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A GENERALIZATION OF A LOCALIZATION PROPERTY OF BESOV SPACES

The notion of a localization property of Besov spaces is introduced by G. Bourdaud, where he has provided that the Besov spaces $B^s_{p,q}(\mathbb{R}^n)$, with $s \in \mathbb{R}$ and $p, q \in [1, +\infty]$ such that $p \neq q$, are not localizable in the ℓ^p norm. Further, he has provided that the Besov spaces $B^s_{p,q}$ are embedded into localized Besov spaces $(B^s_{p,q})_{\ell^p}$ (i.e., $B^s_{p,q} \hookrightarrow (B^s_{p,q})_{\ell^p}$, for $p \geq q$). Also, he has provided that the localized Besov spaces $(B^s_{p,q})_{\ell^p}$ are embedded into the Besov spaces $B^s_{p,q}$ (i.e., $(B^s_{p,q})_{\ell^p} \hookrightarrow B^s_{p,q}$, for $p \leq q$). In particular, $B^s_{p,p}$ is localizable in the ℓ^p norm, where ℓ^p is the space of sequences $(a_k)_k$ such that $\|(a_k)\|_{\ell^p} < \infty$. In this paper, we generalize the Bourdaud theorem of a localization property of Besov spaces $B^s_{p,q}(\mathbb{R}^n)$ on the ℓ^r space, where $r \in [1, +\infty]$. More precisely, we show that any Besov space $B^s_{p,q}$ is embedded into the localized Besov space $(B^s_{p,q})_{\ell^r}$ (i.e., $B^s_{p,q} \hookrightarrow (B^s_{p,q})_{\ell^r}$, for $r \geq \max(p, q)$). Also we show that any localized Besov space $(B^s_{p,q})_{\ell^r}$ is embedded into the Besov space $B^s_{p,q}$ (i.e., $(B^s_{p,q})_{\ell^r} \hookrightarrow B^s_{p,q}$, for $r \leq \min(p, q)$). Finally, we show that the Lizorkin-Triebel spaces $F^s_{p,q}(\mathbb{R}^n)$, where $s \in \mathbb{R}$ and $p \in [1, +\infty)$ and $q \in [1, +\infty]$ are localizable in the ℓ^p norm (i.e., $F^s_{p,q} = (F^s_{p,q})_{\ell^p}$).

Key words and phrases: Besov spaces, Lizorkin-Triebel spaces, localization property.

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INTRODUCTION

Functional calculus is one of the basic theory in functional analysis [5]. It has enabled to study function-analytic in topological (in particular, normed) spaces of functions. For instance, several authors such as Peetre [7], Dahlberg [4], Marcus and Mizel [6] have studied functional calculus in certain Sobolev and Besov spaces. In particular, Bourdaud [1, 2] have established a way of functional calculus in localized Besov spaces. More precisely, in [1] he has proved the following result.

Theorem 1. *Let $p, q \in [1, +\infty]$, $s \in \mathbb{R}$, $B^s_{p,q}$ and $(B^s_{p,q})_{\ell^p}$ are respectively the Besov and localized Besov spaces. Then*

(i) $B^s_{p,q} \hookrightarrow (B^s_{p,q})_{\ell^p}$, *for* $p \geq q$,

(ii) $(B^s_{p,q})_{\ell^p} \hookrightarrow B^s_{p,q}$, *for* $p \leq q$.

In particular, $B^s_{p,p}$ is localizable in the ℓ^p norm, where ℓ^p is the space of sequences $(a_k)_k$ such that $\|(a_k)\|_{\ell^p} = (\sum_{k=0}^{\infty} |a_k|^p)^{\frac{1}{p}} < \infty$.

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In this paper, we generalize this result by proving that it is valid for any ℓ^r space, where $r \in [1, +\infty]$. This paper is organized as follows. In section 1, we recall basic concepts of Besov and Lizorkin-Triebel spaces, the decomposition of Littlewood-Paley, and some notations that will be needed throughout this paper. In section 2, we give a generalization of Bourdaud theorem of a localization property of Besov spaces on the ℓ^r space, where $r \in [1, +\infty]$. Also, we show that the Lizorkin-Triebel spaces are localizable in the ℓ^p norm. Finally, we present some conclusions and discuss future research in section 3.

1 PRELIMINARIES AND NOTATIONS

This section contains the basic definitions and notations that will be needed throughout this paper.

1.1 Notations

We note $(e_1, \ldots, e_n)$ the canonical basis of $\mathbb{R}^n$, $x.y = x_1y_1 + \cdots + x_ny_n$ the scalar product in $\mathbb{R}^n$, and for $\alpha \in \mathbb{N}^n$, $|\alpha| = \alpha_1 + \cdots + \alpha_n$, $\frac{\partial^{|\alpha|} f}{\partial^{\alpha_1} x_1 \ldots \partial^{\alpha_n} x_n}$ the partial derivative of the function f is denoted by $\partial^\alpha f$.

If $f : \mathbb{R}^n \longrightarrow \mathbb{C}$, the support of f denoted by $supp f$. $\mathcal{D}(\mathbb{R}^n)$ is the space of test functions, i.e. of smooth functions which have compact support, $\mathcal{D}'(\mathbb{R}^n)$ is the dual of $\mathcal{D}(\mathbb{R}^n)$. $\mathcal{S}(\mathbb{R}^n)$ is the Schwartz space of functions $\mathcal{C}^\infty(\mathbb{R}^n)$ rapidly decreasing on $\mathbb{R}^n$, the dual $\mathcal{S}'(\mathbb{R}^n)$ is the space of tempered distributions.

If $f \in \mathcal{S}(\mathbb{R}^n)$, then it Fourier transform defined by

$$\mathcal{F}(f(x))(\xi) = \int_{\mathbb{R}^n} \exp(-ix.\xi) f(x) dx$$

and its inverse Fourier transform defined by

$$\mathcal{F}^{-1}(\widehat{f}(\xi))(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} \exp(ix.\xi) \widehat{f}(\xi) d\xi.$$

Let A_1 and A_2 be two spaces, we say that $A_1 \hookrightarrow A_2$ if there exists $c > 0$ such that $\|.\|_{A_2} \leq c\|.\|_{A_1}$. Let p' be the conjugate exponent of p, $\frac{1}{p} + \frac{1}{p'} = 1$ where $p \in [1, +\infty]$.

Let $k \in \mathbb{Z}^n$; τ_k is the translation operator defined by $\tau_k f(\cdot) = f(\cdot - k)$; L^p is the space of the measurable functions f such that $\|f\|_{L^p} = (\int_{\mathbb{R}^n} |f(x)|^p dx)^{\frac{1}{p}} < \infty$; ℓ^q is the space of sequences $(a_k)_k$ such that $\|(a_k)\|_{\ell^q} = (\sum_{k=0}^{\infty} |a_k|^q)^{\frac{1}{q}} < \infty$.

Let $0 < p \leq \infty$, $0 < q \leq \infty$, so

$$\|f_k\|_{\ell^q(L^p)} = (\sum_{k=0}^{\infty} \|f_k(x)\|_p^q)^{\frac{1}{q}} < \infty, \quad \|f_k\|_{L^p(\ell^q)} = \|(\sum_{k=0}^{\infty} |f_k(x)|^q)^{\frac{1}{q}}\|_p < \infty.$$

1.2 The decomposition of Littlewood-Paley

Let $\varphi \in \mathcal{S}(\mathbb{R}^n)$, which satisfy the conditions:

(i) $supp\ \varphi \subset \{\xi \in \mathbb{R}^n : 1 \leq |\xi| \leq 3\}$,

(ii) $\varphi(\xi) > 0$, for $1 \leq |\xi| \leq 3$,

(iii) $\sum_{j \in \mathbb{Z}} \varphi(2^{-j}\xi) = 1$, for $\xi \in \mathbb{R}^n \setminus \{0\}$.

The construction of φ does not pose any difficulty, see for example [3] . We put $\varphi(\xi) = 1 - \sum_{j=1}^{\infty} \varphi(2^{-j}\xi)$, then it follows that the function $\varphi \in \mathcal{C}^{\infty}(\mathbb{R}^n)$, such that $supp\ \varphi \subset \{\xi \in \mathbb{R}^n : |\xi| \leq 3\}$. In the following, we fix the partition of the unit and we obtain:

$$\varphi(\xi) + \Sigma_{j=1}^{\infty}\varphi(2^{-j}\xi) = 1 \qquad (\text{for all } \xi \in \mathbb{R}^n).$$

To this partition we associate a sequence of convolution operators $\Delta_j : \mathcal{S}' \longrightarrow \mathcal{C}^{\infty}$, defined by $\mathcal{F}(\Delta_j f)(\xi) = \varphi(2^{-j}\xi)\widehat{f}(\xi)$, for $j = 1, 2, \ldots$ and $\mathcal{F}(\Delta_0 f)(\xi) = \varphi(\xi)\widehat{f}(\xi)$. Also, we define the operators Q_k by $\mathcal{F}(Q_k f)(\xi) = \varphi(2^{-k}\xi)\widehat{f}(\xi)$, $k = 1, 2, \ldots$, for all $f \in \mathcal{S}'$, the decomposition of f of the Littlewood-Paley type given by

$$f = \sum_{j \geq 0} \Delta_j f. \tag{1}$$

The series (1) converges in the sense of tempered distributions. The series (1) can be written as

$$f = Q_k f + \sum_{j \geq k+1} \Delta_j f,$$

This formula is valid for any $f \in \mathcal{S}'$ and $k \in \mathbb{N}$, such that $Q_k f = \sum_{j \leq k} \Delta_j f$.

Definition 1 ([10]). *Let $f \in \mathcal{S}'$ and $a > 0$. We define the maximal operators associated to the Δ_k and Q_k by*

$$\Delta_k^{*,a} f(x) = \sup_{y \in \mathbb{R}^n} \frac{|\Delta_k f(x-y)|}{(1 + 2^k|y|)^a} \quad \text{and} \quad Q_k^{*,a} f(x) = \sup_{y \in \mathbb{R}^n} \frac{|Q_k f(x-y)|}{(1 + 2^k|y|)^a}.$$

Definition 2 ([8]). *Let $s \in \mathbb{R}, p, q \in [1, +\infty]$. The Besov space $B^s_{p,q}(\mathbb{R}^n)$ is the set of all $f \in \mathcal{S}'(\mathbb{R}^n)$ satisfying*

$$\|f\|_{B^s_{p,q}(\mathbb{R}^n)} = \begin{cases} (\sum_{j \geq 0} (2^{sj} \|\Delta_j f\|_p)^q)^{\frac{1}{q}} < +\infty, & \text{for } q \neq \infty, \\ \sup_{j \geq 0} (2^{sj} \|\Delta_j f\|_p) < +\infty, & \text{for } q = \infty. \end{cases} \tag{2}$$

Definition 3 ([8]). *Let $s \in \mathbb{R}, p \in [1, +\infty[$ and $q \in [1, +\infty]$. The Lizorkin-Triebel space $F^s_{p,q}(\mathbb{R}^n)$ is the set of all $f \in \mathcal{S}'(\mathbb{R}^n)$ satisfying*

$$\|f\|_{F^s_{p,q}(\mathbb{R}^n)} = \begin{cases} \|(\sum_{j \geq 0} (2^{sj} |\Delta_j f|)^q)^{\frac{1}{q}}\|_p < +\infty, & \text{for } q \neq \infty. \\ \|\sup_{j \geq 0} (2^{sj} |\Delta j f|)\|_p < +\infty, & \text{for } q = \infty. \end{cases} \tag{3}$$

Remark 1. *In the formula* (2) *(resp.* (3)*), we can replace Δ_j by $\Delta_j^{*,a}$ with $a > \frac{n}{p}$ (resp. $a > \frac{n}{\min(p,q)}$), and we obtain an equivalent norm in $B^s_{p,q}(\mathbb{R}^n)$ (resp. $F^s_{p,q}(\mathbb{R}^n)$).*

For more details, see Peetre [7] and Triebel [10] .

Proposition 1 ([2]). *Let $s \in \mathbb{R}$.*

(i) For all $\gamma > 1$ there exists $c > 0$ such that for any sequence of functions $(f_j)_{j \geq 0}$, where $supp\ \mathcal{F} f_j \subset \{\xi : \gamma^{-1} 2^j \leq |\xi| \leq \gamma 2^j\}$, we have

$$\|\sum_{j=0}^{\infty} f_j\|_{B^s_{p,q}} \leq c (\sum_{j=0}^{\infty} 2^{sjq} \|f_j\|_p^q)^{\frac{1}{q}}.$$

(ii) For all $a > 1$ there exists $c > 0$ such that for any sequence of functions $(f_j)_{j \in \mathbb{N}}$, where $supp\ \mathcal{F} f_j \subset \{\xi : a^{-1} 2^j \leq |\xi| \leq a 2^j\}$, we have

$$\|\sum_{j=0}^{\infty} f_j\|_{F^s_{p,q}} \leq c \|(\sum_{j=0}^{\infty} 2^{sjq} |f_j|^q)^{\frac{1}{q}}\|_p.$$

2 LOCALIZATION OF BESOV SPACES

In this section, we give a generalization of Bourdaud theorem of a localization property of Besov spaces on the ℓ^r space, where $r \in [1, +\infty]$. Also, we show that the Lizorkin-Triebel spaces are localizable in the ℓ^p norm. We start with these important concepts.

Let E be a Banach space of distributions. We associate on the space E the following hypothesis.

1) Translation invariance; if we denote τ_x the operator given by $\tau_x f(t) = f(x-t)$, then τ_x is an isometric of E;

2) Localization invariance; for all $f \in E$ and $\varphi \in \mathcal{D}(\mathbb{R}^n)$, we have that $\varphi f \in E$.

Let $\varphi \in \mathcal{D}(\mathbb{R}^n)$. The notion of localized is defined by $f_x = \tau_x\varphi \cdot f$, it follows immediately from the hypothesis 1) and 2) that the family $(f_x)_{x\in\mathbb{R}^n}$ is bounded in E. We consider the set A as the class of all the functions $\varphi \in \mathcal{D}(\mathbb{R}^n)$ satisfying

$$\sum_{k\in\mathbb{Z}^n} \varphi(x-k) = 1 \quad \text{for all } x \in \mathbb{R}^n.$$

Definition 4 ([1]). *Let E be a Banach space of distributions, E is localizable in the ℓ^p norm $(1 \le p \le \infty)$, if there exist $\varphi \in A$ and a constant $c \ge 1$, such that*

$$c^{-1}\|f\|_E \le (\sum_{k\in\mathbb{Z}^n} \|\tau_k\varphi.f\|_E^p)^{\frac{1}{p}} \le c\|f\|_E,$$

i.e. $E = (E)_{\ell^p}$, we denote by $(E)_{\ell^p}$ the distribution space of u such that

$$\|u\|_{(E)_{\ell^p}} = \|(\|\tau_k\varphi.u\|_E)_{k\in\mathbb{Z}^n}\|_{\ell^p} < \infty.$$

Proposition 2 ([1]). *Let $\mathcal{S}$ be the Schwartz space, if the function $\theta \in \mathcal{S}$ is not null on the support of φ, then we have*

$$\|u\|_{(E)_{\ell^p}} \sim \|(\|\tau_k\theta.u\|_E)_{k\in\mathbb{Z}^n}\|_{\ell^p}.$$

Proposition 3 ([1]). *Let $B^s_{p,q}$ be a Besov space, and N be a natural number fulfill $N > s$, and $\lambda, \mu \in \mathcal{S}$, such that*

(i) $\mu(\xi) \neq 0$, for $|\xi| \le 3$,

(ii) $\lambda(\xi) \neq 0$, for $1 \le |\xi| \le 3$ and $\lambda^{(\alpha)}(0) = 0$ for $|\alpha| < N$.

We denote by $L_j (j \ge 1)$ the respective symbol operators $\lambda(2^{-j}\xi)$ and by L_0 the symbol operator $\mu(\xi)$, therefore

$$\|u\|_{B^s_{p,q}} \sim \|(2^{js}\|L_j u\|_p)_{j\in\mathbb{N}}\|_{\ell^q}.$$

In the following theorem we give a generalization of Bourdaud theorem of a localization property of Besov spaces on the ℓ^r spaces, by using Proposition 2 and Proposition 3.

Theorem 2. *Let $p, q, r \in [1, +\infty]$, $s \in \mathbb{R}$, and $B^s_{p,q}$ and $(B^s_{p,q})_{\ell^r}$ are respectively the Besov and localized Besov spaces. Then*

(i) $B^s_{p,q} \hookrightarrow (B^s_{p,q})_{\ell^r}$ for $r \ge \max(p, q)$,

(ii) $(B^s_{p,q})_{\ell^r} \hookrightarrow B^s_{p,q}$ *for* $r \le \min(p,q)$.

In particular, $B^s_{p,p}$ *space is localizable in the* ℓ^p *norm.*

Proof. *(i)* We will show that

$$\|u\|_{(B^s_{p,q})_{\ell^r}} \le c\|u\|_{B^s_{p,q}} \quad \text{for } c > 0.$$

By Proposition 1, it follows that

$$\|\sum_{j\ge 0} \tau_k\theta.\Delta_j u\|_{B^s_{p,q}} \le c(\sum_{j\ge 0} 2^{sjq}\|\tau_k\theta.\Delta_j u\|_p^q)^{\frac{1}{q}}.$$

This implies that, $\|\tau_k\theta.u\|^r_{B^s_{p,q}} \le c(\sum_{j=0}^{\infty} 2^{sjq}\|\tau_k\theta.\Delta_j u\|_p^q)^{\frac{r}{q}}$. Then it holds that

$$(\sum_{k\in\mathbb{Z}^n} \|\tau_k\theta.u\|^r_{B^s_{p,q}})^{\frac{1}{r}} \le c(\sum_{k\in\mathbb{Z}^n} (\sum_{j=0}^{\infty} 2^{sjq}\|\tau_k\theta.\Delta_j u\|_p^q)^{\frac{r}{q}})^{\frac{1}{r}}.$$

Consequently

$$(\sum_{k\in\mathbb{Z}^n} \|\tau_k\theta.u\|^r_{B^s_{p,q}})^{\frac{1}{r}} \le c(\|(2^{sj}\|\tau_k\theta.\Delta_j u\|_p)_{k\in\mathbb{Z}^n}\|_{\ell^r(\ell^q)}). \tag{4}$$

Since, $r \ge \max(p,q)$ implies that $q \le r$. Then from Minkowski inequality we have

$$\|(2^{sj}\|\tau_k\theta.\Delta_j u\|_p)_{k\in\mathbb{Z}^n}\|_{\ell^r(\ell^q)} \le c\|(2^{sj}\|\tau_k\theta.\Delta_j u\|_p)_{k\in\mathbb{Z}^n}\|_{\ell^q(\ell^r)}.$$

So, we can see that the inequality (4) becomes as follows

$$(\sum_{k\in\mathbb{Z}^n} \|\tau_k\theta.u\|^r_{B^s_{p,q}})^{\frac{1}{r}} \le c(\|(2^{sj}\|\tau_k\theta.\Delta_j u\|_p)_{k\in\mathbb{Z}^n}\|_{\ell^q(\ell^r)}).$$

Consequently $(\sum_{k\in\mathbb{Z}^n} \|\tau_k\theta.u\|^r_{B^s_{p,q}})^{\frac{1}{r}} \le c(\sum_{j=0}^{\infty} 2^{sjq}(\sum_{k\in\mathbb{Z}^n} \|\tau_k\theta.\Delta_j u\|_p^r)^{\frac{q}{r}})^{\frac{1}{q}}$. Therefore,

$$(\sum_{k\in\mathbb{Z}^n} \|\tau_k\theta.u\|^r_{B^s_{p,q}})^{\frac{1}{r}} \le c(\sum_{j=0}^{\infty} 2^{sjq}(\|\tau_k\theta.\Delta_j u\|_{\ell^r(L^p)})^q)^{\frac{1}{q}}. \tag{5}$$

Also, we have $r \ge \max(p,q)$ implies that $p \le r$, i.e. $\ell^p \hookrightarrow \ell^r$, it follows that $\ell^p(L^p) \hookrightarrow \ell^r(L^p)$. Consequently $\|(\tau_k\theta.\Delta_j u)_{k\in\mathbb{Z}^n}\|_{\ell^r(L^p)} \le c\|(\tau_k\theta.\Delta_j u)_{k\in\mathbb{Z}^n}\|_{\ell^p(L^p)}$. So, we can see that the inequality (5) becomes as follows $(\sum_{k\in\mathbb{Z}^n} \|\tau_k\theta.u\|^r_{B^s_{p,q}})^{\frac{1}{r}} \le c(\sum_{j=0}^{\infty} 2^{sjq}(\|\tau_k\theta.\Delta_j u\|_{\ell^p(L^p)})^q)^{\frac{1}{q}}$. Since L^p is a space localizable in the ℓ^p norm, then it holds that $\|\tau_k\theta.\Delta_j u\|_{\ell^p(L^p)} \backsim \|\Delta_j u\|_p$. Hence,

$$(\sum_{k\in\mathbb{Z}^n} \|\tau_k\theta.u\|^r_{B^s_{p,q}})^{\frac{1}{r}} \le c(\sum_{j=0}^{\infty} 2^{sjq}\|\Delta_j u\|_p^q)^{\frac{1}{q}} \le c\|u\|_{B^s_{p,q}}.$$

Thus, $B^s_{p,q} \hookrightarrow (B^s_{p,q})_{\ell^r}$.

(ii) Now, we will show that

$$\|u\|_{B^s_{p,q}} \le c\|u\|_{(B^s_{p,q})_{\ell^r}} \quad \text{for } c > 0.$$

Let $u \in \mathcal{S}'(\mathbb{R}^n)$. Then it holds that

$$\|L_j(u)\|_p = \|L_j(\sum_{k\in\mathbb{Z}^n} \tau_k\varphi.u)\|_p = \|\sum_{k\in\mathbb{Z}^n} L_j(\tau_k\varphi.u)\|_p \le c(\sum_{k\in\mathbb{Z}^n} \|L_j(\tau_k\varphi.u)\|_p^p)^{\frac{1}{p}}.$$

Since $r \le \min(p,q)$, it holds that $\ell^r \hookrightarrow \ell^p$, i.e.

$$\|(\|L_j(\tau_k\varphi.u)\|_p)_{k\in\mathbb{Z}^n}\|_{\ell^p} \le c\|(\|L_j(\tau_k\varphi.u)\|_p)_{k\in\mathbb{Z}^n}\|_{\ell^r}.$$

So, we have

$$\|L_j(u)\|_p \le c(\sum_{k\in\mathbb{Z}^n} \|L_j(\tau_k\varphi.u)\|_p^p)^{\frac{1}{p}} \le c(\sum_{k\in\mathbb{Z}^n} \|L_j(\tau_k\varphi.u)\|_p^r)^{\frac{1}{r}}.$$

This implies that

$$(\sum_{j=0}^{\infty} 2^{sjq}\|L_j u\|_p^q)^{\frac{1}{q}} \le c(\sum_{j=0}^{\infty} 2^{sjq}(\sum_{k\in\mathbb{Z}^n} \|L_j(\tau_k\varphi.u)\|_p^r)^{\frac{q}{r}})^{\frac{1}{q}}.$$

Consequently

$$(\sum_{j=0}^{\infty} 2^{sjq}\|L_j u\|_p^q)^{\frac{1}{q}} \le c(\|(2^{sj}\|L_j(\tau_k\varphi.u)\|_p)_{k\in\mathbb{Z}^n}\|_{\ell^q(\ell^r)}). \tag{6}$$

Since $r \le \min(p,q)$, it holds that $r \le q$. Then from Minkowski inequality we have

$$\|(2^{sj}\|L_j(\tau_k\varphi.u)\|_p)_{k\in\mathbb{Z}^n}\|_{\ell^q(\ell^r)} \le c\|(2^{sj}\|L_j(\tau_k\varphi.u)\|_p)_{k\in\mathbb{Z}^n}\|_{\ell^r(\ell^q)}.$$

So, we can see that the inequality (6) becomes as follows

$$(\sum_{j=0}^{\infty} 2^{sjq}\|L_j u\|_p^q)^{\frac{1}{q}} \le c(\|(2^{sj}\|L_j(\tau_k\varphi.u)\|_p)_{k\in\mathbb{Z}^n}\|_{\ell^r(\ell^q)}).$$

Consequently

$$(\sum_{j=0}^{\infty} 2^{sjq}\|L_j u\|_p^q)^{\frac{1}{q}} \le c(\sum_{k\in\mathbb{Z}^n}(\sum_{j=0}^{\infty} 2^{sjq}\|L_j(\tau_k\varphi.u)\|_p^q)^{\frac{r}{q}})^{\frac{1}{r}} \le c(\sum_{k\in\mathbb{Z}^n} \|\tau_k\varphi.u\|_{B^s_{p,q}}^r)^{\frac{1}{r}} \le c\|u\|_{(B^s_{p,q})_{\ell^r}}.$$

Thus, $(B^s_{p,q})_{\ell^r} \hookrightarrow B^s_{p,q}$. □

Remark 2. *The generalization of Bourdaud Theorem given by Sickel and Smirnov in 1999* [9] *using the wavelet method, the aim of this work is to generalize the same Theorem of a localization property by using a different method.*

Theorem 3. *Let $p \in [1,+\infty)$, $q \in [1,+\infty]$, $s \in \mathbb{R}$, $F^s_{p,q}$ and $(F^s_{p,q})_{\ell^p}$ are respectively the Lizorkin-Triebel and localized Lizorkin-Triebel spaces. Then the space $F^s_{p,q}$ is localizable in the ℓ^p norm, i.e. $F^s_{p,q} = (F^s_{p,q})_{\ell^p}$.*

Proof. (i) $(F^s_{p,q})_{\ell^p} \hookrightarrow F^s_{p,q}$. We will show that

$$\|f\|_{F^s_{p,q}} \le c\|f\|_{(F^s_{p,q})_{\ell^p}} \quad \text{for } c > 0.$$

From Definition 3, $\|f\|_{F^s_{p,q}} = \|(\sum_{j=0}^{\infty} 2^{sjq}|\Delta_j f|^q)^{\frac{1}{q}}\|_p$. We put $\Delta_j f = \sum_{k\in\mathbb{Z}^n} \tau_k\varphi.\Delta_j f$, it follows that $\|f\|_{F^s_{p,q}} = \|(\sum_{j=0}^{\infty}(\sum_{k\in\mathbb{Z}^n} 2^{sj}|\tau_k\varphi\Delta_j f|)^q)^{\frac{1}{q}}\|_p$. Consequently

$$\|f\|_{F^s_{p,q}} = \|\|\|2^{sj}(\tau_k\varphi\Delta_j f)_{k\in\mathbb{Z}^n}\|_{\ell^q(\ell^1)}\|_p.$$

Since, $1 \leq q$. Then from Minkowski inequality we have

$$\|f\|_{F^s_{p,q}} = \|\|\|2^{sj}(\tau_k\varphi\Delta_j f)_{k\in\mathbb{Z}^n}\|_{\ell^q(\ell^1)}\|_p \leq \|\|\|2^{sj}(\tau_k\varphi\Delta_j f)_{k\in\mathbb{Z}^n}\|_{\ell^1(\ell^q)}\|_p.$$

Consequently

$$\|f\|_{F^s_{p,q}} \leq c\| \sum_{k\in\mathbb{Z}^n} (\sum_{j=0}^{\infty} 2^{sjq}|\tau_k\varphi\Delta_j f|^q)^{\frac{1}{q}}\|_p \leq c(\sum_{k\in\mathbb{Z}^n} \|(\sum_{j=0}^{\infty} 2^{sjq}|\tau_k\varphi\Delta_j f|^q)^{\frac{1}{q}}\|_p^p)^{\frac{1}{p}}.$$

Hence, $\|f\|_{F^s_{p,q}} \leq c(\sum_{k\in\mathbb{Z}^n} \|\tau_k\varphi.f\|^p_{F^s_{p,q}})^{\frac{1}{p}}$. Thus, $(F^s_{p,q})_{\ell^p} \hookrightarrow F^s_{p,q}$.

(ii) $F^s_{p,q} \hookrightarrow (F^s_{p,q})_{\ell^p}$. Now, we will show that

$$\|f\|_{(F^s_{p,q})_{\ell^p}} \leq c\|f\|_{F^s_{p,q}} \quad \text{for } c > 0.$$

Let $p, q \in [1, +\infty]$ and $s \in \mathbb{R}$. Then it holds that

$$(\sum_{k\in\mathbb{Z}^n} \|\tau_k\varphi.f\|^p_{F^s_{p,q}})^{\frac{1}{p}} = (\sum_{k\in\mathbb{Z}^n} \|\tau_k\varphi \sum_{j=0}^{\infty} \Delta_j f\|^p_{F^s_{p,q}})^{\frac{1}{p}} = (\sum_{k\in\mathbb{Z}^n} \| \sum_{j=0}^{\infty} \Delta_j f \tau_k\varphi\|^p_{F^s_{p,q}})^{\frac{1}{p}}.$$

From Proposition 1, it follows that

$$\begin{aligned}(\sum_{k\in\mathbb{Z}^n} \|\tau_k\varphi.f\|^p_{F^s_{p,q}})^{\frac{1}{p}} &\leq c(\sum_{k\in\mathbb{Z}^n} \|(\sum_{j=0}^{\infty} 2^{sjq}|\Delta_j f\tau_k\varphi|^q)^{\frac{1}{q}}\|_p^p)^{\frac{1}{p}} \\ &\leq c(\sum_{k\in\mathbb{Z}^n} \|\tau_k\varphi(\sum_{j=0}^{\infty} 2^{sjq}|\Delta_j f|^q)^{\frac{1}{q}}\|_p^p)^{\frac{1}{p}}.\end{aligned}$$

Since L^p is a space localizable in the ℓ^p norm, then it holds that

$$(\sum_{k\in\mathbb{Z}^n} \|\tau_k\varphi.f\|^p_{F^s_{p,q}})^{\frac{1}{p}} \leq c\|(\sum_{j=0}^{\infty} 2^{sjq}|\Delta_j f|^q)^{\frac{1}{q}}\|_p \leq c\|f\|_{F^s_{p,q}}.$$

Thus, $F^s_{p,q} \hookrightarrow (F^s_{p,q})_{\ell^p}$. □

3 CONCLUSION

In this work, we have generalized the Bourdaud theorem of a lacalization property of Besov spaces $B^s_{p,q}(\mathbb{R}^n)$ on the ℓ^r space, where $s \in \mathbb{R}$, $p, q, r \in [1, +\infty]$. Also, we have provided that the Lizorkin-Triebel spaces are localizable in the ℓ^p norm. In future work, we will investigate the localization property on other functional spaces.

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Поняття локалізаційної властивості просторів Бєсова введене Г. Бурдо, він показав, що простори Бєсова $B^s_{p,q}(\mathbb{R}^n)$, де $s \in \mathbb{R}$ і $p, q \in [1, +\infty]$ такі, що $p \neq q$, є нелокалізовними у нормі ℓ^p. Також він показав, що простори Бєсова $B^s_{p,q}$ вкладені в локалізовані простори Бєсова $(B^s_{p,q})_{\ell^p}$ (тобто $B^s_{p,q} \hookrightarrow (B^s_{p,q})_{\ell^p}$ при $p \geq q$). Також було показано, що локалізовані простори Бєсова $(B^s_{p,q})_{\ell^p}$ вкладені в простори Бєсова $B^s_{p,q}$ (тобто $(B^s_{p,q})_{\ell^p} \hookrightarrow B^s_{p,q}$ при $p \leq q$). Зокрема $B^s_{p,p}$ є локалізовним в нормі ℓ^p, де ℓ^p простір послідовностей $(a_k)_k$ таких, що $\|(a_k)\|_{\ell^p} < \infty$. У цій статті ми узагальнили теорему Бурдо про локалізаційну властивість просторів Бєсова $B^s_{p,q}(\mathbb{R}^n)$ на простір ℓ^r, де $r \in [1, +\infty]$. А точніше ми довели, що будь-який простір Бєсова $B^s_{p,q}$ є вкладений в локалізований простір Бєсова $(B^s_{p,q})_{\ell^r}$ (тобто $B^s_{p,q} \hookrightarrow (B^s_{p,q})_{\ell^r}$ при $r \geq \max(p, q)$). Також ми показали, що будь-який локалізований простір Бєсова $(B^s_{p,q})_{\ell^r}$ вкладений в простір Бєсова $B^s_{p,q}$ (тобто $(B^s_{p,q})_{\ell^r} \hookrightarrow B^s_{p,q}$ при $r \leq \min(p, q)$). І на завершення було показано, що простори Лізоркіна-Трібеля $F^s_{p,q}(\mathbb{R}^n)$, де $s \in \mathbb{R}$ і $p, q \in [1, +\infty]$ є локалізовними в нормі ℓ^p (тобто $F^s_{p,q} = (F^s_{p,q})_{\ell^p}$).

Ключові слова і фрази: простори Бєсова, простори Лізоркіна-Трібеля, локалізаційна властивість.