{-1} = P\overline{p} \cdot \overline{q}P^{-1} = C_1 = \left[\begin{array}{c|ccc} \alpha_{11} & \alpha_{12} & \dots & \alpha_{1n} \\ \hline 0_{n-1,1} & & 0_{n-1,n-1} & \end{array}\right]. \tag{3}$$

It is clear that $\alpha_{11} = c$ is a trace of the matrix C.

Suppose, $c \neq 0$. For the matrix $T_1 = \left[\begin{array}{c|c} 1 & 0_{1,n-1} \\ \hline -\frac{\alpha_{12}}{c} & \\ \vdots & I_{n-1} \\ -\frac{\alpha_{1n}}{c} & \end{array}\right] \in GL(n, \mathrm{F})$ we have

$$T_1^{-1} C_1 T_1 = \operatorname{diag}(c, 0, \dots, 0) = D_1.$$

Thus, if $\operatorname{tr} C = c \neq 0$, then matrices C and D_1 are similar.

Let $\operatorname{tr} C = 0$. From equality (3) it follows

$$C_1 = \left[\begin{array}{c|ccc} 0 & \alpha_{12} & \dots & \alpha_{1n} \\ \hline 0_{n-1,1} & & 0_{n-1,n-1} & \end{array}\right].$$

For elements $\{\alpha_{12}, \alpha_{13}, \ \dots, \ \alpha_{1n}\}$ there exists a matrix $T_0 \in GL(n-1, \mathrm{F})$ such that $\begin{bmatrix} \alpha_{12}, & \alpha_{13}, & \dots, & \alpha_n \end{bmatrix} T_0 = \begin{bmatrix} 1 & 0 & \dots & 0 \end{bmatrix}$. Thus, for the matrix

$$T_2 = \left[\begin{array}{c|c} 1 & 0_{1,n-1} \\ \hline 0_{n-1,1} & T_0 \end{array}\right] \in GL(n, \mathrm{F})$$

we have

$$T_2^{-1} C_1 T_2 = \operatorname{diag}\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \ 0, \ \dots, \ 0 \right) = D_2.$$

Since $\operatorname{tr} C = 0$, matrices C and D_2 are similar. This completes the proof of the Proposition. □

Proof. Let $A, B \in M_{n,n}(\mathrm{F})$ be singular matrices and

$$\operatorname{rank} AB = \operatorname{rank} BA = 1.$$

Suppose $\operatorname{rank} B \geq \operatorname{rank} A = r$. Matrix AB is similar to the matrix

$$C = \begin{bmatrix} B_{11} & B_{12} \\ 0_{n-r,r} & 0_{n-r,n-r} \end{bmatrix},$$

where $B_{11} \in M_{r,r}(\mathrm{F})$ (see equalities (1) and (2)). Similarly BA is similar to the matrix

$$D = \begin{bmatrix} B_{11} & 0_{r,n-r} \\ B_{21} & 0_{n-r,n-r} \end{bmatrix}.$$

Thus, $\operatorname{tr} AB = \operatorname{tr} BA = \operatorname{tr} B_{11}$. Put $\operatorname{tr} B_{11} = c$.

Suppose $c \neq 0$. By Proposition 1 matrices AB and BA are similar to the matrix $D_1 = \operatorname{diag}(c, 0, \dots, 0)$.

If $c = 0$ then by Proposition matrices AB and BA are similar to the matrix $D_2 = \operatorname{diag}\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \ 0, \ \dots, \ 0 \right)$, which completes the proof of the Lemma. □

Corollary 2. *Let $A, B \in M_{n,n}(\mathrm{F})$ be singular matrices and $\operatorname{rank} A = 1$. If $AB \neq 0_{n,n}$ and $BA \neq 0_{n,n}$ then AB and BA are similar.*

Corollary 3. *Let $A, B \in M_{2,2}(\mathrm{F})$. If $AB \neq 0_{2,2}$ and $BA \neq 0_{2,2}$ then AB and BA are similar.*

Theorem 2. *Let $A, B \in M_{n,n}(\mathrm{F})$ and let $\operatorname{rank} A = 2$. If $\operatorname{rank} AB = \operatorname{rank} BA$ then AB and BA are similar.*

Proof. If $\operatorname{rank} AB = \operatorname{rank} BA = 1$ then by Lemma 1 matrices AB and BA are similar. Suppose $\operatorname{rank} AB = \operatorname{rank} BA = 2$. Matrix AB is similar to the matrix $C = \begin{bmatrix} B_{11} & B_{12} \\ 0_{n-2,2} & 0_{n-2,n-2} \end{bmatrix}$, where $B_{11} \in M_{2,2}(\mathrm{F})$ (see equalities (1) and (2)). Similarly, BA is similar to the matrix $D = \begin{bmatrix} B_{11} & 0_{2,n-2} \\ B_{21} & 0_{n-2,n-2} \end{bmatrix}$. Thus, $\operatorname{rank} \begin{bmatrix} B_{11} \\ B_{21} \end{bmatrix} = \operatorname{rank} \begin{bmatrix} B_{11} & B_{12} \end{bmatrix} = 2.$

If $B_{11} = 0_{2,2}$ or $\det B_{11} \neq 0$ then by Theorem 1b or Corollary 1 respectively matrices AB and BA are similar. Let $\operatorname{rank} B_{11} = 1$ and let $\operatorname{tr} B_{11} \neq 0$. For B_{11} there exists a matrix $U_{11} \in GL(2, \mathrm{F})$ such that

$$U_{11} B_{11} U_{11}^{-1} = \begin{bmatrix} \alpha & 0 \\ 0 & 0 \end{bmatrix},$$

where $\alpha = \operatorname{tr} B_{11}$. For the matrix $T_{11} = \begin{bmatrix} U_{11} & 0_{2,n-2} \\ 0_{n-2,2} & I_{n-2} \end{bmatrix}$ we have

$$T_{11} C T_{11}^{-1} = C_{11} = \left[\begin{array}{cc|c} \alpha & 0 & \multirow{2}{*}{$\widetilde{B}_{12}$} \\ 0 & 0 & \\ \hline \multicolumn{2}{c|}{0_{n-2,1}} & 0_{n-2,n-1} \end{array}\right],$$

where $\widetilde{B}_{12} = B_{12} U_1^{-1}$. It is evident that $\operatorname{rank} C_{11} = 2$. It is easy to make sure that if $n = 3$ then $\widetilde{B}_{12} = \begin{bmatrix} c_{13} & c_{23} \end{bmatrix}^T$ and $c_{23} \neq 0$. For $\widetilde{B}_{12}$ the exists a matrix $U_{12} \in GL(n-2, \mathrm{F})$ such that

$$\widetilde{B}_{12} U_{12} = \begin{bmatrix} \alpha_1 & 0 & \dots & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \end{bmatrix}.$$

Thus, for the matrix $T_{12} = \begin{bmatrix} I_2 & 0_{2,n-2} \\ 0_{n-2,2} & U_{12} \end{bmatrix}$ we have

$$T_{12}^{-1} C_{11} T_{12} = C_{12} = \left[\begin{array}{cccc|c} \alpha & 0 & \alpha_1 & 0 & \multirow{2}{*}{$0_{2,n-4}$} \\ 0 & 0 & 0 & 1 & \\ \hline \multicolumn{4}{c|}{0_{n-2,4}} & 0_{n-2,n-4} \end{array}\right].$$

It is obvious that matrix C_{12} is similar to the matrix $C_{13} = \left[\begin{array}{ccc|c} \alpha & 0 & 0 & \multirow{2}{*}{$0_{2,n-3}$} \\ 0 & 0 & 1 & \\ \hline \multicolumn{3}{c|}{0_{n-2,4}} & 0_{n-2,n-3} \end{array}\right]$.

It may be noted that matrices D and D^T are similar. Reasoning similarly we convince ourselves that the matrix $\begin{bmatrix} B_{11}^T & B_{21}^T \\ 0_{n-r,r} & 0_{n-r,n-r} \end{bmatrix}$ is similar to the matrix C_{13}. Thus, in the case when $\operatorname{tr} B_{11} \neq 0$, matrices C and D are similar.

Let us now consider the case when $\operatorname{rank} B_{11} = 1$ and $\operatorname{tr} B_{11} = 0$. For B_{11} there exists a matrix $V_{11} \in GL(2, \mathrm{F})$ such that

$$V_{11} B_{11} V_{11}^{-1} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

For the matrix $S_{11} = \begin{bmatrix} V_{11} & 0_{2,n-2} \\ 0_{n-2,2} & I_{n-2} \end{bmatrix}$ we have

$$S_{11} C S_{11}^{-1} = C_{21} = \left[\begin{array}{cc|c} 0 & 1 & \multirow{2}{*}{$\widehat{B}_{12}$} \\ 0 & 0 & \\ \hline \multicolumn{2}{c|}{0_{n-2,1}} & 0_{n-2,n-1} \end{array}\right], \quad \text{where} \quad \widehat{B}_{12} = B_{12} V_{11}^{-1}.$$

Obviously that $\operatorname{rank} C_{21} = 2$. We note, if $n = 3$ then $\widehat{B}_{12} = \begin{bmatrix} c_{13} & c_{23} \end{bmatrix}^T$ and $c_{23} \neq 0$.

For $\widehat{B}_{12}$ the exists a matrix $V_{12} \in GL(n-2, \mathrm{F})$ such that

$$\widehat{B}_{12}V_{12} = \begin{bmatrix} \beta_1 & 0 & \dots & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \end{bmatrix}.$$

Thus, for the matrix $S_{12} = \begin{bmatrix} I_2 & 0_{2,n-2} \\ 0_{n-2,2} & V_{12} \end{bmatrix}$ we have

$$S_{12}^{-1}C_{21}S_{12} = C_{22} = \left[\begin{array}{cccc|c} 0 & 1 & \beta_1 & 0 & \multirow{2}{*}{$0_{2,n-4}$} \\ 0 & 0 & 0 & 1 & \\ \hline \multicolumn{4}{c|}{0_{n-2,4}} & 0_{n-2,n-4} \end{array}\right].$$

It is evident that matrix C_{22} is similar to the matrix $C_{23} = \left[\begin{array}{ccc|c} 0 & 1 & 0 & \multirow{2}{*}{$0_{2,n-3}$} \\ 0 & 0 & 1 & \\ \hline \multicolumn{3}{c|}{0_{n-2,4}} & 0_{n-2,n-3} \end{array}\right].$

Reasoning similarly, we can prove that matrix $\begin{bmatrix} B_{11}^T & B_{21}^T \\ 0_{n-r,r} & 0_{n-r,n-r} \end{bmatrix}$ is similar to the matrix C_{23}. Thus in the case when $\operatorname{tr} B_{11} = 0$ matrices C and D are similar.

So, we have that matrices AB and BA are similar and the proof of Theorem 2 is complete. □

From Theorem 2 we have the following statement.

Corollary 4. *Let $A, B \in M_{3,3}(\mathrm{F})$. If* $\operatorname{rank} AB = \operatorname{rank} BA$ *then matrices AB and BA are similar.*

ACKNOWLEDGEMENTS

The author wishes to thank the referee for a careful reading of the manuscript and for his comments and suggestions.

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Received 01.07.2018

Прокіп В.М. *Про подібність матриць AB і BA над полем* // Карпатські матем. публ. — 2018. — Т.10, №2. — С. 352–359.

Нехай A і B — $n \times n$ матриці над полем. Вивчення зв'язків між добутками матриць AB і BA має давню історію. Загальновідомо, що матриці AB та BA мають однакові характеристичні многочлени (отже, власні значення, сліди тощо). Один вагомий результат був отриманий Х. Фландерсом у 1951 році. Він вказав зв'язок між елементарними дільниками AB та BA, який можна розглядати як критерій, коли дві матриці C і D можуть бути зображені у вигляді добутків $C = AB$ і $D = BA$. Якщо одна з матриць (A або B) є неособливою, то матриці AB і BA подібні. Якщо ж A і B особливі матриці, то матриці AB і BA не завжди подібні. В статті наведено умови, за яких матриці AB і BA подібні. Поняття рангу відіграє важливу роль у цих дослідженнях.

Ключові слова і фрази: матриця, подібність, ранг.