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ON THE STRUCTURE OF LEAST COMMON MULTIPLE MATRICES FROM SOME CLASS OF MATRICES

For non-singular matrices with some restrictions, we establish the relationships between Smith normal forms and transforming matrices (a invertible matrices that transform the matrix to its Smith normal form) of two matrices with corresponding matrices of their least common right multiple over a commutative principal ideal domains. Thus, for such a class of matrices, given answer to the well-known task of M. Newman. Moreover, for such matrices, received a new method for finding their least common right multiple which is based on the search for its Smith normal form and transforming matrices.

Key words and phrases: Smith normal form, transforming matrices, least common multiple matrices, commutative principal ideal domain .

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INTRODUCTION

Among the different problems and methods of their solutions that are considered in the commutative ring theory the special role is played by those that are similar to integer arithmetic ones, and they are the essential part of rings arithmetics. One of the mentioned problems, that are connected to the elementary divisibility theory, is how one can find the greatest common divisor and the least common multiple of given matrices over some ring and when such objects exist. The research in the area of such problems has started at the beginning of 20th century. Due in essence to E. Cahen and A. Chatelet, C. MacDuffee [4] has proposed elegant method of finding the greatest common divisor and the least common multiple of matrices using their Hermite forms. M. Newman and R. Thompson [10] studied the question: how to find the invariant multipliers of greatest common divisor and least common multiple of matrices over commutative principal ideal domains. The similar researches over the Euclidean domains became rather active in the recent years, as can be seen in the works of V. Nanda [5] , C. Yang, B. Li [1], S. Damkaew, S. Prugsapitak [2], N. Erawaty, M. Bahri, L. Haraynto, A. Amir [3] et al. In the current research author propose a method how to find least common multiple of matrices over commutative principal ideal domains, based on the properties of their Smith normal forms and the invertible matrices that transform these matrices to their Smith normal forms.

Let R be a commutative principal ideal domain with $1 \neq 0$, $M_n(R)$ be a ring $n \times n$ matrices over R. Consider a nonsingular matrix $A \in M_n(R)$. Since R is a principal ideal domain there are invertible matrices P_A, Q_A, such that

$$P_A A Q_A = \mathrm{E} = diag(1, \varepsilon, \ldots, \varepsilon).$$

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The matrix E above is called the Smith normal form or (canonical diagonal form) of matrix A, and matrices P_A and Q_A are left and right transforming matrices of A respectively.

By $\mathbf{P}_A$ we denote the set of all left transforming matrices of matrix A. According to the results [8, 11] we know that $\mathbf{P}_A = \mathbf{G}_{\mathrm{E}} P_A$, where

$$\mathbf{G}_{\mathrm{E}} = \{H \in GL_n(R) \mid \exists H_1 \in GL_n(R) : H\mathrm{E} = \mathrm{E}H_1\}.$$

Note that it is a multiplicative group.

Suppose that the greatest common divisor of minor of size $n-1$ of matrix B equals 1. Then

$$B \sim \Delta = diag(1, \dots, 1, \delta).$$

In the following we will use the set of matrices

$$\mathbf{L}(\mathrm{E}, \Delta) = \{L \in GL_n(R) \mid \exists L_1 \in M_n(R) : L\mathrm{E} = \Delta L_1\},$$

which is called a generating set (introdused by V. Shchedryk [8]).

If $A = BC$, then we will say that B is a left divisor of matrix A and A is a right multiple of B.

Moreover, if $M = AA_1 = BB_1$ then the matrix M is called a common right multiple of matrices A and B. If in addition the matrix M above is a left divisor of any other common right multiple of matrices A and B then we say that M is a **least common right multiple** of A and B. ($[A, B]_r$ in notation).

By the symbols (a, b) and $[a, b]$ we denote the greatest common divisor and the least common multiple of the elements a and b respectively, and the notation $a|b$ means that the element a divides the element b.

1 Main results

Lemma 1. *Let $P_B P_A^{-1} = S = \|s_{ij}\|_1^n$. Then the element $((\varepsilon, \delta), s_{n1})$ is an invariant with respect to transforming matrices P_B and P_A.*

Proof. Let $F_A \in \mathbf{P}_A$ and $F_B \in \mathbf{P}_B$ be some other left transforming matrices of A and B. Then exist matrices $H_A \in \mathbf{G}_{\mathrm{E}}$ and $H_B \in \mathbf{G}_\Delta$ such that $F_A = H_A P_A$, $F_B = H_B P_B$. Consider the following product of the matrices:

$$F_B F_A^{-1} = H_B P_B (H_A P_A)^{-1} = H_B P_B P_A^{-1} H_A^{-1} = H_B S H_A^{-1},$$

where $S = P_B P_A^{-1}$. Let's denote $H_B S = \|k_{ij}\|_1^n$. In view of Corollary 6 [8] H_B is of the form

$$H_B = \left\| \begin{matrix} h_{11} & \dots & h_{1.n-1} & h_{1n} \\ \dots & \dots & \dots & \dots \\ h_{n-1.1} & \dots & h_{n-1.n-1} & h_{n-1.n} \\ \delta h_{n1} & \dots & \delta h_{n.n-1} & h_{nn} \end{matrix} \right\|.$$

Hence,

$$k_{n1} = \left\| \begin{matrix} \delta h_{n1} & \dots & \delta h_{n.n-1} & h_{nn} \end{matrix} \right\| \left\| \begin{matrix} s_{11} \\ \vdots \\ s_{n-1.1} \\ s_{n1} \end{matrix} \right\|$$

$$= \delta(h_{n1}s_{11} + \dots + h_{n.n-1}s_{n-1.1}) + h_{nn}s_{n1} = \delta l + h_{nn}s_{n1}.$$

Consider the following greatest common divisor:

$$((\varepsilon,\ \delta), k_{n1}) = ((\varepsilon,\ \delta),\ \delta l + h_{nn}s_{n1}) = ((\varepsilon,\delta), h_{nn}s_{n1}).$$

The invertibility of H_B implies that $(\delta, h_{nn}) = 1$. Therefore, $((\varepsilon,\delta), h_{nn}) = 1$ and

$$((\varepsilon,\ \delta), k_{n1}) = ((\varepsilon,\delta), s_{n1}).$$

Let's denote $SH_A^{-1} = \|t_{ij}\|_1^n$. Since $H_A^{-1} \in \mathbf{G}_{\mathrm{E}}$ then according to Corollary 6 of [8] the matrix H_A^{-1} has the form

$$H_A^{-1} = \left\| \begin{array}{cccc} v_{11} & v_{12} & \dots & v_{1n} \\ \varepsilon v_{21} & v_{22} & \dots & v_{2n} \\ \dots & \dots & \dots & \dots \\ \varepsilon v_{n1} & v_{n2} & \dots & v_{nn} \end{array} \right\|.$$

Hence,

$$t_{n1} = \left\| \begin{array}{cccc} s_{n1} & s_{n2} & \dots & s_{nn} \end{array} \right\| \left\| \begin{array}{c} v_{11} \\ \varepsilon v_{21} \\ \vdots \\ \varepsilon v_{n1} \end{array} \right\| = s_{n1}v_{11} + \varepsilon(s_{n2}v_{21} + \ldots + s_{nn}v_{n1}).$$

Consider

$$((\varepsilon,\delta), t_{n1}) = ((\varepsilon,\delta), s_{n1}v_{11} + \varepsilon(s_{n2}v_{21} + \ldots + s_{nn}v_{n1})) = ((\varepsilon,\delta), s_{n1}v_{11}).$$

Since $(\varepsilon, v_{11}) = 1$, then $((\varepsilon,\delta), s_{n1}v_{11}) = ((\varepsilon,\delta), s_{n1})$. Hence

$$((\varepsilon,\delta), t_{n1}) = ((\varepsilon,\delta), s_{n1}).$$

Applying the associativity of $M_n(R)$ completes the proof. □

Lemma 2. *Let $S = \|s_{ij}\|_1^n \in GL_n(R)$, $\Omega = diag(\omega_1, \omega_2, \ldots, \omega_n)$, where $\omega_i \mid \omega_{i+1}$, $i = 1, 2, \ldots, n-1$, and $\mathrm{E} \mid \Omega$, $\Delta \mid \Omega$. In order to $SL_A = L_B$, where $L_A \in \mathbf{L}(\Omega, \mathrm{E})$, $L_B \in \mathbf{L}(\Omega, \Delta)$ it is necessary and sufficient that $(a, b) \mid s_{n1}$, where $a = \frac{\varepsilon}{(\varepsilon,\omega_1)}$, $b = \frac{\delta}{(\delta,\omega_1)}$.*

Proof. Necessity. Since $\mathrm{E} \mid \Omega$ then according to Corollary 5 of [8] matrices L_A and L_B are of forms:

$$L_A = \left\| \begin{array}{cccc} p_{11} & p_{12} & \dots & p_{1n} \\ \frac{\varepsilon}{(\varepsilon,\omega_1)}p_{21} & p_{22} & \dots & p_{2n} \\ \vdots & \vdots & \dots & \vdots \\ \frac{\varepsilon}{(\varepsilon,\omega_1)}p_{n1} & p_{n2} & \dots & p_{nn} \end{array} \right\|, \quad L_B = \left\| \begin{array}{cccc} q_{11} & \dots & q_{1.n-1} & q_{1n} \\ q_{21} & \dots & q_{2.n-1} & q_{2n} \\ \vdots & \vdots & \dots & \vdots \\ \frac{\delta}{(\delta,\omega_1)}q_{n1} & \dots & \frac{\delta}{(\delta,\omega_{n-1})}q_{n-1.n-1} & q_{nn} \end{array} \right\|,$$

respectively. Using the Property 4.8 [9], in this case, the set $\mathbf{L}(\Omega, \mathrm{E})$ is a group. Then $S = L_B L_A^{-1}$, where $L_A^{-1} \in \mathbf{L}(\Omega, \mathrm{E})$. It follows that

$$\left(\frac{\varepsilon}{(\varepsilon,\omega_1)}, \frac{\delta}{(\delta,\omega_1)}\right) \mid s_{n1}.$$

If we denote $a = \frac{\varepsilon}{(\varepsilon,\omega_1)}$, $b = \frac{\delta}{(\delta,\omega_1)}$ then we will get that $(a, b) \mid s_{n1}$.

Sufficiency. Let $s_{n1} = (a, b)t$. By Theorem 2.13 [9] there exist some matrices $H_1 \in \mathbf{G}_\Delta$ and $U \in \mathbf{G}_\mathrm{E}$ such that

$$H_1SU = \left\| \begin{matrix} 1 & 0 & \dots & 0 & 0 \\ k_{21} & 1 & \dots & 0 & 0 \\ \vdots & \dots & \ddots & \vdots & \vdots \\ k_{n-1.1} & k_{n-1.2} & \dots & 1 & 0 \\ (a,b)k_{n1} & k_{n2} & \dots & k_{n.n-1} & 1 \end{matrix} \right\| = \left\| \begin{matrix} K_{11} & \mathbf{0} \\ K_{21} & 1 \end{matrix} \right\|.$$

Obviously, K_{11} is invertible. Hence there exists some matrix $H_2 = \left\| \begin{matrix} K_{11}^{-1} & \mathbf{0} \\ \mathbf{0} & 1 \end{matrix} \right\| \in \mathbf{G}_\Delta$ such that

$$H_2H_1SU = \left\| \begin{matrix} 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \dots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \\ (a,b)k_{n1} & k_{n2} & \dots & k_{n.n-1} & 1 \end{matrix} \right\| = K.$$

Since $H_1, H_2 \in \mathbf{G}_\Delta$ then $H_3 = H_2H_1 \in \mathbf{G}_\Delta$. Therefore $K = H_3SU$. Moreover, one can find $v_1, v_2 \in R$ such that

$$(a,b)k_{n1} = (av_1 + bv_2)k_{n1} = av_1k_{n1} + bv_2k_{n1}.$$

If we consider the matrices

$$H_4 = \left\| \begin{matrix} 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 & 0 \\ bv_2k_{n1} & 0 & \dots & 0 & 1 \end{matrix} \right\| \in \mathbf{L}(\Omega, \Delta)$$

and

$$V = \left\| \begin{matrix} 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 & 0 \\ -av_1k_{n1} & -k_{n2} & \dots & -k_{n.n-1} & 1 \end{matrix} \right\| \in \mathbf{L}(\Omega, \mathrm{E}).$$

we obtain that $H_3SUV = H_4$. Then $SUV = H_3^{-1}H_4$. Using Properties 2 and 3 [8] we will have $H_3^{-1}H_4 = L_B \in \mathbf{L}(\Omega, \Delta)$, $UV = L_A \in \mathbf{L}(\Omega, \mathrm{E})$, and so $SL_A = L_B$ which had to be proved. □

Theorem 1. *Let R be a commutative principal ideal domain and let*

$$A \sim diag(1, \varepsilon, \dots, \varepsilon),\ B \sim diag(1, \dots, 1, \delta),$$

$P_BP_A^{-1} = \|s_{ij}\|_1^n$, $P_B \in \mathbf{P}_B$, $P_A \in \mathbf{P}_A$. *Then*

$$[A, B]_r = (L_AP_A)^{-1}\Omega = (L_BP_B)^{-1}\Omega,$$

where

$$\Omega = diag\left(\frac{(\varepsilon, \delta)}{((\varepsilon, \delta), s_{n1})}, \varepsilon, \dots, \varepsilon, [\varepsilon, \delta]\right),$$

L_A , L_B *belong to sets* $\mathbf{L}(\Omega, \mathrm{E})$, $\mathbf{L}(\Omega, \Delta)$ *respectively and satisfy the equality:*

$$(P_BP_A^{-1})L_A = L_B.$$

Proof. Remark that according to Lemma (1), the element $((\varepsilon,\delta),s_{n1})$, and hence the matrix Ω, does not depend on the choice of transforming matrices P_A and P_B.

By Theorem 2 [6] the Smith normal form of the greatest common left divisor of the matrices A and B is of the form

$$(A,B)_l \sim diag(1,\dots,1,(\varepsilon,\delta,s_{n1})).$$

According to Corollary 1.5 [9] we obtain

$$\pm det A det B = det(A,B)_l det[A,B]_r,$$

i.e.

$$det[A,B]_r = \pm\frac{det A det B}{det(A,B)_l} = \pm\frac{\varepsilon^{n-1}\delta}{(\varepsilon,\delta,s_{n1})} = \omega_1\omega_2\dots\omega_{n-1}\omega_n.$$

It follows from [10] that $\omega_n = [\varepsilon,\delta]$ and $\omega_i \mid \varepsilon$, $i = 2,\dots,n-1$. Since $\mathrm{E} \mid \Omega$ then $\varepsilon \mid \omega_i$, for $i = 2,\dots,n-1$, that is $\omega_i = \varepsilon$, $i = 2,\dots,n-1$. Hence,

$$\omega_1 = \pm\frac{\varepsilon^{n-1}\delta(\varepsilon,\delta)}{\varepsilon^{n-2}\varepsilon\delta(\varepsilon,\delta,s_{n1})} = \pm\frac{(\varepsilon,\delta)}{(\varepsilon,\delta,s_{n1})}.$$

Taking into account that the invariant factors of matrix are chosen precisely to the divisors of unit, we obtain that the Smith normal form of the least common right multiple of matrices A and B has the form:

$$\Omega = diag(\frac{(\varepsilon,\delta)}{((\varepsilon,\delta),s_{n1})},\varepsilon,\dots,\varepsilon,[\varepsilon,\delta]).$$

By Lemma 1 [7] we will have

$$(\frac{\varepsilon}{(\varepsilon,\omega_1)},\frac{\delta}{(\delta,\omega_1)}) = (\frac{(\varepsilon,\delta)}{((\varepsilon,\delta),\omega_1)}) = \mu.$$

Since $\omega_1 = \frac{(\varepsilon,\delta)}{(\varepsilon,\delta,s_{n1})}$, then

$$\mu = \frac{(\varepsilon,\delta)}{((\varepsilon,\delta),\frac{(\varepsilon,\delta)}{((\varepsilon,\delta),s_{n1})})} = \frac{(\varepsilon,\delta)((\varepsilon,\delta),s_{n1})}{((\varepsilon,\delta)((\varepsilon,\delta),s_{n1}),(\varepsilon,\delta))} = ((\varepsilon,\delta),s_{n1}).$$

This means that $\mu \mid s_{n1}$. According to Lemma (2) there exist matrices $L_A \in \mathbf{L}(\Omega,\mathrm{E})$, $L_B \in \mathbf{L}(\Omega,\Delta)$ such that $P_B P_A^{-1} L_A = L_B$, so

$$P_A^{-1} L_A \Omega = P_B^{-1} L_B \Omega = M.$$

Since $\mathrm{E} \mid \Omega$ and $\Delta \mid \Omega$, then using Theorem 1 [8] the matrix M is the common right multiple of A and B.

Let N be least common right multiple of matrices A and B. From the above, it follows that $N \sim \Omega$. Hence $N = P_N^{-1}\Omega Q_N^{-1}$. Then $M = P_A^{-1}L_A\Omega = P_M^{-1}\Omega$ is a right multiple of N : $M = NN_1$. According to Theorem 1 [8] this is equivalent to the fact that $P_N = LP_M$, where $L \in \mathbf{L}(\Omega,\Omega)$. Using Property 4.6 [9] we get the equality $\mathbf{L}(\Omega,\Omega) = \mathbf{G}_\Omega$. Then by Corollary 2 [8] the matrices M and N are right associated. Thus, M is the least common right multiples of matrices A and B. The theorem is proved. $\square$

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Для неособливих матриць, при певних обмеженнях, встановлено взаємозв'язки між формами Сміта та перетворювальними матрицями (оборотними матрицями, що зводять матрицю до її форми Сміта) двох матриць з відповідними матрицями їх найменшого спільного правого кратного над комутативними областями головних ідеалів. Тим самим, для такого класу матриць, дано відповідь на відому задачу М. Ньюмена. Більше того, для таких матриць, вказано новий метод знаходження їх найменшого спільного правого кратного, яких ґрунтується на пошуку його форми Сміта та перетворювальних матриць.

Ключові слова і фрази: форма Сміта, перетворювальні матриці, найменше спільне кратне матриць, комутативна область головних ідеалів.