ISSN 2075-9827 e-ISSN 2313-0210
Carpathian Math. Publ. 2018, **10** (2), 303–312
doi:10.15330/cmp.10.2.303-312

http://www.journals.pu.if.ua/index.php/cmp
Карпатські матем. публ. 2018, Т.10, №2, С.303–312

ILASH N.B.

HILBERT POLYNOMIALS OF THE ALGEBRAS OF SL_2-INVARIANTS

We consider one of the fundamental problems of classical invariant theory, the research of Hilbert polynomials for an algebra of invariants of Lie group SL_2. Form of the Hilbert polynomials gives us important information about the structure of the algebra. Besides, the coefficients and the degree of the Hilbert polynomial play an important role in algebraic geometry. It is well known that the Hilbert function of the algebra SL_n-invariants is quasi-polynomial. The Cayley-Sylvester formula for calculation of values of the Hilbert function for algebra of covariants of binary d-form $\mathcal{C}_d = \mathbb{C}[V_d \oplus \mathbb{C}^2]_{SL_2}$ (here V_d is the $d+1$-dimensional space of binary forms of degree d) was obtained by Sylvester. Then it was generalized to the algebra of joint invariants for n binary forms. But the Cayley-Sylvester formula is not expressed in terms of polynomials.

In our article we consider the problem of computing the Hilbert polynomials for the algebras of joint invariants and joint covariants of n linear forms and n quadratic forms. We express the Hilbert polynomials $\mathcal{H}(\mathcal{I}_1^{(n)}, i) = \dim(\mathcal{C}_1^{(n)})_i$, $\mathcal{H}(\mathcal{C}_1^{(n)}, i) = \dim(\mathcal{C}_1^{(n)})_i$, $\mathcal{H}(\mathcal{I}_2^{(n)}, i) = \dim(\mathcal{I}_2^{(n)})_i$, $\mathcal{H}(\mathcal{C}_2^{(n)}, i) = \dim(\mathcal{C}_2^{(n)})_i$ of those algebras in terms of quasi-polynomials. We also present them in the form of Narayana numbers and generalized hypergeometric series.

Key words and phrases: classical invariant theory, invariants, Hilbert function, Hilbert polynomials, Poincaré series, combinatorics.

Khmelnytskyi National University, 11 Instytytska str., 29016, Khmelnytskyi, Ukraine
E-mail: nadyailash@gmail.com

INTRODUCTION

Let $\mathbb{K}$ be a field of characteristic zero. Let V_d be the $d+1$-dimensional module of binary forms of degree d. Let $V_{\mathbf{d}} = V_{d_1} \oplus V_{d_2} \oplus \ldots \oplus V_{d_n}$, $\mathbf{d} := (d_1, d_2, \ldots d_n)$. Denote by $\mathbb{K}[V_{\mathbf{d}}]^{SL_2}$ the algebra of polynomial SL_2-invariant functions on $V_{\mathbf{d}}$. It is well known that $\mathcal{I}_{\mathbf{d}} := \mathbb{K}[V_{\mathbf{d}}]^{SL_2}$ is finitely generated and graded:

$$\mathcal{I}_{\mathbf{d}} := (\mathcal{I}_{\mathbf{d}})_0 \oplus (\mathcal{I}_{\mathbf{d}})_1 \oplus \ldots \oplus (\mathcal{I}_{\mathbf{d}})_i \oplus \ldots,$$

here $(\mathcal{I}_{\mathbf{d}})_i$ is a vector $\mathbb{K}$-space of invariants of degree i. The dimension of the vector space $(\mathcal{I}_{\mathbf{d}})_i$ is called *the Hilbert function* of the algebra $\mathcal{I}_{\mathbf{d}}$. It is defined as a function of the variable i :

$$\mathcal{H}(\mathcal{I}_{\mathbf{d}}, i) = \dim(\mathcal{I}_{\mathbf{d}})_i.$$

It is well known that the Hilbert function of an arbitrary finitely generated graded $\mathbb{K}$-algebra is a quasi-polynomial (starting from some i), see [7, 13, 15]. Since the algebra of invariants $\mathcal{I}_{\mathbf{d}}$ is finitely generated, we have

$$\mathcal{H}(\mathcal{I}_{\mathbf{d}}, i) = h_0(i) i^r + h_1(i) i^{r-1} + \ldots,$$

УДК 512.647
2010 *Mathematics Subject Classification:* 13N15, 13A50, 05A19, 05E40.

where $h_k(i)$ is some periodic function with values in $\mathbb{Q}$. The quasi-polynomial $\mathcal{H}(\mathcal{I}_{\mathbf{d}}, i)$ is called *the Hilbert polynomial* of algebra of invariants $\mathcal{I}_{\mathbf{d}}$.

For the case of one binary form ($n = 1$) there exists classical Cayley-Sylvester formula for calculation of values of Hilbert function of $\mathcal{I}_d$:

$$\mathcal{H}(\mathcal{I}_d, i) = \omega_d(i, 0) - \omega_d(i, 2),$$

where $\omega_d(i, k)$ is the number of non-negative integer solutions of the system:

$$\begin{cases} \alpha_1 + 2\alpha_2 + \ldots + d\alpha_d = \dfrac{di - k}{2}, \\ \alpha_1 + \alpha_2 + \ldots + \alpha_d = i. \end{cases}$$

Also (see [8, 14]) we have

$$\mathcal{H}(\mathcal{I}_d, i) = \left[q^{\frac{id}{2}}\right] \left(\frac{(1 - q^{d+1})\,(1 - q^{d+2}) \ldots (1 - q^{d+i})}{(1 - q^2)\,(1 - q^3) \ldots (1 - q^i)} \right),$$

where $\left[q^{\frac{id}{2}}\right]$ denotes the coefficient of $q^{\frac{id}{2}}$. Generalizations of these formulas to the algebra $\mathcal{I}_{\mathbf{d}}$ was obtained in [1–4].

However, all these results are combinatorial formulas. They are not expressed in terms of Hilbert polynomials in i. Note that, it is hard to calculate for those formulas even for small values of d_k and i.

Although, Maple-procedure for computing of the Hilbert polynomials of the algebras of SL_2-invariants for small values of d was being offered in [5].

A partial characterization of Hilbert polynomials for non-standard graded algebras was obtained in [6].

Consider a direct sum of n linear forms $nV_1 = \underbrace{V_1 \oplus V_1 \oplus \ldots \oplus V_1}_{n \text{ times}}$. In the language of classical invariant theory the algebras $\mathcal{I}_1^{(n)} := \mathbb{C}[nV_1]^{SL_2}$ and $\mathcal{C}_1^{(n)} := \mathbb{C}[nV_1 \oplus \mathbb{C}^2]^{SL_2}$ are called *the algebra of joint invariants* and *the algebra of joint covariants for the n linear forms* respectively. Let V_2 be the complex vector space of quadratic binary forms endowed with the natural action of the special linear group SL_2. Consider the corresponding action of the group SL_2 on the algebras of polynomial functions $\mathbb{C}[nV_2]$ and $\mathbb{C}[nV_2 \oplus \mathbb{C}^2]$, where $nV_2 := \underbrace{V_2 \oplus V_2 \oplus \cdots \oplus V_2}_{n \text{ times}}$. Denote by $\mathcal{I}_2^{(n)} = \mathbb{C}[nV_2]^{SL_2}$ and by $\mathcal{C}_2^{(n)} = \mathbb{C}[nV_2 \oplus \mathbb{C}^2]^{SL_2}$ the corresponding algebras of invariant polynomial functions. In the language of classical invariant theory the algebras $\mathcal{I}_2^{(n)}$ and $\mathcal{C}_2^{(n)}$ are called *the algebra of joint invariants and the algebra of joint covariants for the n quadratic forms* respectively.

The algebras $\mathcal{C}_1^{(n)}, \mathcal{I}_1^{(n)}, \mathcal{C}_2^{(n)}$ and $\mathcal{I}_2^{(n)}$ are graded:

$$\mathcal{C}_1^{(n)} = (\mathcal{C}_1^{(n)})_0 + (\mathcal{C}_1^{(n)})_1 + \cdots + (\mathcal{C}_1^{(n)})_i + \cdots, \quad \mathcal{I}_1^{(n)} = (\mathcal{I}_1^{(n)})_0 + (\mathcal{I}_1^{(n)})_1 + \cdots + (\mathcal{I}_1^{(n)})_i + \cdots,$$
$$\mathcal{C}_2^{(n)} = (\mathcal{C}_2^{(n)})_0 + (\mathcal{C}_2^{(n)})_1 + \cdots + (\mathcal{C}_2^{(n)})_i + \cdots, \quad \mathcal{I}_2^{(n)} = (\mathcal{I}_2^{(n)})_0 + (\mathcal{I}_2^{(n)})_1 + \cdots + (\mathcal{I}_2^{(n)})_i + \cdots,$$

where each of the subspaces $(\mathcal{C}_1^{(n)})_i, (\mathcal{I}_1^{(n)})_i, (\mathcal{C}_2^{(n)})_i$ and $(\mathcal{I}_2^{(n)})_i$ is finite dimensional. The functions

$$\mathcal{H}(\mathcal{C}_1^{(n)}, i) = \dim(\mathcal{C}_1^{(n)})_i, \quad \mathcal{H}(\mathcal{I}_1^{(n)}, i) = \dim(\mathcal{I}_1^{(n)})_i,$$
$$\mathcal{H}(\mathcal{C}_2^{(n)}, i) = \dim(\mathcal{C}_2^{(n)})_i, \quad \mathcal{H}(\mathcal{I}_2^{(n)}, i) = \dim(\mathcal{I}_2^{(n)})_i$$

are called the Hilbert polynomials of the algebra of joint covariants for the n linear forms, the Hilbert polynomial of the algebra of joint invariants for the n linear forms, the Hilbert polynomial of the algebra of joint covariants for the n quadratic forms and the Hilbert polynomial of the algebra of joint invariants for the n quadratic forms, respectively. The formal power series

$$\mathcal{P}(\mathcal{C}_1^{(n)},z)=\sum_{i=0}^{\infty}\mathcal{H}(\mathcal{C}_1^{(n)},i)\,z^i,\quad \mathcal{P}(\mathcal{I}_1^{(n)},z)=\sum_{i=0}^{\infty}\mathcal{H}(\mathcal{I}_1^{(n)},i)\,z^i,$$
$$\mathcal{P}(\mathcal{C}_2^{(n)},z)=\sum_{i=0}^{\infty}\mathcal{H}(\mathcal{C}_2^{(n)},i)\,z^i,\ \mathcal{P}(\mathcal{I}_2^{(n)},z)=\sum_{i=0}^{\infty}\mathcal{H}(\mathcal{I}_2^{(n)},i)\,z^i$$

are called the Poincaré series of the algebras $\mathcal{C}_1^{(n)},\mathcal{I}_1^{(n)},\mathcal{C}_2^{(n)}$ and $\mathcal{I}_2^{(n)}$ respectively.

In the present paper we obtain explicit formulas for computation of the Hilbert polynomial of those algebras. We present some results in terms of *generalized hypergeometric functions.* A generalized hypergeometric function is given by a hypergeometric series, i.e., a series for which the ratio of successive terms can be written as follows

$${}_pF_q\left[\begin{matrix}a_1,\ldots,a_p\\ b_1,\ldots,b_q\end{matrix}\middle| z\right]=\sum_{k=0}^{\infty}\frac{\prod_{h=1}^{p}(a_h)_k z^k}{\prod_{j=1}^{q}(b_j)_k k!},$$

where $(a)_k=a(a+1)\ldots(a+k-1)$ is the Pochhammer symbol or rising factorial.

If any a_j is a non-positive integer $(0,-1,-2,\ldots)$, then the series has only a finite number of terms and in fact is a polynomial of degree a_j. If any b_k is a non-positive integer (excepting the previous case with $b_k<a_j$), then the denominators become 0 and the series is undefined.

In the present paper we compute the Hilbert polynomials of the algebras of joint covariants and invariants for the n linear and quadratic forms:

$$\mathcal{H}(\mathcal{I}_1^{(n)},i)=\begin{cases}N_{n+k-1,\,k+1}, & \text{if } i=2k,\\ 0, & \text{if } i=2k+1,\end{cases}$$
$$\mathcal{H}(\mathcal{C}_1^{(n)},i)=\begin{cases}\binom{n+k-1}{k}^2, & \text{if } i=2k,\\ \\ nN_{n+k,\,k+1}, & \text{if } i=2k+1,\end{cases}$$

$$\mathcal{H}(\mathcal{I}_2^{(n)},i)=\begin{cases}\sum\limits_{k=0}^{[\frac{i}{2}]}\binom{n+k-1}{k}^2\binom{n+i-2k-2}{n-2}\dfrac{3k-i+1}{k+1}, & \text{if } i>1,\\ 1, & \text{if } i=1,\end{cases}$$
$$\mathcal{H}(\mathcal{C}_2^{(n)},i)=\sum_{k=0}^{[\frac{i}{2}]}\binom{n+k-1}{k}^2\binom{n+i-2k-1}{n-1},$$

where $N_{n,k}=\frac{1}{n}\binom{n}{k}\binom{n}{k-1}, (1\le k\le n)$ is the Narayana number.

We also express the Hilbert polynomials $\mathcal{H}(\mathcal{I}_2^{(n)},i), \mathcal{H}(\mathcal{C}_2^{(n)},i)$ in terms of generalized hypergeometric function:

$$\mathcal{H}(\mathcal{I}_2^{(n)},i) = (1-n)\binom{n+i-2}{i}{}_5F_4\left[\begin{matrix} n, n, -\frac{i}{2}, -\frac{i-1}{2}, -\frac{i}{3}+\frac{4}{3} \\ 1, -\frac{n+i-2}{2}, -\frac{n+i-3}{2}, -\frac{i}{3}+\frac{1}{3} \end{matrix}\middle| 1\right], \text{if } n>3, i>1,$$

$$\mathcal{H}(\mathcal{C}_2^{(n)},i) = \binom{n+i-1}{i}{}_4F_3\left[\begin{matrix} n, n, -\frac{i}{2}, -\frac{i-1}{2}, -\frac{n}{3} \\ 1, -\frac{n+i-1}{2}, -\frac{n+i-2}{2} \end{matrix}\middle| 1\right], \text{if } n>2.$$

1 Hilbert polynomials of the algebras of joint invariants and covariants of n linear forms

Poincaré series for the algebras of joint invariants and covariants of n linear forms was derived by L. Bedratyuk in [2]. Using them, author found the following explicit formula for Poincaré series those algebras in [11]:

$$\mathcal{P}(\mathcal{I}_1^{(n)},z) = \frac{N_{n-2}(z^2)}{(1-z^2)^{2n-3}} \quad \text{and} \quad \mathcal{P}(\mathcal{C}_1^{(n)},z) = \frac{W_{n-1}(z^2)+nzN_{n-1}(z^2)}{(1-z^2)^{2n-1}},$$

where

$$N_n(z) = \sum_{k=1}^{n}\frac{1}{k}\binom{n-1}{k-1}\binom{n}{k-1}z^{k-1} \quad \text{and} \quad W_n(z) = \sum_{k=0}^{n}\binom{n}{k}^2 z^k$$

are the Narayana polynomials. Let us use these formulas to obtain the Hilbert polynomials of the algebras $\mathcal{I}_1^{(n)}$ and $\mathcal{C}_1^{(n)}$.

To prove Theorem 1, we need the following lemma.

Lemma 1 ([9, 11, 16])**.** *Let m, k, s be non-negative integers. Then the generalized Le Jen Shoo identity holds:*

$$\sum_{i=0}^{\min\{k,m\}}\binom{m}{i}\binom{m+2s}{i+s}\binom{k-i+2m+2s}{2m+2s} = \binom{m+k+s}{m+s}\binom{m+k+2s}{m+s}.$$

Theorem 1. *The following formulas hold*

$$(i) \quad \mathcal{H}(\mathcal{I}_1^{(n)},i) = \begin{cases} N_{n+k-1,\,k+1}, & \text{if } i=2k, \\ 0, & \text{if } i=2k+1, \end{cases}$$

$$(ii) \quad \mathcal{H}(\mathcal{C}_1^{(n)},i) = \begin{cases} \binom{n+k-1}{k}^2, & \text{if } i=2k, \\ nN_{n+k,\,k+1}, & \text{if } i=2k+1, \end{cases}$$

where $N_{n,\,k} = \frac{1}{n}\binom{n}{k}\binom{n}{k-1}$ are the Narayana numbers.

Proof. (i) Let us expand function

$$\mathcal{P}(\mathcal{I}_1^{(n)},z) = \frac{N_{n-2}(z^2)}{(1-z^2)^{2n-3}}$$

into the Taylor series about $z = 0$:

$$\begin{aligned}\mathcal{P}(\mathcal{I}_1^{(n)}, z) &= \sum_{k=0}^{n-3} \binom{n-3}{k}\binom{n-2}{k}\frac{z^{2k}}{k+1}\sum_{i=0}^{\infty}\binom{(2n-3)+i-1}{i} z^{2i}\\ &= \sum_{k=0}^{\infty}\sum_{i=0}^{\min\{k,n-3\}}\binom{n-3}{i}\binom{n-2}{i}\binom{2n+k-i-4}{k-i}\frac{1}{i+1}z^{2k}\\ &= \sum_{k=0}^{\infty}\frac{z^{2k}}{n-1}\sum_{i=0}^{\min\{k,n-3\}}\binom{n-3}{i}\binom{n-1}{i+1}\binom{2n+k-i-4}{k-i}.\end{aligned}$$

Using Lemma 1 ($m = n-3$ and $s = 1$), we have:

$$\mathcal{P}(\mathcal{I}_1^{(n)}, z) = \sum_{k=0}^{\infty}\frac{1}{n-1}\binom{n+k-2}{n-2}\binom{n+k-1}{n-2}z^{2k}.$$

Statement (i) follows immediately from the definitions of Poincaré series, Hilbert polynomials and Narayana numbers.

Note that the identity $\mathcal{P}(\mathcal{I}_1^{(n)}, z) = \frac{N_{n-2}(z^2)}{(1-z^2)^{2n-3}}$ holds for $n \geq 3$. Then statement (i) holds for $n \geq 3$. Consider the case $n = 2$.We obtain that (x_1, y_1) are coordinates for the first V_1 and (x_2, y_2) are coordinates for the second one, both with respect to the canonical representation of SL_2. There is a single quadratic invariant $y_1x_2 - x_1y_2$. Hence

$$\mathcal{P}(\mathcal{I}_1^{(2)}, z) = \frac{1}{1-z^2} = 1 + z^2 + z^4 + z^6 + \dots.$$

We have

$$\mathcal{H}(\mathcal{I}_1^{(2)}, i) = \cos^2\frac{\pi i}{2} = N_{2+[\frac{i}{2}]-1,\,2-1}\cos^2\frac{i\pi}{2}.$$

This proves that statement (i) holds for $n \geq 2$.

(ii) As above we use Poincaré series of the algebra $\mathcal{C}_1^{(n)}$ $(n > 1)$:

$$\begin{aligned}\mathcal{P}(\mathcal{C}_1^{(n)}, z) &= \frac{\sum_{k=0}^{n-1}\binom{n-1}{k}^2 z^{2k}}{(1-z^2)^{2n-1}} + \frac{\sum_{k=0}^{n-2}\binom{n-2}{k}\binom{n}{k+1}z^{2k+1}}{(1-z^2)^{2n-1}}\\ &= \sum_{k=0}^{n-1}\binom{n-1}{k}^2 z^{2k}\sum_{k=0}^{\infty}\binom{(2n-1)+i-1}{i}z^{2i}\\ &\quad + \sum_{k=0}^{n-2}\binom{n-2}{k}\binom{n}{k+1}z^{2k+1}\sum_{k=0}^{\infty}\binom{(2n-1)+i-1}{i}z^{2i}\\ &= \sum_{k=0}^{\infty}\sum_{i=0}^{\min\{k,n-1\}}\binom{n-1}{i}^2\binom{2n+k-i-2}{k-i}z^{2k}\\ &\quad + \sum_{k=0}^{\infty}\sum_{i=0}^{\min\{k,n-2\}}\binom{n-2}{i}\binom{n}{i+1}\binom{k-i+2n-2}{2n-2}z^{2k+1}.\end{aligned}$$

Using Lemma 1, we get:

$$\mathcal{P}(\mathcal{C}_1^{(n)}, z) = \sum_{k=0}^{\infty}\binom{n+k-1}{k}^2 z^{2k} + \sum_{k=0}^{\infty}\binom{n+k-1}{n-1}\binom{n+k}{n-1}z^{2k+1}.$$

This proves (ii) for $n>1$. Using a Maple-procedure for computing the Hilbert polynomials of the algebra $\mathcal{C}_1^{(1)}$, see [5], we get $\mathcal{H}(\mathcal{C}_1^{(1)},i)=1$. By formulas (ii) we have $\mathcal{H}(\mathcal{C}_1^{(1)},i)=1$, too. Hence, (ii) holds for $n\geq 1$. □

Corollary 1.

$$(i)\quad \mathcal{H}(\mathcal{I}_1^{(n)},i)=\frac{1}{(n-1)!(n-2)!}\sum_{m=1}^{n-1}\begin{bmatrix}n-1\\m\end{bmatrix}\left[\frac{i}{2}\right]^{m-1}\sum_{j=1}^{n-1}\begin{bmatrix}n-1\\j\end{bmatrix}\left(\left[\frac{i}{2}\right]+1\right)^{j-1},\ n>1,$$

$$(ii)\quad \mathcal{H}(\mathcal{C}_1^{(n)},i)=\frac{1}{(n-1)!^2}\sum_{m=1}^{n}\sum_{j=1}^{n}\begin{bmatrix}n\\m\end{bmatrix}\begin{bmatrix}n\\j\end{bmatrix}\frac{i^{m+j-2}\cos^2\frac{i\pi}{2}+(i+1)^{m-1}(i-1)^{j-1}\sin^2\frac{i\pi}{2}}{2^{m+j-2}},$$

where $\begin{bmatrix}n\\m\end{bmatrix}$ *are the unsigned Stirling numbers of the first kind.*

Proof. (i) Let us express the Narayana numbers in terms of the unsigned Stirling numbers of the first kind:

$$\begin{aligned}N_{n+[\frac{i}{2}]-1,\,n-1}&=\frac{1}{n-1}\binom{n+[\frac{i}{2}]-2}{n-2}\binom{n+[\frac{i}{2}]-1}{n-2}=\frac{\left([\frac{i}{2}]\right)_{n-1}\left([\frac{i}{2}]+1\right)_{n-1}}{(n-1)!(n-2)![\frac{i}{2}]([\frac{i}{2}]+1)}\\&=\frac{1}{(n-1)!(n-2)!}\sum_{m=1}^{n-1}\begin{bmatrix}n-1\\m\end{bmatrix}\left[\frac{i}{2}\right]^{m-1}\sum_{j=1}^{n-1}\begin{bmatrix}n-1\\j\end{bmatrix}\left(\left[\frac{i}{2}\right]+1\right)^{j-1}.\end{aligned}$$

In Theorem 1(i), we proved that $\mathcal{H}(\mathcal{I}_1^{(n)},i)=N_{n+[\frac{i}{2}]-1,\,n-1}\cos^2\frac{i\pi}{2}$ as $n>1$. Since $\cos^2\frac{i\pi}{2}=0$ as i is odd, it follows that

$$\begin{aligned}\mathcal{H}(\mathcal{I}_1^{(n)},i)&=\frac{1}{(n-1)!(n-2)!}\sum_{m=1}^{n-1}\sum_{j=1}^{n-1}\begin{bmatrix}n-1\\m\end{bmatrix}\begin{bmatrix}n-1\\j\end{bmatrix}\left(\left[\frac{i}{2}\right]+1\right)^{j-1}\left[\frac{i}{2}\right]^{m-1}\cos^2\frac{i\pi}{2}\\&=\frac{1}{(n-1)!(n-2)!}\sum_{m=1}^{n-1}\sum_{j=1}^{n-1}\begin{bmatrix}n-1\\m\end{bmatrix}\begin{bmatrix}n-1\\j\end{bmatrix}\frac{i^{m-1}(i+2)^{j-1}}{2^{m+j-2}}\cos^2\frac{i\pi}{2},\text{ if } n>1.\end{aligned}$$

(ii) The proof of (ii) is completely analogous to that of (i). □

2 HILBERT POLYNOMIALS OF THE ALGEBRAS OF JOINT INVARIANTS AND COVARIANTS OF n QUADRATIC FORMS

The Poincaré series of the algebras of joint invariants and covariants of n quadratic forms are needed for the sequel. They were derived by L.Bedratyuk in [2]. Using them, the author obtained the following formulas in [12]:

$$\mathcal{P}(\mathcal{C}_2^{(n)},z)=\frac{W_{n-1}(z^2)}{(1-z)^n(1-z^2)^{2n-1}}\quad\text{and}\quad\mathcal{P}(\mathcal{I}_2^{(n)},z)=\frac{W_{n-1}(z^2)-nzN_{n-1}(z^2)}{(1-z)^n(1-z^2)^{2n-1}}.$$

Theorem 2. *Hilbert polynomials of the algebras of joint invariants and covariants of* n *quadratic forms are calculated by the following formula:*

$$(i)\quad \mathcal{H}(\mathcal{C}_2^{(n)},i)=\sum_{k=0}^{[\frac{i}{2}]}\binom{n+k-1}{k}^2\binom{n+i-2k-1}{n-1},$$

$$(ii)\quad \mathcal{H}(\mathcal{I}_2^{(n)},i)=\begin{cases}\displaystyle\sum_{k=0}^{[\frac{i}{2}]}\binom{n+k-1}{k}^2\binom{n+i-2k-2}{n-2}\frac{3k-i+1}{k+1}, & \text{if } i>1,\\ 1, & \text{if } i=1,\end{cases}\quad \text{where } n>1.$$

Proof. This theorem can be proved basically in the same way as Theorem 1.

(i) Let us expand the Poincaré series of the algebra $\mathcal{C}_2^{(n)}$ into the Taylor series about $z=0$. We have:

$$\mathcal{P}(\mathcal{C}_2^{(n)},z)=\sum_{k=0}^{\infty}\binom{n+k-1}{k}^2 z^{2k}\sum_{i=0}^{\infty}\binom{n+i-1}{i}z^i$$
$$=\sum_{i=0}^{\infty}\sum_{k=0}^{[\frac{i}{2}]}\binom{n+k-1}{k}^2\binom{n+i-2k-1}{i-2k}z^i.$$

(ii) Using Theorem 1 (ii), we get

$$\mathcal{P}(\mathcal{I}_2^{(n)},z)=\left(\sum_{k=0}^{\infty}\binom{n+k-1}{k}^2 z^{2k}-\sum_{k=0}^{\infty}\binom{n+k-1}{n-1}\binom{n+k}{n-1}z^{2k+1}\right)\sum_{i=0}^{\infty}\binom{n+i-1}{i}z^i$$
$$=\sum_{i=0}^{\infty}\sum_{k=0}^{[i/2]}\binom{n+k-1}{k}^2\binom{n+i-2k-1}{n-1}z^i-\sum_{i=0}^{\infty}\sum_{k=0}^{[i/2]}\binom{n+k-1}{k}\binom{n+k}{n-1}\binom{n+i-2k-1}{n-1}z^{i+1}$$
$$=\sum_{i=0}^{\infty}\left(\sum_{k=0}^{[i/2]}\binom{n+k-1}{k}^2\binom{n+i-2k-1}{n-1}-\sum_{k=0}^{[(i-1)/2]}\binom{n+k-1}{k}\binom{n+k}{n-1}\binom{n+i-2k-2}{n-1}\right)z^i.$$

By the definitions of Poincaré series and Hilbert polynomials,

$$\mathcal{H}(\mathcal{I}_2^{(n)},i)=\sum_{k=0}^{[i/2]}\binom{n+k-1}{k}^2\binom{n+i-2k-1}{n-1}-\sum_{k=0}^{[(i-1)/2]}\binom{n+k-1}{k}\binom{n+k}{n-1}\binom{n+i-2k-2}{n-1}.$$

Note that $\binom{n+i-2k-2}{n-1}=0$, as $k>\left[\frac{i-1}{2}\right]$. We have

$$\mathcal{H}(\mathcal{I}_2^{(n)},i)=\sum_{k=0}^{[i/2]}\binom{n+k-1}{k}^2\binom{n+i-2k-1}{n-1}-\sum_{k=0}^{[i/2]}\binom{n+k-1}{k}\binom{n+k}{n-1}\binom{n+i-2k-2}{n-1}$$
$$=\sum_{k=0}^{[i/2]}\binom{n+k-1}{k}^2\binom{n+i-2k-2}{n-2}\frac{3k-i+1}{k+1}.$$

We used the Poincaré series $\mathcal{I}_2^{(n)}$ and $\mathcal{C}_2^{(n)}$ for $n>1$. Using Maple-procedure for computing the Hilbert polynomials of the algebras $\mathcal{C}_2^{(1)}$ and $\mathcal{I}_2^{(1)}$ (see [5]), we get

$$\mathcal{H}(\mathcal{I}_2^{(1)},i)=\frac{1}{2}\cos(\pi i)+\frac{1}{2}=\cos\left(\frac{\pi i}{2}\right),$$
$$\mathcal{H}(\mathcal{C}_2^{(1)},i)=\frac{i}{2}+\frac{1}{4}\cos(\pi i)+\frac{3}{4}=\left[\frac{i}{2}\right]+1.$$

This completes the proof of Theorem 2. ☐

Let us express $\mathcal{H}(\mathcal{I}_2^{(n)},i)$ in terms of a polynomial.

Corollary 2.

$$(i)\quad \mathcal{H}(\mathcal{C}_2^{(n)},i)=\frac{1}{(n-1)!}\sum_{k=0}^{[\frac{i}{2}]}\sum_{m=1}^{n-1}\sum_{j=1}^{m-1}\binom{n+k-1}{k}^2\binom{m-1}{j}\begin{bmatrix}n-1\\ m\end{bmatrix}i^j(-2k)^{m-j-1},$$

$$(ii)\quad \mathcal{H}(\mathcal{I}_2^{(n)},i)=\frac{1}{(n-1)!}\sum_{k=0}^{[\frac{i}{2}]}\sum_{m=1}^{n-2}\binom{n+k-1}{k}\binom{n+k-1}{k+1}\begin{bmatrix}n-2\\ m\end{bmatrix}(i-2k)^{m-1}(3k-i+1).$$

Proof. (i) By Theorem $2(i)$ we have:

$$\mathcal{H}(\mathcal{C}_2^{(n)},i)=\sum_{k=0}^{[\frac{i}{2}]}\binom{n+k-1}{k}^2\binom{n+i-2k-1}{i-2k}.$$

Let us express $\binom{n+i-2k-1}{i-2k}$ in terms of the unsigned Stirling numbers of the first kind:

$$\begin{aligned}\binom{n+i-2k-1}{i-2k}&=\frac{\sum_{m=1}^{n-1}\left[{n-1 \atop m}\right](i-2k)^{m-1}}{(n-1)!}\\&=\frac{1}{(n-1)!}\sum_{m=1}^{n-1}\sum_{j=0}^{m-1}\left[{n-1 \atop m}\right]\binom{m-1}{j}(-2k)^{m-j-1}i^j.\end{aligned}$$

(ii) The proof of (ii) is completely analogous to that of (i). □

Let us express the Hilbert polynomial of the algebras of joint covariants and invariants for n quadratic forms in terms of generalized hypergeometric function:

Corollary 3.

$$(i)\ \ \mathcal{H}(\mathcal{C}_2^{(n)},i)=\binom{n+i-1}{i}{}_4F_3\left[\begin{matrix}n,n,-\frac{i}{2},-\frac{i-1}{2},-\frac{n}{3}\\1,-\frac{n+i-1}{2},-\frac{n+i-2}{2}\end{matrix}\middle|\,1\right],\quad \text{if } n>2,$$

$$(ii)\ \ \mathcal{H}(\mathcal{I}_2^{(n)},i)=(1-n)\binom{n+i-2}{i}{}_5F_4\left[\begin{matrix}n,n,-\frac{i}{2},-\frac{i-1}{2},-\frac{i}{3}+\frac{4}{3}\\1,-\frac{n+i-2}{2},-\frac{n+i-3}{2},-\frac{i}{3}+\frac{1}{3}\end{matrix}\middle|\,1\right],\quad \text{if } n>3 \text{ and } i>1.$$

Proof. (i) By the above

$$\mathcal{H}(\mathcal{C}_2^{(n)},i)=\sum_{k=0}^{[\frac{i}{2}]}\binom{n+k-1}{k}^2\binom{n+i-2k-1}{i-2k}.$$

Let us remark that $\binom{n+i-2k-1}{i-2k}=0$ as $2k>i$. It means that:

$$\mathcal{H}(\mathcal{C}_2^{(n)},i)=\sum_{k=0}^{\infty}\binom{n+k-1}{k}^2\binom{n+i-2k-1}{i-2k}.$$

Let us express $\sum_{k=0}^{\infty}\binom{n+k-1}{k}^2\binom{n+i-2k-1}{i-2k}$ in terms of a generalized hypergeometric function in a way analogous to that used in [10]. Let us denote

$$a_k=\binom{n+k-1}{k}^2\binom{n+i-2k-1}{i-2k}.$$

We have

$$\begin{aligned}a_0&=\binom{n+i-1}{i},\\\frac{a_{k+1}}{a_k}&=\frac{(k+n)^2(k-\frac{i}{2})(k-\frac{i-1}{2})}{(k+1)^2(k-\frac{n+i-1}{2})(k-\frac{n+i-2}{2})}.\end{aligned}$$

It now follows that

$$\mathcal{H}(\mathcal{C}_2^{(n)}, i) = \binom{n+i-1}{i} {}_4F_3\left[\begin{matrix} n, n, -\frac{i}{2}, -\frac{i-1}{2} \\ 1, -\frac{n+i-1}{2}, -\frac{n+i-2}{2} \end{matrix} \middle| 1\right].$$

(ii) The proof of (ii) is completely analogous to that of (i). □

REFERENCES

[1] Bedratyuk L. *The Poincaré series of the algebras of simultaneous invariants and covariants of two binary forms.* Linear Multilinear Algebra. 2010, **58**(6), 789–803. doi:10.1080/03081080903127262

[2] Bedratyuk L. *Weitzenböck derivations and the classical invariant theory, I:Poincaré series.* Serdica Math. J. 2010, **36** (2), 99–120.

[3] Bedratyuk L. *The Poincaré series of the covariants of binary forms.* International Journal of Algebra. 2010, **4**(25), 1201–1207.

[4] Bedratyuk L. *Analogue of the Cayley-Sylvester formula and the Poincaré series for the algebra of invariants of n-ary form.* Linear Multilinear Algebra 2011, **59**(11), 1189–1199. doi:10.1080/03081081003621303

[5] Bedratyuk L. *Hilbert polynomials of the algebras of SL_2-invariants. preprint 2011.–arXiv:1102.3290v1.*

[6] Bruns W., Ichim. B. *On the coefficients of Hilbert quasipolynomials.* Proc. Amer. Math. Soc. 2007, **135**(5), 1305–1308.

[7] Eisenbud D. The geometry of syzygies. A second course in commutative algebra and algebraic geometry. Springer, NY, 2005.

[8] Hilbert D. Theory of algebraic invariants. Cambridge Univ. Press, 1993.

[9] Graham R., Riordan J. *The Solution of a Certain Recurrence.* Amer. Math. Monthly, 1966, **73**(6), 604–608.

[10] Graham R.L., Knuth D.E., Patashnik O. Concrete Mathematics - A foundation for computer science. In: Reading. Addison-Wesley Professional, MA. USA, 1993.

[11] Ilash N. *The Poincaré series for the algebras of joint invariants and covariants of n linear forms.* C. R. Acad. Bulg. Sci. 2015, **68** (6), 715–724.

[12] Ilash N. *Poincaré series for the algebras of joint invariants and covariants of n quadratic forms* Carpathian Math. Publ. 2017, **9**(1), 57–62. doi:10.15330/cmp.9.1.57-62

[13] Robbiano L. *Introduction to the Theory of Hilbert Function.* Queen's Papers in Pure and Applied Mathematics 1990, **85**, 1–26.

[14] Springer T. Invariant theory. In: Lecture Notes in Mathematics, vol. 585. Springer-Verlag, Berlin and New York, 1977.

[15] Stanley R. *Hilbert functions of graded algebras.* Adv. Math. 1978, **28**, 57–83.

[16] Székely L. *Common origin of cubic binomial identities; a generalization of Surányi's proof on Le Jen Shoo's formula.* J. Comb. Theory. Ser. A. 1985, **40**, 171–174.

Received 14.03.2018

Іланш Н.Б. *Многочлени Гільберта алгебр SL_2-інваріантів* // Карпатські матем. публ. — 2018. — Т.10, №2. — С. 303–312.

Ми розглядаємо одну з фундаментальних проблем класичної теорії інваріантів — дослідження многочленів Гільберта алгебри інваріантів групи Лі SL_2. Форма многочленів Гільберта несе важливу інформацію про структуру цієї алгебри. Крім того коефіцієнти і степінь многочленів Гільберта відіграють важливу роль в алгебраїчній геометрії. Відомо, що починаючи з деякого i функція Гільберта алгебри SL_n-інваріантів є квазімногочленом. Формула Келлі-Сільвестра для обчислення значень функції Гільберта алгебри коваріантів бінарної d-форми $\mathcal{C}_d = \mathbb{C}[V_d \oplus \mathbb{C}^2]^{SL_2}$ (тут V_d — комплексний $d+1$-вимірний векторний простір бінарних форм степеня d) була запропонована ще Сільвестром і пізніше узагальнена на алгебри спільних інваріантів скінченої кількості бінарних форм. Проте ці формули не виражають функції Гільберта як многочлен від i.

В нашій статті ми розглядаємо задачу обчислення в явній формі многочленів Гільберта алгебр спільних інваріантів та спільних коваріантів n лінійних форм і n квадратичних форм. Ми виразили многочлени Гільберта цих алгебр $\mathcal{H}(\mathcal{I}_1^{(n)}, i) = \dim(\mathcal{C}_1^{(n)})_i$, $\mathcal{H}(\mathcal{C}_1^{(n)}, i) = \dim(\mathcal{C}_1^{(n)})_i$, $\mathcal{H}(\mathcal{I}_2^{(n)}, i) = \dim(\mathcal{I}_2^{(n)})_i$, $\mathcal{H}(\mathcal{C}_2^{(n)}, i) = \dim(\mathcal{C}_2^{(n)})_i$ у вигяді квазімногочленів від i, а також подали їх у термінах відомих комбінаторних структур, таких як число Нараяна та узагальнений гіпергеометричний ряд.

Ключові слова і фрази: класична теорія інваріантів, інваріанти, функція Гільберта, многочлени Гільберта, квазімногочлени, ряди Пуанкаре, комбінаторика.