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ADVANCEMENT ON THE STUDY OF GROWTH ANALYSIS OF DIFFERENTIAL POLYNOMIAL AND DIFFERENTIAL MONOMIAL IN THE LIGHT OF SLOWLY INCREASING FUNCTIONS

Study of the growth analysis of entire or meromorphic functions has generally been done through their Nevanlinna's characteristic function in comparison with those exponential functions. But if one is interested to compare the growth rates of any entire or meromorphic function with respect to another, the concepts of relative growth indicators will come. The field of study in this area may be more significant through the intensive applications of the theories of slowly increasing functions which actually means that $L(ar) \sim L(r)$ as $r \to \infty$ for every positive constant a, i.e. $\lim\limits_{r\to\infty} \frac{L(ar)}{L(r)} = 1$, where $L \equiv L(r)$ is a positive continuous function increasing slowly. Actually in the present paper, we establish some results depending on the comparative growth properties of composite entire and meromorphic functions using the idea of relative ${}_pL^*$-order, relative ${}_pL^*$- type, relative ${}_pL^*$-weak type and differential monomials, differential polynomials generated by one of the factors which extend some earlier results, where ${}_pL^*$ is nothing but a weaker assumption of L.

Key words and phrases: entire function, meromorphic function, relative ${}_pL^*$-order, relative ${}_pL^*$-type, relative ${}_pL^*$-weak type, growth, differential monomial, differential polynomial, slowly increasing function.

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INTRODUCTION, DEFINITIONS AND NOTATIONS

Let us consider that the reader is familiar with the fundamental results and the standard notations of the Nevanlinna theory of meromorphic functions which are available in [13, 16, 22, 23]. We also use the standard notations and definitions of the theory of entire functions which are available in [24] and therefore we do not explain those in details.

For $x \in [0, \infty)$ and $k \in \mathbb{N}$, we define the following functions $\exp^{[k]} x = \exp\left(\exp^{[k-1]} x\right)$ and $\log^{[k]} x = \log\left(\log^{[k-1]} x\right)$, where $\mathbb{N}$ be the set of all positive integers.

Let f be an entire function defined in the open complex plane $\mathbb{C}$. The maximum modulus function $M_f(r)$ corresponding to f is defined on $|z| = r$ as $M_f(r) = \max|z| = r\,|f(z)|$. In this connection the following definition is relevant.

Definition 1 ([4]). *A non-constant entire function f is said have the Property (A) if for any $\sigma > 1$ and for all sufficiently large r, $\left[M_f(r)\right]^2 \leq M_f(r^\sigma)$ holds.*

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For examples of functions with or without the Property (A) we refer the reader to [4].

When f is meromorphic, one may introduce another function $T_f(r)$ known as Nevanlinna's characteristic function of f, playing the same role as $M_f(r)$.

Now we just recall the following properties of meromorphic functions which will be needed in the sequel.

Let $n_{0j}, n_{1j}, \ldots, n_{kj}$ $(k \geq 1)$ be non-negative integers such that for each j the following inequality holds $\sum\limits_{i=0}^{k} n_{ij} \geq 1$. For a non-constant meromorphic function f, we call $M_j[f] = A_j(f)^{n_{0j}}\left(f^{(1)}\right)^{n_{1j}} \cdots \left(f^{(k)}\right)^{n_{kj}}$, where $T(r, A_j) = S(r, f)$ to be a differential monomial generated by f. The numbers $\gamma_{Mj} = \sum\limits_{i=0}^{k} n_{ij}and\Gamma_{Mj} = \sum\limits_{i=0}^{k}(i+1)n_{ij}$ are called the degree and weight of $M_j[f]$ respectively [6, 19]. The expression $P[f] = \sum\limits_{j=1}^{s} M_j[f]$ is called a differential polynomial generated by f. The numbers $\gamma_P = \max\limits_{1\leq j\leq s} \gamma_{Mj}$ and $\Gamma_P = \max\limits_{1\leq j\leq s} \Gamma_{Mj}$ are called the degree and weight of $P[f]$ respectively [6, 19]. Also we call the numbers $\underline{\gamma_P} = \min\limits_{1\leq j\leq s} \gamma_{Mj}$ and k (the order of the highest derivative of f) the lower degree and the order of $P[f]$ respectively. If $\underline{\gamma_p} = \gamma_P$, $P[f]$ is called a homogeneous differential polynomial. Throughout the paper, we consider only the non-constant differential polynomials and we denote by $P_0[f]$ a differential polynomial not containing f, i.e. for which $n_{0j} = 0$ for $j = 1, 2, \ldots, s$. We consider only those $P[f]$, $P_0[f]$ singularities of whose individual terms do not cancel each other. We also denote by $M[f]$ a differential monomial generated by a transcendental meromorphic function f.

However, the Nevanlinna's Characteristic function of a meromorphic function f is defined as

$$T_f(r) = N_f(r) + m_f(r),$$

wherever the function $N_f(r,a)$ $\left(\overline{N}_f(r,a)\right)$ known as counting function of a-points (distinct a-points) of meromorphic f is defined as follows:

$$N_f(r,a) = \int_0^r \frac{n_f(t,a) - n_f(0,a)}{t} dt + n_f(0,a)\log r$$

$$\left(\overline{N}_f(r,a) = \int_0^r \frac{\overline{n}_f(t,a) - \overline{n}_f(0,a)}{t} dt + \overline{n}_f(0,a)\log r\right),$$

in addition we represent by $n_f(r,a)$ $\left(\overline{n}_f(r,a)\right)$ the number of a-points (distinct a-points) of f in $|z| \leq r$ and an ∞-point is a pole of f. In many occasions $N_f(r,\infty)$ and $\overline{N}_f(r,\infty)$ are symbolized by $N_f(r)$ and $\overline{N}_f(r)$ respectively.

On the other hand, the function $m_f(r,\infty)$ alternatively indicated by $m_f(r)$ known as the proximity function of f is defined as follows

$$m_f(r) = \frac{1}{2\pi}\int_0^{2\pi} \log^+ \left|f\left(re^{i\theta}\right)\right| d\theta, \text{ where } \log^+ x = \max(\log x, 0) \text{ for all } x \geqslant 0.$$

Also we may employ $m\left(r, \frac{1}{f-a}\right)$ by $m_f(r,a)$.

If f is entire, then the Nevanlinna's Characteristic function $T_f(r)$ of f is defined as

$$T_f(r) = m_f(r).$$

Moreover for any non-constant entire function f, $T_f(r)$ is strictly increasing and continuous functions of r. Also its inverse $T_f^{-1} : (|T_f(0)|, \infty) \to (0,\infty)$ exists, where $\lim\limits_{s\to\infty} T_f^{-1}(s) = \infty$.

In this connection we immediately remind the following definition which is relevant.

Definition 2. *Let a be a complex number, finite or infinite. The Nevanlinna's deficiency and the Valiron deficiency of a with respect to a meromorphic function f are defined as*

$$\delta(a;f) = 1 - \overline{\lim_{r\to\infty}} \frac{N_f(r,a)}{T_f(r)} = \underline{\lim_{r\to\infty}} \frac{m_f(r,a)}{T_f(r)}$$

and

$$\Delta(a;f) = 1 - \underline{\lim_{r\to\infty}} \frac{N_f(r,a)}{T_f(r)} = \overline{\lim_{r\to\infty}} \frac{m_f(r,a)}{T_f(r)}.$$

Definition 3. *The quantity $\Theta(a;f)$ of a meromorphic function f is defined as follows*

$$\Theta(a;f) = 1 - \overline{\lim_{r\to\infty}} \frac{\overline{N}_f(r,a)}{T_f(r)}.$$

Definition 4 ([21])**.** *For $a \in \mathbb{C} \cup \{\infty\}$, we denote by $n_{f|=1}(r,a)$, the number of simple zeros of $f-a$ in $|z| \leq r$. $N_{f|=1}(r,a)$ is defined in terms of $n_{f|=1}(r,a)$ in the usual way. We put*

$$\delta_1(a;f) = 1 - \overline{\lim_{r\to\infty}} \frac{N_{f|=1}(r,a)}{T_f(r)},$$

the deficiency of a corresponding to the simple a-points of f, i.e. simple zeros of $f-a$.

Yang [20] proved that there exists at most a denumerable number of complex numbers $a \in \mathbb{C} \cup \{\infty\}$ for which $\delta_1(a;f) > 0$ and $\sum\limits_{a\in\mathbb{C}\cup\{\infty\}} \delta_1(a;f) \leq 4$.

Definition 5 ([14])**.** *For $a \;\varepsilon\; \mathbb{C} \cup \{\infty\}$, let $n_p(r,a;f)$ denotes the number of zeros of $f-a$ in $|z| \leq r$, where a zero of multiplicity $< p$ is counted according to its multiplicity and a zero of multiplicity $\geq p$ is counted exactly p times and $N_p(r,a;f)$ is defined in terms of $n_p(r,a;f)$ in the usual way. We define*

$$\delta_p(a;f) = 1 - \overline{\lim_{r\to\infty}} \frac{N_p(r,a;f)}{T_f(r)}.$$

Definition 6 ([1])**.** *$P[f]$ is said to be admissible if*

(i) $P[f]$ is homogeneous, or

(ii) $P[f]$ is non homogeneous and $m_f(r) = S_f(r)$.

However in case of any two meromorphic functions f and g, the ratio $\frac{T_f(r)}{T_g(r)}$ as $r \to \infty$ is called as the growth of f with respect to g in terms of their Nevanlinna's Characteristic functions. Further the concept of the growth measuring tools such as order and lower order which are conventional in complex analysis and the growth of entire or meromorphic functions can be studied in terms of their orders and lower orders are normally defined in terms of their growth with respect to the exp function which are shown in the following definition.

Definition 7. *The order ρ_f (the lower order λ_f) of a meromorphic function f is defined as*

$$\rho_f = \overline{\lim_{r\to\infty}} \frac{\log T_f(r)}{\log T_{\exp z}(r)} = \overline{\lim_{r\to\infty}} \frac{\log T_f(r)}{\log\left(\frac{r}{\pi}\right)} = \overline{\lim_{r\to\infty}} \frac{\log T_f(r)}{\log(r) + O(1)}$$

$$\left(\lambda_f = \underline{\lim_{r\to\infty}} \frac{\log T_f(r)}{\log T_{\exp z}(r)} = \underline{\lim_{r\to\infty}} \frac{\log T_f(r)}{\log\left(\frac{r}{\pi}\right)} = \underline{\lim_{r\to\infty}} \frac{\log T_f(r)}{\log(r) + O(1)}\right).$$

If f is entire, then

$$\rho_f = \overline{\lim_{r\to\infty}} \frac{\log\log M_f(r)}{\log\log M_{\exp z}(r)} = \overline{\lim_{r\to\infty}} \frac{\log^{[2]} M_f(r)}{\log r}$$

$$\left(\lambda_f = \underline{\lim_{r\to\infty}} \frac{\log\log M_f(r)}{\log\log M_{\exp z}(r)} = \underline{\lim_{r\to\infty}} \frac{\log^{[2]} M_f(r)}{\log r}\right).$$

Somasundaram and Thamizharasi [18] introduced the notions of L-order and L-type for entire functions, where $L \equiv L(r)$ is a positive continuous function increasing slowly, i.e. $L(ar) \sim L(r)$ as $r \to \infty$ for every positive constant a. The more generalized concept of L-order and L-type of meromorphic functions are L^*-order and L^*-type (resp. L^*- lower type) respectively which are as follows.

Definition 8 ([18]). *The L^*-order $\rho_f^{L^*}$ and the L^*-lower order $\lambda_f^{L^*}$ of a meromorphic function f are defined by*

$$\rho_f^{L^*} = \overline{\lim_{r\to\infty}} \frac{\log T_f(r)}{\log\left[re^{L(r)}\right]} \quad \text{and} \quad \lambda_f^{L^*} = \underline{\lim_{r\to\infty}} \frac{\log T_f(r)}{\log\left[re^{L(r)}\right]},$$

where $L \equiv L(r)$ is a positive continuous function increasing slowly.

If f is entire, then

$$\rho_f^{L^*} = \overline{\lim_{r\to\infty}} \frac{\log^{[2]} M_f(r)}{\log\left[re^{L(r)}\right]} \quad \text{and} \quad \lambda_f^{L^*} = \underline{\lim_{r\to\infty}} \frac{\log^{[2]} M_f(r)}{\log\left[re^{L(r)}\right]}.$$

Definition 9 ([18]). *The L^*-type $\sigma_f^{L^*}$ and L^*-lower type $\overline{\sigma}_f^{L^*}$ of a meromorphic function f such that $0 < \rho_f^{L^*} < \infty$ are defined as*

$$\sigma_f^{L^*} = \overline{\lim_{r\to\infty}} \frac{T_f(r)}{\left[re^{L(r)}\right]^{\rho_f^{L^*}}} \quad \text{and} \quad \overline{\sigma}_f^{L^*} = \underline{\lim_{r\to\infty}} \frac{T_f(r)}{\left[re^{L(r)}\right]^{\rho_f^{L^*}}},$$

where $L \equiv L(r)$ is a positive continuous function increasing slowly.

If f is entire, then

$$\sigma_f^{L^*} = \overline{\lim_{r\to\infty}} \frac{\log M_f(r)}{\left[re^{L(r)}\right]^{\rho_f^{L^*}}} \quad \text{and} \quad \overline{\sigma}_f^{L^*} = \underline{\lim_{r\to\infty}} \frac{\log M_f(r)}{\left[re^{L(r)}\right]^{\rho_f^{L^*}}}.$$

Analogously in order to determine the relative growth of two meromorphic functions having same non zero finite L^*-lower order one may introduce the definition of L^*-weak type of meromorphic functions having finite positive L^*-lower order in the following way.

Definition 10. *The L^*-weak type denoted by $\tau_f^{L^*}$ of a meromorphic function f having $0 < \lambda_f^{L^*} < \infty$ is defined as follows*

$$\tau_f^{L^*} = \underline{\lim}_{r\to\infty} \frac{T_f(r)}{\left[re^{L(r)}\right]^{\lambda_f^{L^*}}} ,$$

where $L \equiv L(r)$ is a positive continuous function increasing slowly.

Similarly the growth indicator $\overline{\tau}_f^{L^*}$ is defined as

$$\overline{\tau}_f^{L^*} = \overline{\lim}_{r\to\infty} \frac{T_f(r)}{\left[re^{L(r)}\right]^{\lambda_f^{L^*}}}, \quad \text{where} \quad 0 < \lambda_f^{L^*} < \infty .$$

If f is entire, then

$$\tau_f^{L^*} = \underline{\lim}_{r\to\infty} \frac{\log M_f(r)}{\left[re^{L(r)}\right]^{\lambda_f^{L^*}}} \quad \text{and} \quad \overline{\tau}_f^{L^*} = \overline{\lim}_{r\to\infty} \frac{\log M_f(r)}{\left[re^{L(r)}\right]^{\lambda_f^{L^*}}}, \quad \text{where} \quad 0 < \lambda_f^{L^*} < \infty .$$

Extending the notion of Somasundaram and Thamizharasi [18], one may introduce concept of ${}_pL^*$-order, ${}_pL^*$-type and ${}_pL^*$-weak type of a meromorphic function f as follows.

Definition 11. *For any positive integer p, the ${}_pL^*$-order $\rho_p^{L^*}(f)$ and the ${}_pL^*$-lower order $\lambda_p^{L^*}(f)$ of a meromorphic function f are defined by*

$$\rho_p^{L^*}(f) = \overline{\lim}_{r\to\infty} \frac{\log T_f(r)}{\log\left[r\exp^{[p]} L(r)\right]} \quad \text{and} \quad \lambda_p^{L^*}(f) = \underline{\lim}_{r\to\infty} \frac{\log T_f(r)}{\log\left[r\exp^{[p]} L(r)\right]} ,$$

where $L \equiv L(r)$ is a positive continuous function increasing slowly.

If f is entire, then

$$\rho_p^{L^*}(f) = \overline{\lim}_{r\to\infty} \frac{\log^{[2]} M_f(r)}{\log\left[r\exp^{[p]} L(r)\right]} \quad \text{and} \quad \lambda_p^{L^*}(f) = \underline{\lim}_{r\to\infty} \frac{\log^{[2]} M_f(r)}{\log\left[r\exp^{[p]} L(r)\right]} .$$

Definition 12. *For any positive integer p, the ${}_pL^*$-type $\sigma_p^{L^*}(f)$ and ${}_pL^*$-lower type $\overline{\sigma}_p^{L^*}(f)$ of a meromorphic function f such that $0 < \rho_p^{L^*}(f) < \infty$ are defined by*

$$\sigma_p^{L^*}(f) = \overline{\lim}_{r\to\infty} \frac{T_f(r)}{\left[r\exp^{[p]} L(r)\right]^{\rho_p^{L^*}(f)}} \quad \text{and} \quad \overline{\sigma}_p^{L^*}(f) = \underline{\lim}_{r\to\infty} \frac{T_f(r)}{\left[r\exp^{[p]} L(r)\right]^{\rho_p^{L^*}(f)}} ,$$

where $L \equiv L(r)$ is a positive continuous function increasing slowly.

If f is entire, then

$$\sigma_p^{L^*}(f) = \overline{\lim}_{r\to\infty} \frac{\log M_f(r)}{\left[r\exp^{[p]} L(r)\right]^{\rho_p^{L^*}(f)}} \quad \text{and} \quad \overline{\sigma}_p^{L^*}(f) = \underline{\lim}_{r\to\infty} \frac{\log M_f(r)}{\left[r\exp^{[p]} L(r)\right]^{\rho_p^{L^*}(f)}} .$$

Definition 13. *For any positive integer p, the ${}_pL^*$-weak type denoted by $\tau_p^{L^*}(f)$ of a meromorphic function f having $0 < \lambda_p^{L^*}(f) < \infty$ is defined by*

$$\tau_p^{L^*}(f) = \underline{\lim}_{r\to\infty} \frac{T_f(r)}{\left[r\exp^{[p]} L(r)\right]^{\lambda_p^{L^*}(f)}} ,$$

where $L \equiv L(r)$ is a positive continuous function increasing slowly.

Similarly the growth indicator $\overline{\tau}_p^{L^*}(f)$ is defined by

$$\overline{\tau}_p^{L^*}(f) = \overline{\lim_{r\to\infty}} \frac{T_f(r)}{\left[r\exp^{[p]} L(r)\right]^{\lambda_p^{L^*}(f)}}, \quad where \quad 0 < \lambda_p^{L^*}(f) < \infty .$$

If f is entire, then for $0 < \lambda_p^{L^*}(f) < \infty$,

$$\tau_p^{L^*}(f) = \underline{\lim_{r\to\infty}} \frac{\log M_f(r)}{\left[r\exp^{[p]} L(r)\right]^{\lambda_p^{L^*}(f)}} \quad and \quad \overline{\tau}_p^{L^*}(f) = \overline{\lim_{r\to\infty}} \frac{\log M_f(r)}{\left[r\exp^{[p]} L(r)\right]^{\lambda_p^{L^*}(f)}} .$$

Lahiri and Banerjee [17] introduced the following definition of relative order of a meromorphic function with respect to an entire function.

Definition 14 ([17])**.** *Let f be meromorphic and g be entire functions. The relative order of f with respect to g denoted by $\rho_g(f)$ is defined as*

$$\begin{aligned}\rho(f,g) &= \inf\left\{\mu > 0 : T_f(r) < T_g(r^\mu) \text{ for all sufficiently large } r\right\}\\ &= \overline{\lim_{r\to\infty}} \frac{\log T_g^{-1} T_f(r)}{\log r}.\end{aligned}$$

The definition coincides with the classical one [17] if $g(z) = \exp z$.

Similarly one can define the relative lower order of a meromorphic function f with respect to an entire g denoted by $\lambda_g(f)$ in the following manner

$$\lambda(f,g) = \underline{\lim_{r\to\infty}} \frac{\log T_g^{-1} T_f(r)}{\log r}.$$

In order to make some progress in the study of relative order, now we introduce relative ${}_pL^*$-order and relative ${}_pL^*$- lower order of a meromorphic function f with respect to an entire function g.

Definition 15. *The relative ${}_pL^*$-order denoted as $\rho_p^{L^*}(f,g)$ and relative ${}_pL^*$- lower order denoted as $\lambda_p^{L^*}(f,g)$ of a meromorphic function f with respect to an entire g are defined as*

$$\rho_p^{L^*}(f,g) = \overline{\lim_{r\to\infty}} \frac{\log T_g^{-1} T_f(r)}{\log\left[r\exp^{[p]} L(r)\right]} \quad and \quad \lambda_p^{L^*}(f,g) = \underline{\lim_{r\to\infty}} \frac{\log T_g^{-1} T_f(r)}{\log\left[r\exp^{[p]} L(r)\right]} ,$$

where p is any positive integers and $L \equiv L(r)$ is a positive continuous function increasing slowly.

Further to compare the relative growth of two meromorphic functions having same non zero finite relative ${}_pL^*$-order with respect to another entire function, one may introduce the definitions of relative ${}_pL^*$-type and relative ${}_pL^*$-lower type in the following manner.

Definition 16. *The relative ${}_pL^*$-type and relative ${}_pL^*$-lower type denoted respectively by $\sigma_p^{L^*}(f,g)$ and $\overline{\sigma}_p^{L^*}(f,g)$ of a meromorphic function f with respect to an entire function g such that $0 < \rho_p^{L^*}(f,g) < \infty$ are respectively defined by*

$$\sigma_p^{L^*}(f,g) = \overline{\lim_{r\to\infty}} \frac{T_g^{-1} T_f(r)}{\left[r\exp^{[p]} L(r)\right]^{\rho_p^{L^*}(f,g)}} \quad and \quad \overline{\sigma}_p^{L^*}(f,g) = \underline{\lim_{r\to\infty}} \frac{T_g^{-1} T_f(r)}{\left[r\exp^{[p]} L(r)\right]^{\rho_p^{L^*}(f,g)}} ,$$

where $L \equiv L(r)$ is a positive continuous function increasing slowly.

Analogously to determine the relative growth of two meromorphic functions having same non zero finite relative ${}_pL^*$-lower order with respect to an entire function one may introduce the definition of relative ${}_pL^*$-weak type in the following way.

Definition 17. *The relative ${}_pL^*$-weak type denoted by $\tau_p^{L^*}(f,g)$ of a meromorphic function f with respect to an entire function g such that $0 < \lambda_p^{L^*}(f,g) < \infty$ is defined by*

$$\tau_p^{L^*}(f,g) = \underline{\lim}_{r\to\infty} \frac{T_g^{-1}T_f(r)}{\left[r\exp^{[p]}L(r)\right]^{\lambda_p^{L^*}(f,g)}},$$

where $L \equiv L(r)$ is a positive continuous function increasing slowly.

Similarly one may define the growth indicator $\overline{\tau}_p^{L^*}(f,g)$ of a meromorphic function f with respect to an entire function g as follows

$$\overline{\tau}_p^{L^*}(f,g) = \overline{\lim}_{r\to\infty} \frac{T_g^{-1}T_f(r)}{\left[r\exp^{[p]}L(r)\right]^{\lambda_p^{L^*}(f,g)}}, \quad 0 < \lambda_p^{L^*}(f,g) < \infty.$$

In the paper we establish some new results depending on the comparative growth properties of composite entire or meromorphic functions using relative ${}_pL^*$-order, relative ${}_pL^*$- type, relative ${}_pL^*$-weak type and differential monomials, differential polynomials generated by one of the factors which in fact extend and improve some results of [9] and [10].

1 Lemmas

In this section we present some lemmas which will be needed in the sequel.

Lemma 1 ([7])**.** *Let f be a meromorphic function either of finite order or of non-zero lower order such that $\Theta(\infty;f) = \sum_{a\neq\infty}\delta_p(a;f) = 1$ or $\delta(\infty;f) = \sum_{a\neq\infty}\delta(a;f) = 1$ and h be an entire function with regular growth and non zero finite type. Also let $\Theta(\infty;h) = \sum_{a\neq\infty}\delta_p(a;h) = 1$ or $\delta(\infty;h) = \sum_{a\neq\infty}\delta(a;h) = 1$. Then for homogeneous $P_0[f]$ and $P_0[g]$,*

$$\lim_{r\to\infty} \frac{T_{P_0[h]}^{-1}T_{P_0[f]}(r)}{T_h^{-1}T_f(r)} = \left(\frac{\gamma_{P_0[f]}}{\gamma_{P_0[h]}}\right)^{\frac{1}{\rho_h}}.$$

Lemma 2 ([8])**.** *Let f be a transcendental meromorphic function of finite order or of non-zero lower order and $\sum_{a\in\mathbb{C}\cup\{\infty\}}\delta_1(a;f) = 4$ and h be a transcendental entire function with regular growth and non zero finite type. Also let $\sum_{a\in\mathbb{C}\cup\{\infty\}}\delta_1(a;h) = 4$. Then*

$$\lim_{r\to\infty} \frac{T_{P[h]}^{-1}T_{P[f]}(r)}{T_h^{-1}T_f(r)} = \left(\frac{\Gamma_{M[f]} - (\Gamma_{M[f]} - \gamma_{M[f]})\Theta(\infty;f)}{\Gamma_{M[h]} - (\Gamma_{M[h]} - \gamma_{M[h]})\Theta(\infty;h)}\right)^{\frac{1}{\rho_h}},$$

where

$$\Theta(\infty;f) = 1 - \overline{\lim}_{r\to\infty}\frac{\overline{N}_f(r)}{T_f(r)} \quad \text{and} \quad \Theta(\infty;h) = 1 - \overline{\lim}_{r\to\infty}\frac{\overline{N}_h(r)}{T_h(r)}.$$

Lemma 3 ([5])**.** *Let f be a meromorphic function either of finite order or of non-zero lower order such that $\Theta(\infty;f)=\sum_{a\neq\infty}\delta_p(a;f)=1$ or $\delta(\infty;f)=\sum_{a\neq\infty}\delta(a;f)=1$ and h be an entire function with regular growth having non zero finite order and $\Theta(\infty;h)=\sum_{a\neq\infty}\delta_p(a;h)=1$ or $\delta(\infty;h)=\sum_{a\neq\infty}\delta(a;h)=1$. Then for any positive integer p, the relative ${}_pL^*$-order and relative ${}_pL^*$-lower order of $P_0[f]$ with respect to $P_0[h]$ are same as those of f with respect to h for homogeneous $P_0[f]$ and $P_0[h]$.*

Lemma 4 ([5])**.** *Let f be a transcendental meromorphic function of finite order or of non-zero lower order and $\sum_{a\in\mathbb{C}\cup\{\infty\}}\delta_1(a;f)=4$ and h be a transcendental entire function with regular growth and non zero finite order. Also let $\sum_{a\in\mathbb{C}\cup\{\infty\}}\delta_1(a;h)=4$. Then for any positive integer p, the relative ${}_pL^*$-order and relative ${}_pL^*$-lower order of $M[f]$ with respect to $M[h]$ are same as those of f with respect to h., i.e.*

$$\rho_p^{L^*}(M[f],M[h])=\rho_p^{L^*}(f,h)\quad\text{and}\quad\lambda_p^{L^*}(M[f],M[h])=\lambda_p^{L^*}(f,h).$$

Lemma 5. *Let f be a meromorphic function either of finite order or of non-zero lower order such that $\Theta(\infty;f)=\sum_{a\neq\infty}\delta_p(a;f)=1$ or $\delta(\infty;f)=\sum_{a\neq\infty}\delta(a;f)=1$ and h be an entire function of regular growth having non zero finite type and $\Theta(\infty;h)=\sum_{a\neq\infty}\delta_p(a;h)=1$ or $\delta(\infty;h)=\sum_{a\neq\infty}\delta(a;h)=1$. Then for any positive integer p, the relative ${}_pL^*$-type and relative ${}_pL^*$-lower type of $P_0[f]$ with respect to $P_0[h]$ are $\left(\frac{\gamma_{P_0[f]}}{\gamma_{P_0[h]}}\right)^{\frac{1}{\rho_h}}$ times that of f with respect to h if $\rho_p^{L^*}(f,h)$ is positive finite, where $P_0[f]$ and $P_0[h]$ are homogeneous.*

Proof. By Lemma 3 and Lemma 1 and above we get that

$$\begin{aligned}\sigma_p^{L^*}(P_0[f],P_0[h])&=\overline{\lim_{r\to\infty}}\frac{T^{-1}_{P_0[h]}T_{P_0[f]}(r)}{\left[r\exp^{[p]}L(r)\right]^{\rho_p^{L^*}(P_0[f],P_0[h])}}\\&=\lim_{r\to\infty}\frac{T^{-1}_{P_0[h]}T_{P_0[f]}(r)}{T_h^{-1}T_f(r)}\cdot\overline{\lim_{r\to\infty}}\frac{T_h^{-1}T_f(r)}{\left[r\exp^{[p]}L(r)\right]^{\rho_p^{L^*}(f,h)}}=\left(\frac{\gamma_{P_0[f]}}{\gamma_{P_0[h]}}\right)^{\frac{1}{\rho_g}}\cdot\sigma_p^{L^*}(f,h).\end{aligned}$$

Similarly $\overline{\sigma}_p^{L^*}(P_0[f],P_0[h])=\left(\frac{\gamma_{P_0[f]}}{\gamma_{P_0[h]}}\right)^{\frac{1}{\rho_h}}\cdot\overline{\sigma}_p^{L^*}(f,h)$. □

In the line of Lemma 5 we may state the following lemma without its proof.

Lemma 6. *Let f be a meromorphic function either of finite order or of non-zero lower order such that $\Theta(\infty;f)=\sum_{a\neq\infty}\delta_p(a;f)=1$ or $\delta(\infty;f)=\sum_{a\neq\infty}\delta(a;f)=1$ and h be an entire function of regular growth having non zero finite type and $\Theta(\infty;h)=\sum_{a\neq\infty}\delta_p(a;h)=1$ or $\delta(\infty;h)=\sum_{a\neq\infty}\delta(a;h)=1$. Then $\tau_p^{L^*}(P_0[f],P_0[h])$ and $\overline{\tau}_p^{L^*}(P_0[f],P_0[h])$ are $\left(\frac{\gamma_{P_0[f]}}{\gamma_{P_0[h]}}\right)^{\frac{1}{\rho_h}}$ times that of f with respect to h, i.e.*

$$\tau_p^{L^*}(P_0[f],P_0[h])=\left(\frac{\gamma_{P_0[f]}}{\gamma_{P_0[h]}}\right)^{\frac{1}{\rho_h}}\cdot\tau_p^{L^*}(f,h)\quad\text{and}\quad\overline{\tau}_p^{L^*}(P_0[f],P_0[h])=\left(\frac{\gamma_{P_0[f]}}{\gamma_{P_0[h]}}\right)^{\frac{1}{\rho_h}}\cdot\overline{\tau}_p^{L^*}(f,h),$$

when $\lambda_p^{L^*}(f,h)$ is positive finite and $P_0[f]$, $P_0[h]$ are homogeneous.

In the line of Lemma 5 and with the help of Lemma 2 and Lemma 4, we may state the following two lemmas without their proofs.

Lemma 7. *Let f be a transcendental meromorphic function of finite order or of non-zero lower order and $\sum\limits_{a\in\mathbb{C}\cup\{\infty\}}\delta_1(a;f) = 4$ and h be a transcendental entire function of regular growth having non zero finite type and $\sum\limits_{a\in\mathbb{C}\cup\{\infty\}}\delta_1(a;h) = 4$. Then for any positive integer p, the relative ${}_pL^*$-type and relative ${}_pL^*$-lower type of $M[f]$ with respect to $M[h]$ are $\left(\frac{\Gamma_{M[f]}-(\Gamma_{M[f]}-\gamma_{M[f]})\Theta(\infty;f)}{\Gamma_{M[h]}-(\Gamma_{M[h]}-\gamma_{M[h]})\Theta(\infty;h)}\right)^{\frac{1}{\rho_h}}$ times that of f with respect to h if $\rho_p^{L^*}(f,h)$ is positive finite, where*

$$\Theta(\infty;f) = 1 - \overline{\lim_{r\to\infty}}\frac{\overline{N}_f(r)}{T_f(r)} \quad \text{and} \quad \Theta(\infty;h) = 1 - \overline{\lim_{r\to\infty}}\frac{\overline{N}_h(r)}{T_h(r)}.$$

Lemma 8. *Let f be a transcendental meromorphic function of finite order or of non-zero lower order and $\sum\limits_{a\in\mathbb{C}\cup\{\infty\}}\delta_1(a;f) = 4$ and h be a transcendental entire function of regular growth having non zero finite type and $\sum\limits_{a\in\mathbb{C}\cup\{\infty\}}\delta_1(a;h) = 4$. Then $\tau_p^{L^*}(M[f], M[h])$ and $\overline{\tau}_p^{L^*}(M[f], M[h])$ are $\left(\frac{\Gamma_{M[f]}-(\Gamma_{M[f]}-\gamma_{M[f]})\Theta(\infty;f)}{\Gamma_{M[h]}-(\Gamma_{M[h]}-\gamma_{M[h]})\Theta(\infty;h)}\right)^{\frac{1}{\rho_h}}$ times that of f with respect to h, i.e.*

$$\tau_p^{L^*}(M[f], M[h]) = \left(\frac{\Gamma_{M[f]} - (\Gamma_{M[f]} - \gamma_{M[f]})\Theta(\infty;f)}{\Gamma_{M[h]} - (\Gamma_{M[h]} - \gamma_{M[h]})\Theta(\infty;h)}\right)^{\frac{1}{\rho_h}} \cdot \tau_p^{L^*}(f,h)$$

$$\text{and} \quad \overline{\tau}_p^{L^*}(M[f], M[h]) = \left(\frac{\Gamma_{M[f]} - (\Gamma_{M[f]} - \gamma_{M[f]})\Theta(\infty;f)}{\Gamma_{M[h]} - (\Gamma_{M[h]} - \gamma_{M[h]})\Theta(\infty;h)}\right)^{\frac{1}{\rho_h}} \cdot \overline{\tau}_p^{L^*}(f,h),$$

when $\lambda_p^{L^*}(f,h)$ is positive finite and

$$\Theta(\infty;f) = 1 - \overline{\lim_{r\to\infty}}\frac{\overline{N}_f(r)}{T_f(r)} \quad \text{and} \quad \Theta(\infty;h) = 1 - \overline{\lim_{r\to\infty}}\frac{\overline{N}_h(r)}{T_h(r)}.$$

Lemma 9 ([2]). *If f is a meromorphic function and g is an entire function then for all sufficiently large values of r we have*

$$T_{f\circ g}(r) \le \{1 + o(1)\}\frac{T_g(r)}{\log M_g(r)} T_f(M_g(r)).$$

Lemma 10 ([3]). *Let f be meromorphic function and g be entire function and suppose that $0 < \mu < \rho_g \le \infty$. Then for a sequence of values of r tending to infinity*

$$T_{f\circ g}(r) \ge T_f(\exp(r^\mu)).$$

Lemma 11 ([15]). *Let f be meromorphic function and g be entire function such that $0 < \rho_g < \infty$ and $0 < \lambda_f$. Then for a sequence of values of r tending to infinity*

$$T_{f\circ g}(r) > T_g(\exp(r^\mu)),$$

where $0 < \mu < \rho_g$.

Lemma 12 ([11]). *Let f be a meromorphic function and g be an entire function such that $\lambda_g < \mu < \infty$ and $0 < \lambda_f \leq \rho_f < \infty$. Then for a sequence of values of r tending to infinity*

$$T_{f\circ g}(r) < T_f\left(\exp\left(r^{\mu}\right)\right).$$

Lemma 13 ([11]). *Let f be a meromorphic function of finite order and g be an entire function such that $0 < \lambda_g < \mu < \infty$. Then for a sequence of values of r tending to infinity*

$$T_{f\circ g}(r) < T_g\left(\exp\left(r^{\mu}\right)\right).$$

Lemma 14 ([12]). *Let f be an entire function which satisfies the Property (A), $\beta > 0$, $\delta > 1$ and $\alpha > 2$. Then*

$$\beta T_f(r) < T_f\left(\alpha r^{\delta}\right).$$

2 THEOREMS

In this section we present the main results of the paper. It is needless to mention that in the paper, the admissibility and homogeneity of $P_0[f]$ for meromorphic f will be needed as per the requirements of the theorems.

Theorem 1. *Let the meromorphic function f and entire function h satisfy the conditions of Lemma 3. Also let g be an entire function and $0 < \lambda_p^{L^*}(f,h) < \infty$, $\sigma_p^{L^*}(g) < \infty$, where p is any positive integer. If h satisfy the Property (A) and $\exp^{[p-1]} L\left(M_g(r)\right) = o\left(\left[r\exp L(r)\right]^{\beta}\right)$ as $r\to\infty$ and for some positive $\beta < \rho_p^{L^*}(g)$, then*

$$\varliminf_{r\to\infty}\frac{\log T_h^{-1}T_{f\circ g}(r)}{\log T_{P_0[h]}^{-1}T_{P_0[f]}\left(\exp\left[r\exp L(r)\right]^{\rho_p^{L^*}(g)}\right)} \leq \sigma_p^{L^*}(g).$$

Proof. Let us consider that $\alpha > 2$ and $\delta \to 1^+$ in Lemma 14. Since $T_h^{-1}(r)$ is an increasing function of r, it follows from Lemma 9, Lemma 14 and the inequality $T_g(r) \leq \log M_g(r)$ ([13]) for a sequence of values of r tending to infinity that

$$
\begin{aligned}
& T_h^{-1}T_{f\circ g}(r) \leqslant T_h^{-1}\left[\{1+o(1)\}T_f\left(M_g(r)\right)\right],\\
& \text{i.e. } T_h^{-1}T_{f\circ g}(r) \leqslant \alpha\left[T_h^{-1}T_f\left(M_g(r)\right)\right],\\
& \text{i.e. } \log T_h^{-1}T_{f\circ g}(r) \leqslant \log T_h^{-1}T_f\left(M_g(r)\right)+O(1),\\
& \text{i.e. } \log T_h^{-1}T_{f\circ g}(r) \leqslant \left(\lambda_p^{L^*}(f,h)+\varepsilon\right)\left[\log M_g(r)+\exp^{[p-1]}L\left(M_g(r)\right)\right]+O(1), \qquad (1)\\
& \text{i.e. } \log T_h^{-1}T_{f\circ g}(r) \leqslant \left(\lambda_p^{L^*}(f,h)+\varepsilon\right)\\
& \qquad\qquad \times\left[\left(\sigma_p^{L^*}(g)+\varepsilon\right)\left[r\exp^{[p]}L(r)\right]^{\rho_p^{L^*}(g)}+\exp^{[p-1]}L\left(M_g(r)\right)\right]+O(1).
\end{aligned}
$$

Further in view of Lemma 3, we obtain for all sufficiently large values of r that

$$
\begin{aligned}
& \log T_{P_0[h]}^{-1}T_{P_0[f]}\left(\exp\left[r\exp L(r)\right]^{\rho_p^{L^*}(g)}\right)\\
& \qquad \geq \left(\lambda_p^{L^*}(P_0[f],P_0[h])-\varepsilon\right)\left[\left[r\exp L(r)\right]^{\rho_p^{L^*}(g)}+\exp^{[p-1]}L\left(\exp\left[r\exp L(r)\right]^{\rho_p^{L^*}(g)}\right)\right],
\end{aligned}
$$

i.e. $\log T^{-1}_{P_0[h]} T_{P_0[f]} \left(\exp\left[r \exp L(r)\right]^{\rho_p^{L^*}(g)}\right) \geq \left(\lambda_p^{L^*}(f,h) - \varepsilon\right) \cdot \left[r \exp L(r)\right]^{\rho_p^{L^*}(g)}.$

Now from (1) and above it follows for a sequence of values of r tending to infinity that

$$\frac{\log T_h^{-1} T_{f \circ g}(r)}{\log T^{-1}_{P_0[h]} T_{P_0[f]} \left(\exp\left[r \exp L(r)\right]^{\rho_p^{L^*}(g)}\right)}$$
$$\leq \frac{O(1) + \left(\lambda_p^{L^*}(f,h) + \varepsilon\right) \cdot \left[\left(\sigma_p^{L^*}(g) + \varepsilon\right)\left[r \exp L(r)\right]^{\rho_p^{L^*}(g)} + \exp^{[p-1]} L\left(M_g(r)\right)\right]}{\left(\lambda_p^{L^*}(f,h) - \varepsilon\right) \cdot \left[r \exp L(r)\right]^{\rho_p^{L^*}(g)}},$$

$$\frac{\log T_h^{-1} T_{f \circ g}(r)}{\log T^{-1}_{P_0[h]} T_{P_0[f]} \left(\exp\left[r \exp L(r)\right]^{\rho_p^{L^*}(g)}\right)} \leq \frac{O(1)}{\left(\lambda_p^{L^*}(f,h) - \varepsilon\right) \cdot \left[r \exp L(r)\right]^{\rho_p^{L^*}(g)}} \tag{2}$$
$$+ \frac{\left(\lambda_p^{L^*}(f,h) + \varepsilon\right) \cdot \left[\left(\sigma_p^{L^*}(g) + \varepsilon\right) + \frac{\exp^{[p-1]} L\left(M_g(r)\right)}{\left[r \exp L(r)\right]^{\rho_p^{L^*}(g)}}\right]}{\left(\lambda_p^{L^*}(f,h) - \varepsilon\right)}.$$

As $\beta < \rho_p^{L^*}(g)$ and $\exp^{[p-1]} L\left(M_g(r)\right) = o\left(\left[r \exp L(r)\right]^{\beta}\right)$ as $r \to \infty$, we obtain that

$$\lim_{r \to \infty} \frac{\exp^{[p-1]} L\left(M_g(r)\right)}{\left[r \exp L(r)\right]^{\rho_p^{L^*}(g)}} = 0. \tag{3}$$

Since $\varepsilon\,(> 0)$ is arbitrary, it follows from (2) and (3) that

$$\underline{\lim}_{r \to \infty} \frac{\log T_h^{-1} T_{f \circ g}(r)}{\log T^{-1}_{P_0[h]} T_{P_0[f]} \left(\exp\left[r \exp L(r)\right]^{\rho_p^{L^*}(g)}\right)} \leq \sigma_p^{L^*}(g).$$

Thus the theorem is established. □

Remark 1. *In Theorem 1 the condition* $0 < \lambda_p^{L^*}(f,h) < \infty$ *can be replaced by the condition* $0 < \rho_p^{L^*}(f,h) < \infty$. *If we will replace this condition by* $0 < \lambda_p^{L^*}(f,h) \leq \rho_p^{L^*}(f,h) < \infty$, *then*

$$\overline{\lim}_{r \to \infty} \frac{\log T_h^{-1} T_{f \circ g}(r)}{\log T^{-1}_{P_0[h]} T_{P_0[f]} \left(\exp\left[r \exp L(r)\right]^{\rho_p^{L^*}(g)}\right)} \leq \frac{\rho_p^{L^*}(f,h) \cdot \sigma_p^{L^*}(g)}{\lambda_p^{L^*}(f,h)},$$

and if in addition we will replace the condition $\sigma_p^{L^*}(g) < \infty$ *by* $\overline{\sigma}_p^{L^*}(g) < \infty$ *then*

$$\underline{\lim}_{r \to \infty} \frac{\log T_h^{-1} T_{f \circ g}(r)}{\log T^{-1}_{P_0[h]} T_{P_0[f]} \left(\exp\left[r \exp L(r)\right]^{\rho_p^{L^*}(g)}\right)} \leq \frac{\rho_p^{L^*}(f,h) \cdot \overline{\sigma}_p^{L^*}(g)}{\lambda_p^{L^*}(f,h)}.$$

In the line of Theorem 1 and with the help of Lemma 4, one can easily prove the following theorem and therefore its proof is omitted.

Theorem 2. *Let the meromorphic function f and entire function h satisfy the conditions of Lemma 4. Also let g be an entire function and* $0 < \lambda_p^{L^*}(f,h) < \infty$, $\sigma_p^{L^*}(g) < \infty$, *where p is any positive integer. If h satisfy the Property (A) and* $\exp^{[p-1]} L\left(M_g(r)\right) = o\left(\left[r \exp L(r)\right]^{\beta}\right)$ *as $r \to \infty$ and for some positive* $\beta < \rho_p^{L^*}(g)$, *then*

$$\underline{\lim}_{r \to \infty} \frac{\log T_h^{-1} T_{f \circ g}(r)}{\log T^{-1}_{M[h]} T_{M[f]} \left(\exp\left[r \exp L(r)\right]^{\rho_p^{L^*}(g)}\right)} \leq \sigma_p^{L^*}(g).$$

Remark 2. *In Theorem 2 the condition* $0 < \lambda_p^{L^*}(f,h) < \infty$ *can be replaced by the condition* $0 < \rho_p^{L^*}(f,h) < \infty$. *If we will replace this condition by* $0 < \lambda_p^{L^*}(f,h) \leq \rho_p^{L^*}(f,h) < \infty$, *then*

$$\overline{\lim_{r\to\infty}} \frac{\log T_h^{-1} T_{f\circ g}(r)}{\log T_{M[h]}^{-1} T_{M[f]}\left(\exp\left[r\exp L(r)\right]^{\rho_p^{L^*}(g)}\right)} \leq \frac{\rho_p^{L^*}(f,h)\cdot \sigma_p^{L^*}(g)}{\lambda_p^{L^*}(f,h)},$$

and if in addition we will replace the condition $\sigma_p^{L^*}(g) < \infty$ *by* $\overline{\sigma}_p^{L^*}(g) < \infty$ *then*

$$\underline{\lim_{r\to\infty}} \frac{\log T_h^{-1} T_{f\circ g}(r)}{\log T_{M[h]}^{-1} T_{M[f]}\left(\exp\left[r\exp L(r)\right]^{\rho_p^{L^*}(g)}\right)} \leq \frac{\rho_p^{L^*}(f,h)\cdot \overline{\sigma}_p^{L^*}(g)}{\lambda_p^{L^*}(f,h)}.$$

Now we state the following theorem without proof as it can be carried out in the line of Theorem 1.

Theorem 3. *Let* g *be an entire function either of finite order or of non-zero lower order such that* $\Theta(\infty;g) = \sum_{a\neq\infty} \delta_p(a;g) = 1$ *or* $\delta(\infty;g) = \sum_{a\neq\infty} \delta(a;g) = 1$ *and* k *be an entire function with regular growth having non zero finite order and* $\Theta(\infty;k) = \sum_{a\neq\infty} \delta_p(a;k) = 1$ *or* $\delta(\infty;k) = \sum_{a\neq\infty} \delta(a;k) = 1$. *Also let* f *be a meromorphic function and* h *be an entire function such that* $\lambda_p^{L^*}(f,h) < \infty$, $\lambda_p^{L^*}(g,k) > 0$ *and* $\sigma_p^{L^*}(g) < \infty$, *where* p *is any positive integer. If* h *satisfy the Property (A) and* $\exp^{[p-1]} L(M_g(r)) = o\left(\left[r\exp L(r)\right]^{\beta}\right)$ *as* $r\to\infty$ *and for some positive* $\beta < \rho_p^{L^*}(g)$, *then*

$$\underline{\lim_{r\to\infty}} \frac{\log T_h^{-1} T_{f\circ g}(r)}{\log T_{P_0[k]}^{-1} T_{P_0[g]}\left(\exp\left[r\exp L(r)\right]^{\rho_p^{L^*}(g)}\right)} \leq \frac{\lambda_p^{L^*}(f,h)\cdot \sigma_p^{L^*}(g)}{\lambda_p^{L^*}(g,k)}. \quad (4)$$

Remark 3. *In Theorem 3, if we will replace the conditions* $\lambda_p^{L^*}(f,h) < \infty$ *and* $\lambda_p^{L^*}(g,k) > 0$ *by* $\rho_p^{L^*}(f,h) < \infty$ *and* $\rho_p^{L^*}(g,k) > 0$ *respectively, then is need to go the same replacement in right part of* (4). *Also if we will replace only the condition* $\lambda_p^{L^*}(f,h) < \infty$ *by* $\rho_p^{L^*}(f,h) < \infty$ *in Theorem 3, then*

$$\overline{\lim_{r\to\infty}} \frac{\log T_h^{-1} T_{f\circ g}(r)}{\log T_{P_0[k]}^{-1} T_{P_0[g]}\left(\exp\left[r\exp L(r)\right]^{\rho_p^{L^*}(g)}\right)} \leq \frac{\rho_p^{L^*}(f,h)\cdot \sigma_p^{L^*}(g)}{\lambda_p^{L^*}(g,k)}.$$

Remark 4. *In Theorem 3, if we will replace the conditions* $\lambda_p^{L^*}(f,h) < \infty$ *and* $\sigma_p^{L^*}(g) < \infty$ *by* $\rho_p^{L^*}(f,h) < \infty$ *and* $\overline{\sigma}_p^{L^*}(g) < \infty$ *respectively, then is need to go the same replacement in right part of* (4).

In the line of Theorem 3 and with the help of Lemma 4, one can easily prove the following theorem and therefore its proof is omitted.

Theorem 4. *Let* g *be a transcendental entire function of finite order or of non-zero lower order such that* $\sum_{a\in\mathbb{C}\cup\{\infty\}} \delta_1(a;g) = 4$ *and* k *be a transcendental entire function with regular growth* *and non zero finite order and* $\sum_{a \in \mathbb{C} \cup \{\infty\}} \delta_1(a;k) = 4$*. Also let* f *be a meromorphic function and* h *be an entire function such that* $\lambda_p^{L^*}(f,h) < \infty$, $\lambda_p^{L^*}(g,k) > 0$ *and* $\sigma_p^{L^*}(g) < \infty$*, where* p *is any positive integer. If* h *satisfy the Property (A) and* $\exp^{[p-1]} L(M_g(r)) = o\left([r \exp L(r)]^{\beta}\right)$ *as* $r \to \infty$ *and for some positive* $\beta < \rho_p^{L^*}(g)$*, then*

$$\underline{\lim}_{r\to\infty} \frac{\log T_h^{-1} T_{f\circ g}(r)}{\log T_{M[k]}^{-1} T_{M[g]}\left(\exp [r \exp L(r)]^{\rho_p^{L^*}(g)}\right)} \leq \frac{\lambda_p^{L^*}(f,h) \cdot \sigma_p^{L^*}(g)}{\lambda_p^{L^*}(g,k)}. \tag{5}$$

Remark 5. *In Theorem 4, if we will replace the conditions* $\lambda_p^{L^*}(f,h) < \infty$ *and* $\lambda_p^{L^*}(g,k) > 0$ *by* $\rho_p^{L^*}(f,h) < \infty$ *and* $\rho_p^{L^*}(g,k) > 0$ *respectively, then is need to go the same replacement in right part of* (5)*. Also if we will replace only the condition* $\lambda_p^{L^*}(f,h) < \infty$ *by* $\rho_p^{L^*}(f,h) < \infty$ *in Theorem 4, then*

$$\overline{\lim}_{r\to\infty} \frac{\log T_h^{-1} T_{f\circ g}(r)}{\log T_{M[k]}^{-1} T_{M[g]}\left(\exp [r \exp L(r)]^{\rho_p^{L^*}(g)}\right)} \leq \frac{\rho_p^{L^*}(f,h) \cdot \sigma_p^{L^*}(g)}{\lambda_p^{L^*}(g,k)}.$$

Remark 6. *In Theorem 4, if we will replace the conditions* $\lambda_p^{L^*}(f,h) < \infty$ *and* $\sigma_p^{L^*}(g) < \infty$ *by* $\rho_p^{L^*}(f,h) < \infty$ *and* $\overline{\sigma}_p^{L^*}(g) < \infty$ *respectively, then is need to go the same replacement in right part of* (5)*.*

Further we state the following two theorems which are based on ${}_pL^*$-weak type.

Theorem 5. *Let the meromorphic function* f *and entire function* h *satisfy the conditions of Lemma 3. Let* g *be an entire function and* $0 < \lambda_p^{L^*}(f,h) \leq \rho_p^{L^*}(f,h) < \infty$, $\tau_p^{L^*}(g) < \infty$*, where* p *is any positive integer. If* h *satisfy the Property (A) and* $\exp^{[p-1]} L(M_g(r)) = o\left([r \exp L(r)]^{\beta}\right)$ *as* $r \to \infty$ *and for some positive* $\beta < \lambda_p^{L^*}(g)$*, then*

$$\underline{\lim}_{r\to\infty} \frac{\log T_h^{-1} T_{f\circ g}(r)}{\log T_{P_0[h]}^{-1} T_{P_0[f]}\left(\exp [r \exp L(r)]^{\lambda_p^{L^*}(g)}\right)} \leq \frac{\rho_p^{L^*}(f,h) \cdot \tau_p^{L^*}(g)}{\lambda_p^{L^*}(f,h)}.$$

Theorem 6. *Let* g *be an entire function either of finite order or of non-zero lower order such that* $\Theta(\infty;g) = \sum_{a\neq\infty} \delta_p(a;g) = 1$ *or* $\delta(\infty;g) = \sum_{a\neq\infty} \delta(a;g) = 1$ *and* k *be an entire function with regular growth having non zero finite order and* $\Theta(\infty;k) = \sum_{a\neq\infty} \delta_p(a;k) = 1$ *or* $\delta(\infty;k) = \sum_{a\neq\infty} \delta(a;k) = 1$*. Also let* f *be a meromorphic function and* h *be an entire function such that* $\rho_p^{L^*}(f,h) < \infty$, $\lambda_p^{L^*}(g,k) > 0$ *and* $\tau_p^{L^*}(g) < \infty$*, where* p *is any positive integer. If* h *satisfy the Property (A) and* $\exp^{[p-1]} L(M_g(r)) = o\left([r \exp L(r)]^{\beta}\right)$ *as* $r \to \infty$ *and for some positive* $\beta < \lambda_p^{L^*}(g)$*, then*

$$\underline{\lim}_{r\to\infty} \frac{\log T_h^{-1} T_{f\circ g}(r)}{\log T_{P_0[k]}^{-1} T_{P_0[g]}\left(\exp [r \exp L(r)]^{\lambda_p^{L^*}(g)}\right)} \leq \frac{\rho_p^{L^*}(f,h) \cdot \tau_p^{L^*}(g)}{\lambda_p^{L^*}(g,k)}.$$

The proofs of the above two theorems can be carried out in the line of Theorem 1 and Theorem 3 respectively and therefore their proofs are omitted.

In the line of Theorem 5 and Theorem 6 respectively and with the help of Lemma 4, one can easily prove the following two theorems and therefore their proofs are omitted.

Theorem 7. *Let meromorphic function f and entire function h satisfy the conditions of Lemma 4. Also let g be an entire function and $0 < \lambda_p^{L^*}(f,h) \leq \rho_p^{L^*}(f,h) < \infty$, $\tau_p^{L^*}(g) < \infty$, where p is any positive integer. If h satisfy the Property (A) and $\exp^{[p-1]} L\left(M_g(r)\right) = o\left([r \exp L(r)]^{\beta}\right)$ as $r \to \infty$ and for some positive $\beta < \lambda_p^{L^*}(g)$, then*

$$\underline{\lim}_{r\to\infty} \frac{\log T_h^{-1} T_{f\circ g}(r)}{\log T_{M[h]}^{-1} T_{M[f]}\left(\exp[r \exp L(r)]^{\lambda_p^{L^*}(g)}\right)} \leq \frac{\rho_p^{L^*}(f,h) \cdot \tau_p^{L^*}(g)}{\lambda_p^{L^*}(f,h)}.$$

Theorem 8. *Let g be a transcendental entire function of finite order or of non-zero lower order such that $\sum_{a\in\mathbb{C}\cup\{\infty\}} \delta_1(a;g) = 4$ and k be a transcendental entire function with regular growth and non zero finite order and $\sum_{a\in\mathbb{C}\cup\{\infty\}} \delta_1(a;k) = 4$. Also let f be a meromorphic function and h be an entire function such that $\rho_p^{L^*}(f,h) < \infty$, $\lambda_p^{L^*}(g,k) > 0$ and $\tau_p^{L^*}(g) < \infty$, where p is any positive integer. If h satisfy the Property (A) and $\exp^{[p-1]} L\left(M_g(r)\right) = o\left([r \exp L(r)]^{\beta}\right)$ as $r \to \infty$ and for some positive $\beta < \lambda_p^{L^*}(g)$, then*

$$\underline{\lim}_{r\to\infty} \frac{\log T_h^{-1} T_{f\circ g}(r)}{\log T_{M[k]}^{-1} T_{M[g]}\left(\exp[r \exp L(r)]^{\lambda_p^{L^*}(g)}\right)} \leq \frac{\rho_p^{L^*}(f,h) \cdot \tau_p^{L^*}(g)}{\lambda_p^{L^*}(g,k)}.$$

Using the concept of the growth indicator $\overline{\tau}_p^{L^*}(g)$ of an entire function g, we may state the subsequent two theorems without their proofs since those can be carried out in the line of Theorem 1 and Theorem 3 respectively.

Theorem 9. *Let the meromorphic function f and entire function h satisfy the conditions of Lemma 3. Also let g be an entire function and $0 < \lambda_p^{L^*}(f,h) < \infty$, $\overline{\tau}_p^{L^*}(g) < \infty$, where p is any positive integer. If h satisfy the Property (A) and $\exp^{[p-1]} L\left(M_g(r)\right) = o\left([r \exp L(r)]^{\beta}\right)$ as $r \to \infty$ and for some positive $\beta < \lambda_p^{L^*}(g)$, then*

$$\underline{\lim}_{r\to\infty} \frac{\log T_h^{-1} T_{f\circ g}(r)}{\log T_{P_0[h]}^{-1} T_{P_0[f]}\left(\exp[r \exp L(r)]^{\lambda_p^{L^*}(g)}\right)} \leq \overline{\tau}_p^{L^*}(g).$$

Remark 7. *In Theorem 9 the condition $0 < \lambda_p^{L^*}(f,h) < \infty$ can be replaced by the condition $0 < \rho_p^{L^*}(f,h) < \infty$. If we will replace this condition by $0 < \lambda_p^{L^*}(f,h) \leq \rho_p^{L^*}(f,h) < \infty$, then*

$$\overline{\lim}_{r\to\infty} \frac{\log T_h^{-1} T_{f\circ g}(r)}{\log T_{P_0[h]}^{-1} T_{P_0[f]}\left(\exp[r \exp L(r)]^{\lambda_p^{L^*}(g)}\right)} \leq \frac{\rho_p^{L^*}(f,h) \cdot \overline{\tau}_p^{L^*}(g)}{\lambda_p^{L^*}(f,h)}.$$

Theorem 10. *Let entire functions g and k satisfy the conditions of Theorem 3. Let f be a meromorphic function and h be an entire function such that $\lambda_p^{L^*}(f,h) < \infty$, $\lambda_p^{L^*}(g,k) > 0$*

and $\overline{\tau}_p^{L^*}(g) < \infty$, *where* p *is any positive integer. If* h *satisfy the Property (A) and* $\exp^{[p-1]} L\left(M_g(r)\right) = o\left([r \exp L(r)]^{\beta}\right)$ *as* $r \to \infty$ *and for some positive* $\beta < \lambda_p^{L^*}(g)$, *then*

$$\underline{\lim}_{r\to\infty} \frac{\log T_h^{-1} T_{f\circ g}(r)}{\log T_{P_0[k]}^{-1} T_{P_0[g]}\left(\exp\left[r \exp L(r)\right]^{\lambda_p^{L^*}(g)}\right)} \leq \frac{\lambda_p^{L^*}(f,h) \cdot \overline{\tau}_p^{L^*}(g)}{\lambda_p^{L^*}(g,k)}. \tag{6}$$

Remark 8. *In Theorem 10, if we will replace the condition* $\lambda_p^{L^*}(f,h) < \infty$ *by* $\rho_p^{L^*}(f,h) < \infty$, *then is need to go the same replacement in right part of* (6).

Remark 9. *In Theorem 10, if we will replace the conditions* $\lambda_p^{L^*}(f,h) < \infty$ *and* $\lambda_p^{L^*}(g,k) > 0$ *by* $\rho_p^{L^*}(f,h) < \infty$ *and* $\rho_p^{L^*}(g,k) > 0$ *respectively, then is need to go the same replacement in right part of* (6).

In the line of Theorem 9 and Theorem 10 respectively, one can easily prove the following six theorems and therefore their proofs are omitted.

Theorem 11. *Let the meromorphic function* f *and entire function* h *satisfy the conditions of Lemma 4. Also let* g *be an entire function and* $0 < \lambda_p^{L^*}(f,h) < \infty$, $\overline{\tau}_p^{L^*}(g) < \infty$, *where* p *is any positive integer. If* h *satisfy the Property (A) and* $\exp^{[p-1]} L\left(M_g(r)\right) = o\left([r \exp L(r)]^{\beta}\right)$ *as* $r \to \infty$ *and for some positive* $\beta < \lambda_p^{L^*}(g)$, *then*

$$\underline{\lim}_{r\to\infty} \frac{\log T_h^{-1} T_{f\circ g}(r)}{\log T_{M[h]}^{-1} T_{M[f]}\left(\exp\left[r \exp L(r)\right]^{\lambda_p^{L^*}(g)}\right)} \leq \overline{\tau}_p^{L^*}(g).$$

Remark 10. *In Theorem 11 the condition* $0 < \lambda_p^{L^*}(f,h) < \infty$ *can be replaced by the condition* $0 < \rho_p^{L^*}(f,h) < \infty$. *If we will replace this condition by* $0 < \lambda_p^{L^*}(f,h) \leq \rho_p^{L^*}(f,h) < \infty$, *then*

$$\overline{\lim}_{r\to\infty} \frac{\log T_h^{-1} T_{f\circ g}(r)}{\log T_{M[h]}^{-1} T_{M[f]}\left(\exp\left[r \exp L(r)\right]^{\lambda_p^{L^*}(g)}\right)} \leq \frac{\rho_p^{L^*}(f,h) \cdot \overline{\tau}_p^{L^*}(g)}{\lambda_p^{L^*}(f,h)}.$$

Theorem 12. *Let the entire functions* g *and* k *satisfy the conditions of Theorem 4. Let* f *be a meromorphic function and* h *be an entire function such that* $\lambda_p^{L^*}(f,h) < \infty$, $\lambda_p^{L^*}(g,k) > 0$ *and* $\overline{\tau}_p^{L^*}(g) < \infty$, *where* p *is any positive integer. If* h *satisfy the Property (A) and* $\exp^{[p-1]} L\left(M_g(r)\right) = o\left([r \exp L(r)]^{\beta}\right)$ *as* $r \to \infty$ *and for some positive* $\beta < \lambda_p^{L^*}(g)$, *then*

$$\underline{\lim}_{r\to\infty} \frac{\log T_h^{-1} T_{f\circ g}(r)}{\log T_{M[k]}^{-1} T_{M[g]}\left(\exp\left[r \exp L(r)\right]^{\lambda_p^{L^*}(g)}\right)} \leq \frac{\lambda_p^{L^*}(f,h) \cdot \overline{\tau}_p^{L^*}(g)}{\lambda_p^{L^*}(g,k)}. \tag{7}$$

Remark 11. *In Theorem 12, if we will replace the condition* $\lambda_p^{L^*}(f,h) < \infty$ *by* $\rho_p^{L^*}(f,h) < \infty$, *then is need to go the same replacement in right part of* (7).

Remark 12. *In Theorem 12, if we will replace the conditions* $\lambda_p^{L^*}(f,h) < \infty$ *and* $\lambda_p^{L^*}(g,k) > 0$ *by* $\rho_p^{L^*}(f,h) < \infty$ *and* $\rho_p^{L^*}(g,k) > 0$ *respectively, then is need to go the same replacement in right part of* (7).

Theorem 13. *Let the meromorphic function f and entire function h satisfy the conditions of Lemma 5. Also let g be an entire function and $0 < \rho_p^{L^*}(f,h) < \rho_g$, $\sigma_p^{L^*}(f,h) > 0$, where p is any positive integer. If $\exp^{[p-1]} L\left(\exp\left(re^{L(r)}\right)^{\beta}\right) = o\left(\left[r\exp^{[p]} L(r)\right]^{\beta}\right)$ $(r \to \infty)$ for any $\beta > 0$, then*

$$\varliminf_{r\to\infty} \frac{\log T_h^{-1} T_{f\circ g}\left(re^{L(r)}\right)}{T_{P_0[h]}^{-1} T_{P_0[f]}(r)} \geq \frac{\lambda_p^{L^*}(f,h)}{\left(\frac{\gamma_{P_0[f]}}{\gamma_{P_0[h]}}\right)^{\frac{1}{\rho_h}} \cdot \sigma_p^{L^*}(f,h)}.$$

Proof. From the definition of relative ${}_pL^*$- type of meromorphic function and in view of Lemma 5, we obtain for all sufficiently large values of r that

$$T_{P_0[h]}^{-1} T_{P_0[f]}(r) \leq \left(\sigma_p^{L^*}(P_0[f], P_0[h]) + \varepsilon\right)\left[r\exp^{[p]} L(r)\right]^{\rho_p^{L^*}(P_0[f],P_0[h])},$$

$$\text{i.e.} \quad T_{P_0[h]}^{-1} T_{P_0[f]}(r) \leq \left(\left(\frac{\gamma_{P_0[f]}}{\gamma_{P_0[h]}}\right)^{\frac{1}{\rho_h}} \cdot \sigma_p^{L^*}(f,h) + \varepsilon\right)\left[r\exp^{[p]} L(r)\right]^{\rho_p^{L^*}(f,h)}. \tag{8}$$

As $0 < \rho_p^{L^*}(f,h) < \rho_g$, we obtain in view of Lemma 10 for a sequence of values of r tending to infinity that

$$\log T_h^{-1} T_{f\circ g}\left(re^{L(r)}\right) \geq \log T_h^{-1} T_f\left(\exp\left(re^{L(r)}\right)^{\rho_p^{L^*}(f,h)}\right), \text{ i.e.}$$

$$\log T_h^{-1} T_{f\circ g}\left(re^{L(r)}\right) \geq \left(\lambda_p^{L^*}(f,h) - \varepsilon\right)\left[\left[re^{L(r)}\right]^{\rho_p^{L^*}(f,h)} + \exp^{[p-1]} L\left(\exp\left(re^{L(r)}\right)^{\rho_p^{L^*}(f,h)}\right)\right].$$

Therefore from (8) and above, it follows for a sequence of values of r tending to infinity that

$$\frac{\log T_h^{-1} T_{f\circ g}\left(re^{L(r)}\right)}{T_{P_0[h]}^{-1} T_{P_0[f]}(r)} \geq \frac{\left(\lambda_p^{L^*}(f,h) - \varepsilon\right)\left[\left[re^{L(r)}\right]^{\rho_p^{L^*}(f,h)} + \exp^{[p-1]} L\left(\exp\left(re^{L(r)}\right)^{\rho_p^{L^*}(f,h)}\right)\right]}{\left(\left(\frac{\gamma_{P_0[f]}}{\gamma_{P_0[h]}}\right)^{\frac{1}{\rho_h}} \cdot \sigma_p^{L^*}(f,h) + \varepsilon\right)\left[r\exp^{[p]} L(r)\right]^{\rho_p^{L^*}(f,h)}}.$$

Since $\lim\limits_{r\to\infty} \frac{\exp^{[p-1]} L\left(\exp\left(re^{L(r)}\right)^{\rho_p^{L^*}(f,h)}\right)}{\left[r\exp^{[p]} L(r)\right]^{\rho_p^{L^*}(f,h)}} = 0$ as $\exp^{[p-1]} L\left(\exp\left(re^{L(r)}\right)^{\beta}\right) = o\left(\left[r\exp^{[p]} L(r)\right]^{\beta}\right)$ $(r \to \infty)$ for any $\alpha > 0$, we obtain from above that

$$\varliminf_{r\to\infty} \frac{\log T_h^{-1} T_{f\circ g}\left(re^{L(r)}\right)}{T_{P_0[h]}^{-1} T_{P_0[f]}(r)} \geq \frac{\lambda_p^{L^*}(f,h)}{\left(\frac{\gamma_{P_0[f]}}{\gamma_{P_0[h]}}\right)^{\frac{1}{\rho_h}} \cdot \sigma_p^{L^*}(f,h)}.$$

Thus the theorem follows. □

Remark 13. *If we take $\overline{\tau}_p^{L^*}(f,h) > 0$ instead of $\sigma_p^{L^*}(f,h) > 0$ and the other conditions remain the same, then with the help of Lemma 6, one can easily verify that the conclusion of Theorem 13 remains valid with $\sigma_p^{L^*}(f,h)$ replaced by $\overline{\tau}_p^{L^*}(f,h)$.*

In the line of Theorem 13 and in view of Lemma 7, one can easily prove the following theorem and therefore its proofs is omitted.

Theorem 14. *Let the meromorphic function f and entire function h satisfy the conditions of Lemma 7. Also let g be an entire function and $0 < \rho_p^{L^*}(f,h) < \rho_g$, $\sigma_p^{L^*}(f,h) > 0$, where p is any positive integer. If $\exp^{[p-1]} L\left(\exp\left(re^{L(r)}\right)^{\beta}\right) = o\left(\left[r\exp^{[p]} L(r)\right]^{\beta}\right)$ $(r \to \infty)$ for any $\beta > 0$, then*

$$\varliminf_{r\to\infty} \frac{\log T_h^{-1} T_{f\circ g}\left(re^{L(r)}\right)}{T_{M[h]}^{-1} T_{M[f]}(r)} \geq \frac{\lambda_p^{L^*}(f,h)}{\left(\frac{\Gamma_{M[f]}-(\Gamma_{M[f]}-\gamma_{M[f]})\Theta(\infty;f)}{\Gamma_{M[h]}-(\Gamma_{M[h]}-\gamma_{M[h]})\Theta(\infty;h)}\right)^{\frac{1}{\rho_h}} \cdot \sigma_p^{L^*}(f,h)}.$$

Remark 14. *If we take $\overline{\tau}_p^{L^*}(f,h) > 0$ instead of $\sigma_p^{L^*}(f,h) > 0$ and the other conditions remain the same, then with the help of Lemma 8, one can easily verify that the conclusion of Theorem 14 remains valid with $\sigma_p^{L^*}(f,h)$ replaced by $\overline{\tau}_p^{L^*}(f,h)$.*

Theorem 15. *Let the meromorphic function f and entire function h satisfy the conditions of Lemma 5. Also let g be an entire function, h satisfy the Property (A), $\rho_p^{L^*}(f,h) = \rho_p^{L^*}(g)$, $\sigma_p^{L^*}(g) < \infty$ and $\overline{\sigma}_p^{L^*}(f,h) > 0$, where p is any positive integer.*

(a) If $\exp^{[p-1]} L(M_g(r)) = o\left\{T_{P_0[h]}^{-1} T_{P_0[f]}(r)\right\}$ then

$$\varlimsup_{r\to\infty} \frac{\log T_h^{-1} T_{f\circ g}(r)}{T_{P_0[h]}^{-1} T_{P_0[f]}(r) + \exp^{[p-1]} L(M_g(r))} \leq \frac{\rho_p^{L^*}(f,h)\cdot\sigma_p^{L^*}(g)}{\left(\frac{\gamma_{P_0[f]}}{\gamma_{P_0[h]}}\right)^{\frac{1}{\rho_h}} \cdot \overline{\sigma}_p^{L^*}(f,h)}.$$

(b) If $T_{P_0[h]}^{-1} T_{P_0[f]}(r) = o\left\{\exp^{[p-1]} L(M_g(r))\right\}$ then

$$\varlimsup_{r\to\infty} \frac{\log T_h^{-1} T_{f\circ g}(r)}{T_{P_0[h]}^{-1} T_{P_0[f]}(r) + \exp^{[p-1]} L(M_g(r))} \leq \rho_p^{L^*}(f,h).$$

Proof. Let us consider that $\alpha > 2$ and $\delta \to 1^+$ in Lemma 14. Since $T_h^{-1}(r)$ is an increasing function of r, it follows from Lemma 9, Lemma 14 and the inequality $T_g(r) \leq \log M_g(r)$ (cf. [13]) for all sufficiently large values of r that

$$\begin{aligned}
&T_h^{-1} T_{f\circ g}(r) \leqslant T_h^{-1}\left[\{1+o(1)\}\, T_f(M_g(r))\right],\\
&\text{i.e.}\quad T_h^{-1} T_{f\circ g}(r) \leqslant \alpha\left[T_h^{-1} T_f(M_g(r))\right],\\
&\text{i.e.}\quad \log T_h^{-1} T_{f\circ g}(r) \leqslant \log T_h^{-1} T_f(M_g(r)) + O(1),\\
&\text{i.e.}\quad \log T_h^{-1} T_{f\circ g}(r) \leqslant \left(\rho_p^{L^*}(f,h)+\varepsilon\right)\left(\log M_g(r) + \exp^{[p-1]} L(M_g(r))\right) + O(1),\\
&\text{i.e.}\quad \log T_h^{-1} T_{f\circ g}(r) \leqslant \left(\rho_p^{L^*}(f,h)+\varepsilon\right)\left(\sigma_p^{L^*}(g)+\varepsilon\right)\left[r\exp^{[p]} L(r)\right]^{\rho_p^{L^*}(g)}\\
&\qquad\qquad + \left(\rho_p^{L^*}(f,h)+\varepsilon\right)\exp^{[p-1]} L(M_g(r)) + O(1).
\end{aligned}$$

In view of condition (ii) we obtain from above for all sufficiently large values of r that

$$\begin{aligned}
\log T_h^{-1} T_{f\circ g}(r) \leqslant{}& \left(\rho_p^{L^*}(f,h)+\varepsilon\right)\left(\sigma_p^{L^*}(g)+\varepsilon\right)\left[r\exp^{[p]} L(r)\right]^{\rho_p^{L^*}(f,h)}\\
&+ \left(\rho_p^{L^*}(f,h)+\varepsilon\right)\exp^{[p-1]} L(M_g(r)) + O(1).
\end{aligned} \tag{9}$$

Again from the definition of relative ${}_pL^*$-lower type we get in view of Lemma 5, for all sufficiently large values of r that

$$
\begin{aligned}
&T^{-1}_{P_0[h]}T_{P_0[f]}(r) \geq \left(\overline{\sigma}^{L^*}_p(P_0[f], P_0[h]) - \varepsilon\right)\left[r\exp^{[p]}L(r)\right]^{\rho^{L^*}_p(P_0[f],P_0[h])}, \text{ i.e.}\\
&T^{-1}_{P_0[h]}T_{P_0[f]}(r) \geq \left(\left(\frac{\gamma_{P_0[f]}}{\gamma_{P_0[h]}}\right)^{\frac{1}{\rho_h}} \cdot \overline{\sigma}^{L^*}_p(f,h) - \varepsilon\right)\left[r\exp^{[p]}L(r)\right]^{\rho^{L^*}_p(f,h)}, \text{ i.e.}\\
&\left[r\exp^{[p]}L(r)\right]^{\rho^{L^*}_p(f,h)} \leq \frac{T^{-1}_{P_0[h]}T_{P_0[f]}(r)}{\left(\left(\frac{\gamma_{P_0[f]}}{\gamma_{P_0[h]}}\right)^{\frac{1}{\rho_h}} \cdot \overline{\sigma}^{L^*}_p(f,h) - \varepsilon\right)}.
\end{aligned}
\tag{10}
$$

Now from (9) and (10), it follows for all sufficiently large values of r that

$$
\begin{aligned}
\log T^{-1}_h T_{f\circ g}(r) \leqslant{}& \left(\rho^{L^*}_p(f,h) + \varepsilon\right)\left(\sigma^{L^*}_p(g) + \varepsilon\right)\frac{T^{-1}_{P_0[h]}T_{P_0[f]}(r)}{\left(\left(\frac{\gamma_{P_0[f]}}{\gamma_{P_0[h]}}\right)^{\frac{1}{\rho_h}} \cdot \overline{\sigma}^{L^*}_p(f,h) - \varepsilon\right)}\\
&+ \left(\rho^{L^*}_p(f,h) + \varepsilon\right)\exp^{[p-1]}L\left(M_g(r)\right) + O(1),
\end{aligned}
$$

$$
\begin{aligned}
\text{i.e. } \frac{\log T^{-1}_h T_{f\circ g}(r)}{T^{-1}_{P_0[h]}T_{P_0[f]}(r) + \exp^{[p-1]}L\left(M_g(r)\right)} \leq{}& \frac{O(1)}{T^{-1}_{P_0[h]}T_{P_0[f]}(r) + \exp^{[p-1]}L\left(M_g(r)\right)}\\
&+ \frac{\frac{\left(\rho^{L^*}_p(f,h)+\varepsilon\right)\left(\sigma^{L^*}_p(g)+\varepsilon\right)}{\left(\left(\frac{\gamma_{P_0[f]}}{\gamma_{P_0[h]}}\right)^{\frac{1}{\rho_h}} \cdot \overline{\sigma}^{L^*}_p(f,h)-\varepsilon\right)}}{1 + \frac{\exp^{[p-1]}L\left(M_g(r)\right)}{T^{-1}_{P_0[h]}T_{P_0[f]}(r)}} + \frac{\left(\rho^{L^*}_p(f,h) + \varepsilon\right)}{1 + \frac{T^{-1}_{P_0[h]}T_{P_0[f]}(r)}{\exp^{[p-1]}L\left(M_g(r)\right)}}.
\end{aligned}
\tag{11}
$$

If $\exp^{[p-1]}L\left(M_g(r)\right) = o\left\{T^{-1}_{P_0[h]}T_{P_0[f]}(r)\right\}$ then from (11) we get that

$$
\overline{\lim_{r\to\infty}} \frac{\log T^{-1}_h T_{f\circ g}(r)}{T^{-1}_{P_0[h]}T_{P_0[f]}(r) + \exp^{[p-1]}L\left(M_g(r)\right)} \leq \frac{\left(\rho^{L^*}_p(f,h) + \varepsilon\right)\left(\sigma^{L^*}_p(g) + \varepsilon\right)}{\left(\left(\frac{\gamma_{P_0[f]}}{\gamma_{P_0[h]}}\right)^{\frac{1}{\rho_h}} \cdot \overline{\sigma}^{L^*}_p(f,h) - \varepsilon\right)}.
$$

Since $\varepsilon\,(>0)$ is arbitrary, it follows from above that

$$
\overline{\lim_{r\to\infty}} \frac{\log T^{-1}_h T_{f\circ g}(r)}{T^{-1}_{P_0[h]}T_{P_0[f]}(r) + \exp^{[p-1]}L\left(M_g(r)\right)} \leq \frac{\rho^{L^*}_p(f,h)\cdot\sigma^{L^*}_p(g)}{\left(\frac{\gamma_{P_0[f]}}{\gamma_{P_0[h]}}\right)^{\frac{1}{\rho_h}} \cdot \overline{\sigma}^{L^*}_p(f,h)}.
$$

Thus the first part of the theorem follows.

Since $\varepsilon\,(>0)$ is arbitrary, and if $T^{-1}_{P_0[h]}T_{P_0[f]}(r) = o\left\{\exp^{[p-1]}L\left(M_g(r)\right)\right\}$ then from (11) it follows that

$$
\overline{\lim_{r\to\infty}} \frac{\log T^{-1}_h T_{f\circ g}(r)}{T^{-1}_{P_0[h]}T_{P_0[f]}(r) + \exp^{[p-1]}L\left(M_g(r)\right)} \leq \rho^{L^*}_p(f,h).
$$

Thus the second part of the theorem is established. □

Theorem 16. *Let the meromorphic function f and entire function h satisfy the conditions of Lemma 5. Also let g be an entire function, h satisfy the Property (A), $\rho_p^{L^*}(f,h) = \rho_p^{L^*}(g)$, $\lambda_p^{L^*}(f,h) < \infty$, $\overline{\sigma}_p^{L^*}(f,h) > 0$ and $\sigma_p^{L^*}(g) < \infty$, where p is any positive integer.*
(a) If $\exp^{[p-1]} L(M_g(r)) = o\left\{T_{P_0[h]}^{-1} T_{P_0[f]}(r)\right\}$ then

$$\underline{\lim}_{r\to\infty} \frac{\log T_h^{-1} T_{f\circ g}(r)}{T_{P_0[h]}^{-1} T_{P_0[f]}(r) + \exp^{[p-1]} L(M_g(r))} \le \frac{\lambda_p^{L^*}(f,h)\cdot \sigma_p^{L^*}(g)}{\left(\frac{\gamma_{P_0[f]}}{\gamma_{P_0[h]}}\right)^{\frac{1}{\rho_h}} \cdot \overline{\sigma}_p^{L^*}(f,h)}.$$

(b) If $T_{P_0[h]}^{-1} T_{P_0[f]}(r) = o\left\{\exp^{[p-1]} L(M_g(r))\right\}$ then

$$\underline{\lim}_{r\to\infty} \frac{\log T_h^{-1} T_{f\circ g}(r)}{T_{P_0[h]}^{-1} T_{P_0[f]}(r) + \exp^{[p-1]} L(M_g(r))} \le \lambda_p^{L^*}(f,h).$$

We omit the proof of the above theorem as it can be carried out in the line of Theorem 15.

Remark 15. *In Theorem 16, if we take $\rho_p^{L^*}(f,h) = \rho_p^{L^*}(g)$, $\sigma_p^{L^*}(f,h) > 0$ and $\sigma_p^{L^*}(g) < \infty$ instead of $\rho_p^{L^*}(f,h) = \rho_p^{L^*}(g)$, $\lambda_p^{L^*}(f,h) < \infty$, $\overline{\sigma}_p^{L^*}(f,h) > 0$ and $\sigma_p^{L^*}(g) < \infty$ and the other conditions remain the same, then one can easily verify that the conclusion of Theorem 16 remains valid with $\lambda_p^{L^*}(f,h)$ replaced by $\rho_p^{L^*}(f,h)$ and $\overline{\sigma}_p^{L^*}(f,h)$ replaced by $\sigma_p^{L^*}(f,h)$ respectively.*

Remark 16. *In Theorem 16, if we take $\rho_p^{L^*}(f,h) = \rho_p^{L^*}(g)$, $\overline{\sigma}_p^{L^*}(f,h) > 0$ and $\overline{\sigma}_p^{L^*}(g) < \infty$ instead of $\rho_p^{L^*}(f,h) = \rho_p^{L^*}(g)$, $\lambda_p^{L^*}(f,h) < \infty$, $\overline{\sigma}_p^{L^*}(f,h) > 0$ and $\sigma_p^{L^*}(g) < \infty$ and the other conditions remain the same, then one can easily verify that the conclusion of Theorem 16 remains valid with $\lambda_p^{L^*}(f,h)$ replaced by $\rho_p^{L^*}(f,h)$ and $\sigma_p^{L^*}(g)$ replaced by $\overline{\sigma}_p^{L^*}(g)$ respectively.*

Similarly using the concept of the growth indicator $\tau_p^{L^*}(f,h)$ and $\overline{\tau}_p^{L^*}(g)$ we may state the subsequent two theorems without their proofs since those can be carried out in view of Lemma 6 and in the line of Theorem 15 and Theorem 16 respectively.

Theorem 17. *Let the meromorphic function f and entire function h satisfy the conditions of Lemma 6. Also let g be an entire function, h satisfy the Property (A), $\rho_p^{L^*}(f,h) < \infty$, $\lambda_p^{L^*}(f,h) = \lambda_p^{L^*}(g)$, $\overline{\tau}_p^{L^*}(g) < \infty$ and $\tau_p^{L^*}(f,h) > 0$, where p is any positive integer.*
(a) If $\exp^{[p-1]} L(M_g(r)) = o\left\{T_{P_0[h]}^{-1} T_{P_0[f]}(r)\right\}$ then

$$\overline{\lim}_{r\to\infty} \frac{\log T_h^{-1} T_{f\circ g}(r)}{T_{P_0[h]}^{-1} T_{P_0[f]}(r) + \exp^{[p-1]} L(M_g(r))} \le \frac{\rho_p^{L^*}(f,h)\cdot \overline{\tau}_p^{L^*}(g)}{\left(\frac{\gamma_{P_0[f]}}{\gamma_{P_0[h]}}\right)^{\frac{1}{\rho_h}} \cdot \tau_p^{L^*}(f,h)}.$$

(b) If $T_{P_0[h]}^{-1} T_{P_0[f]}(r) = o\left\{\exp^{[p-1]} L(M_g(r))\right\}$ then

$$\overline{\lim}_{r\to\infty} \frac{\log T_h^{-1} T_{f\circ g}(r)}{T_{P_0[h]}^{-1} T_{P_0[f]}(r) + \exp^{[p-1]} L(M_g(r))} \le \rho_p^{L^*}(f,h).$$

Remark 17. *In Theorem 17, if we replace the condition $\lambda_p^{L^*}(f,h) = \lambda_p^{L^*}(g)$ and $\overline{\tau}_p^{L^*}(g) < \infty$ by $\lambda_p^{L^*}(f,h) = \rho_p^{L^*}(g)$ and $\sigma_p^{L^*}(g) < \infty$ and the other conditions remain the same, then one can easily verify that the conclusion of Theorem 17 remains valid with $\overline{\tau}_p^{L^*}(g)$ replaced by $\sigma_p^{L^*}(g)$.*

Remark 18. *In Theorem 17, if we take* $\rho_p^{L^*}(f,h) = \lambda_p^{L^*}(g)$, $\overline{\tau}_p^{L^*}(g) < \infty$ *and* $\overline{\sigma}_p^{L^*}(f,h) > 0$ *instead of* $\rho_p^{L^*}(f,h) < \infty$, $\lambda_p^{L^*}(f,h) = \lambda_p^{L^*}(g)$, $\overline{\tau}_p^{L^*}(g) < \infty$ *and* $\tau_p^{L^*}(f,h) > 0$ *and the other conditions remain the same, then one can easily verify that the conclusion of Theorem 17 remains valid with* $\tau_p^{L^*}(f,h)$ *replaced by* $\overline{\sigma}_p^{L^*}(f,h)$.

Theorem 18. *Let the meromorphic function* f *and entire function* h *satisfy the conditions of Lemma 6. Also let* g *be an entire function,* h *satisfy the Property (A),* $\lambda_p^{L^*}(f,h) = \lambda_p^{L^*}(g)$, $\overline{\tau}_p^{L^*}(g) < \infty$ *and* $\tau_p^{L^*}(f,h) > 0$, *where* p *is any positive integer.*
(a) *If* $\exp^{[p-1]} L(M_g(r)) = o\left\{T_{P_0[h]}^{-1} T_{P_0[f]}(r)\right\}$ *then*

$$\varliminf_{r\to\infty} \frac{\log T_h^{-1} T_{f\circ g}(r)}{T_{P_0[h]}^{-1} T_{P_0[f]}(r) + \exp^{[p-1]} L(M_g(r))} \leq \frac{\lambda_p^{L^*}(f,h) \cdot \overline{\tau}_p^{L^*}(g)}{\left(\frac{\gamma_{P_0[f]}}{\gamma_{P_0[h]}}\right)^{\frac{1}{\rho_h}} \cdot \tau_p^{L^*}(f,h)}.$$

(b) *If* $T_{P_0[h]}^{-1} T_{P_0[f]}(r) = o\left\{\exp^{[p-1]} L(M_g(r))\right\}$ *then*

$$\varliminf_{r\to\infty} \frac{\log T_h^{-1} T_{f\circ g}(r)}{T_{P_0[h]}^{-1} T_{P_0[f]}(r) + \exp^{[p-1]} L(M_g(r))} \leq \lambda_p^{L^*}(f,h).$$

Remark 19. *In Theorem 18, if we take* $\rho_p^{L^*}(f,h) < \infty$, $\lambda_p^{L^*}(f,h) = \lambda_p^{L^*}(g)$, $\overline{\tau}_p^{L^*}(g) < \infty$ *and* $\overline{\tau}_p^{L^*}(f,h) > 0$ *instead of* $\lambda_p^{L^*}(f,h) = \lambda_p^{L^*}(g)$, $\overline{\tau}_p^{L^*}(g) < \infty$ *and* $\tau_p^{L^*}(f,h) > 0$ *and the other conditions remain the same, then one can easily verify that the conclusion of Theorem 18 remains valid with* $\lambda_p^{L^*}(f,h)$ *replaced by* $\rho_p^{L^*}(f,h)$ *and* $\tau_p^{L^*}(f,h)$ *replaced by* $\overline{\tau}_p^{L^*}(f,h)$ *respectively.*

Remark 20. *In Theorem 18, if we take* $\rho_p^{L^*}(f,h) < \infty$, $\lambda_p^{L^*}(f,h) = \lambda_p^{L^*}(g)$, $\tau_p^{L^*}(g) < \infty$ *and* $\tau_p^{L^*}(f,h) > 0$ *instead of* $\lambda_p^{L^*}(f,h) = \lambda_p^{L^*}(g)$, $\overline{\tau}_p^{L^*}(g) < \infty$ *and* $\tau_p^{L^*}(f,h) > 0$ *and the other conditions remain the same, then one can easily verify that the conclusion of Theorem 18 remains valid with* $\lambda_p^{L^*}(f,h)$ *replaced by* $\rho_p^{L^*}(f,h)$ *and* $\overline{\tau}_p^{L^*}(g)$ *replaced by* $\tau_p^{L^*}(g)$ *respectively.*

Remark 21. *In Theorem 18, if we replace the conditions* $\lambda_p^{L^*}(f,h) = \lambda_p^{L^*}(g)$ *and* $\overline{\tau}_p^{L^*}(g) < \infty$ *by* $\lambda_p^{L^*}(f,h) = \rho_p^{L^*}(g)$ *and* $\sigma_p^{L^*}(g) < \infty$ *and the other conditions remain the same, then one can easily verify that the conclusion of Theorem 18 remains valid with* $\overline{\tau}_p^{L^*}(g)$ *replaced by* $\sigma_p^{L^*}(g)$.

Remark 22. *In Theorem 18, if we take* $\rho_p^{L^*}(f,h) = \lambda_p^{L^*}(g)$, $\overline{\tau}_p^{L^*}(g) < \infty$ *and* $\overline{\sigma}_p^{L^*}(f,h) > 0$ *instead of* $\lambda_p^{L^*}(f,h) = \lambda_p^{L^*}(g)$, $\overline{\tau}_p^{L^*}(g) < \infty$ *and* $\tau_p^{L^*}(f,h) > 0$ *and the other conditions remain the same, then one can easily verify that the conclusion of Theorem 18 remains valid with* $\tau_p^{L^*}(f,h)$ *replaced by* $\overline{\sigma}_p^{L^*}(f,h)$.

In the line of Theorem 15, Theorem 16, Theorem 17 and Theorem 18 and in view of Lemma 7 and Lemma 8, one can easily prove the following four theorems and therefore their proofs are omitted.

Theorem 19. *Let the meromorphic function* f *and entire function* h *satisfy the conditions of Lemma 7. Also let* g *be an entire function,* h *satisfy the Property (A),* $\rho_p^{L^*}(f,h) = \rho_p^{L^*}(g)$,

$\sigma_p^{L^*}(g) < \infty$ *and* $\overline{\sigma}_p^{L^*}(f,h) > 0$, *where* p *is any positive integer.*
(a) *If* $\exp^{[p-1]} L(M_g(r)) = o\left\{T_{P_0[h]}^{-1} T_{P_0[f]}(r)\right\}$ *then*

$$\overline{\lim_{r\to\infty}} \frac{\log T_h^{-1} T_{f\circ g}(r)}{T_{M[h]}^{-1} T_{M[f]}(r) + \exp^{[p-1]} L(M_g(r))} \le \frac{\rho_p^{L^*}(f,h) \cdot \sigma_p^{L^*}(g)}{\left(\frac{\Gamma_{M[f]} - (\Gamma_{M[f]} - \gamma_{M[f]})\Theta(\infty;f)}{\Gamma_{M[h]} - (\Gamma_{M[h]} - \gamma_{M[h]})\Theta(\infty;h)}\right)^{\frac{1}{\rho_h}} \cdot \overline{\sigma}_p^{L^*}(f,h)}.$$

(b) *If* $T_{P_0[h]}^{-1} T_{P_0[f]}(r) = o\left\{\exp^{[p-1]} L(M_g(r))\right\}$ *then*

$$\overline{\lim_{r\to\infty}} \frac{\log T_h^{-1} T_{f\circ g}(r)}{T_{M[h]}^{-1} T_{M[f]}(r) + \exp^{[p-1]} L(M_g(r))} \le \rho_p^{L^*}(f,h).$$

Theorem 20. *Let the meromorphic function* f *and entire function* h *satisfy the conditions of Lemma 7. Also let* g *be an entire function,* h *satisfy the Property (A),* $\lambda_p^{L^*}(f,h) < \infty$, $\rho_p^{L^*}(f,h) = \rho_p^{L^*}(g)$, $\sigma_p^{L^*}(g) < \infty$ *and* $\overline{\sigma}_p^{L^*}(f,h) > 0$, *where* p *is any positive integer.*
(a) *If* $\exp^{[p-1]} L(M_g(r)) = o\left\{T_{P_0[h]}^{-1} T_{P_0[f]}(r)\right\}$ *then*

$$\underline{\lim_{r\to\infty}} \frac{\log T_h^{-1} T_{f\circ g}(r)}{T_{M[h]}^{-1} T_{M[f]}(r) + \exp^{[p-1]} L(M_g(r))} \le \frac{\lambda_p^{L^*}(f,h) \cdot \sigma_p^{L^*}(g)}{\left(\frac{\Gamma_{M[f]} - (\Gamma_{M[f]} - \gamma_{M[f]})\Theta(\infty;f)}{\Gamma_{M[h]} - (\Gamma_{M[h]} - \gamma_{M[h]})\Theta(\infty;h)}\right)^{\frac{1}{\rho_h}} \cdot \overline{\sigma}_p^{L^*}(f,h)}.$$

(b) *If* $T_{P_0[h]}^{-1} T_{P_0[f]}(r) = o\left\{\exp^{[p-1]} L(M_g(r))\right\}$ *then*

$$\underline{\lim_{r\to\infty}} \frac{\log T_h^{-1} T_{f\circ g}(r)}{T_{M[h]}^{-1} T_{M[f]}(r) + \exp^{[p-1]} L(M_g(r))} \le \lambda_p^{L^*}(f,h).$$

Remark 23. *In Theorem 20, if we take* $\rho_p^{L^*}(f,h) = \rho_p^{L^*}(g)$, $\sigma_p^{L^*}(f,h) > 0$ *and* $\sigma_p^{L^*}(g) < \infty$ *instead of* $\rho_p^{L^*}(f,h) = \rho_p^{L^*}(g)$, $\lambda_p^{L^*}(f,h) < \infty$, $\overline{\sigma}_p^{L^*}(f,h) > 0$ *and* $\sigma_p^{L^*}(g) < \infty$ *and the other conditions remain the same, then one can easily verify that the conclusion of Theorem 20 remains valid with* $\lambda_p^{L^*}(f,h)$ *replaced by* $\rho_p^{L^*}(f,h)$ *and* $\overline{\sigma}_p^{L^*}(f,h)$ *replaced by* $\sigma_p^{L^*}(f,h)$ *respectively.*

Remark 24. *In Theorem 20, if we take* $\rho_p^{L^*}(f,h) = \rho_p^{L^*}(g)$, $\overline{\sigma}_p^{L^*}(f,h) > 0$ *and* $\overline{\sigma}_p^{L^*}(g) < \infty$ *instead of* $\rho_p^{L^*}(f,h) = \rho_p^{L^*}(g)$, $\lambda_p^{L^*}(f,h) < \infty$, $\overline{\sigma}_p^{L^*}(f,h) > 0$ *and* $\sigma_p^{L^*}(g) < \infty$ *and the other conditions remain the same, then one can easily verify that the conclusion of Theorem 20 remains valid with* $\lambda_p^{L^*}(f,h)$ *replaced by* $\rho_p^{L^*}(f,h)$ *and* $\sigma_p^{L^*}(g)$ *replaced by* $\overline{\sigma}_p^{L^*}(g)$ *respectively.*

Theorem 21. *Let the meromorphic function* f *and entire function* h *satisfy the conditions of Lemma 8. Also let* g *be an entire function,* h *satisfy the Property (A),* $\rho_p^{L^*}(f,h) < \infty$, $\lambda_p^{L^*}(f,h) = \lambda_p^{L^*}(g)$, $\overline{\tau}_p^{L^*}(g) < \infty$ *and* $\tau_p^{L^*}(f,h) > 0$, *where* p *is any positive integer.*
(a) *If* $\exp^{[p-1]} L(M_g(r)) = o\left\{T_{P_0[h]}^{-1} T_{P_0[f]}(r)\right\}$ *then*

$$\overline{\lim_{r\to\infty}} \frac{\log T_h^{-1} T_{f\circ g}(r)}{T_{M[h]}^{-1} T_{M[f]}(r) + \exp^{[p-1]} L(M_g(r))} \le \frac{\rho_p^{L^*}(f,h) \cdot \overline{\tau}_p^{L^*}(g)}{\left(\frac{\Gamma_{M[f]} - (\Gamma_{M[f]} - \gamma_{M[f]})\Theta(\infty;f)}{\Gamma_{M[h]} - (\Gamma_{M[h]} - \gamma_{M[h]})\Theta(\infty;h)}\right)^{\frac{1}{\rho_h}} \cdot \tau_p^{L^*}(f,h)}.$$

(b) *If* $T^{-1}_{P_0[h]}T_{P_0[f]}(r) = o\left\{\exp^{[p-1]} L\left(M_g(r)\right)\right\}$ *then*

$$\overline{\lim_{r\to\infty}} \frac{\log T_h^{-1} T_{f\circ g}(r)}{T^{-1}_{M[h]}T_{M[f]}(r) + \exp^{[p-1]} L\left(M_g(r)\right)} \leq \rho_p^{L^*}(f,h).$$

Remark 25. *In Theorem 21, if we replace the condition* $\lambda_p^{L^*}(f,h) = \lambda_p^{L^*}(g)$ *and* $\overline{\tau}_p^{L^*}(g) < \infty$ *by* $\lambda_p^{L^*}(f,h) = \rho_p^{L^*}(g)$ *and* $\sigma_p^{L^*}(g) < \infty$ *and the other conditions remain the same, then one can easily verify that the conclusion of Theorem 21 remains valid with* $\overline{\tau}_p^{L^*}(g)$ *replaced by* $\sigma_p^{L^*}(g)$.

Remark 26. *In Theorem 21, if we take* $\rho_p^{L^*}(f,h) = \lambda_p^{L^*}(g)$, $\overline{\tau}_p^{L^*}(g) < \infty$ *and* $\overline{\sigma}_p^{L^*}(f,h) > 0$ *instead of* $\rho_p^{L^*}(f,h) < \infty$, $\lambda_p^{L^*}(f,h) = \lambda_p^{L^*}(g)$, $\overline{\tau}_p^{L^*}(g) < \infty$ *and* $\tau_p^{L^*}(f,h) > 0$ *and the other conditions remain the same, then one can easily verify that the conclusion of Theorem 21 remains valid with* $\tau_p^{L^*}(f,h)$ *replaced by* $\overline{\sigma}_p^{L^*}(f,h)$.

Theorem 22. *Let the meromorphic function* f *and entire function* h *satisfy the conditions of Lemma 8. Also let* g *be an entire function,* h *satisfy the Property (A),* $\lambda_p^{L^*}(f,h) = \lambda_p^{L^*}(g)$, $\overline{\tau}_p^{L^*}(g) < \infty$ *and* $\tau_p^{L^*}(f,h) > 0$, *where* p *is any positive integer.*
(a) *If* $\exp^{[p-1]} L\left(M_g(r)\right) = o\left\{T^{-1}_{P_0[h]}T_{P_0[f]}(r)\right\}$ *then*

$$\underline{\lim}_{r\to\infty} \frac{\log T_h^{-1} T_{f\circ g}(r)}{T^{-1}_{M[h]}T_{M[f]}(r) + \exp^{[p-1]} L\left(M_g(r)\right)} \leq \frac{\lambda_p^{L^*}(f,h)\cdot \overline{\tau}_p^{L^*}(g)}{\left(\frac{\Gamma_{M[f]} - (\Gamma_{M[f]} - \gamma_{M[f]})\Theta(\infty;f)}{\Gamma_{M[h]} - (\Gamma_{M[h]} - \gamma_{M[h]})\Theta(\infty;h)}\right)^{\frac{1}{\rho_h}} \cdot \tau_p^{L^*}(f,h)}.$$

(b) *If* $T^{-1}_{P_0[h]}T_{P_0[f]}(r) = o\left\{\exp^{[p-1]} L\left(M_g(r)\right)\right\}$ *then*

$$\underline{\lim}_{r\to\infty} \frac{\log T_h^{-1} T_{f\circ g}(r)}{T^{-1}_{M[h]}T_{M[f]}(r) + \exp^{[p-1]} L\left(M_g(r)\right)} \leq \lambda_p^{L^*}(f,h).$$

Remark 27. *In Theorem 22, if we take* $\rho_p^{L^*}(f,h) < \infty$, $\lambda_p^{L^*}(f,h) = \lambda_p^{L^*}(g)$, $\overline{\tau}_p^{L^*}(g) < \infty$ *and* $\overline{\tau}_p^{L^*}(f,h) > 0$ *instead of* $\lambda_p^{L^*}(f,h) = \lambda_p^{L^*}(g)$, $\overline{\tau}_p^{L^*}(g) < \infty$ *and* $\tau_p^{L^*}(f,h) > 0$ *and the other conditions remain the same, then one can easily verify that the conclusion of Theorem 22 remains valid with* $\lambda_p^{L^*}(f,h)$ *replaced by* $\rho_p^{L^*}(f,h)$ *and* $\tau_p^{L^*}(f,h)$ *replaced by* $\overline{\tau}_p^{L^*}(f,h)$ *respectively.*

Remark 28. *In Theorem 22, if we take* $\rho_p^{L^*}(f,h) < \infty$, $\lambda_p^{L^*}(f,h) = \lambda_p^{L^*}(g)$, $\tau_p^{L^*}(g) < \infty$ *and* $\tau_p^{L^*}(f,h) > 0$ *instead of* $\lambda_p^{L^*}(f,h) = \lambda_p^{L^*}(g)$, $\overline{\tau}_p^{L^*}(g) < \infty$ *and* $\tau_p^{L^*}(f,h) > 0$ *and the other conditions remain the same, then one can easily verify that the conclusion of Theorem 22 remains valid with* $\lambda_p^{L^*}(f,h)$ *replaced by* $\rho_p^{L^*}(f,h)$ *and* $\overline{\tau}_p^{L^*}(g)$ *replaced by* $\tau_p^{L^*}(g)$ *respectively.*

Remark 29. *In Theorem 22, if we replace the condition* $\lambda_p^{L^*}(f,h) = \lambda_p^{L^*}(g)$ *and* $\overline{\tau}_p^{L^*}(g) < \infty$ *by* $\lambda_p^{L^*}(f,h) = \rho_p^{L^*}(g)$ *and* $\sigma_p^{L^*}(g) < \infty$ *and the other conditions remain the same, then one can easily verify that the conclusion of Theorem 22 remains valid with* $\overline{\tau}_p^{L^*}(g)$ *replaced by* $\sigma_p^{L^*}(g)$.

Remark 30. *In Theorem 22, if we take* $\rho_p^{L^*}(f,h) = \lambda_p^{L^*}(g)$, $\overline{\tau}_p^{L^*}(g) < \infty$ *and* $\overline{\sigma}_p^{L^*}(f,h) > 0$ *instead of* $\lambda_p^{L^*}(f,h) = \lambda_p^{L^*}(g)$, $\overline{\tau}_p^{L^*}(g) < \infty$ *and* $\tau_p^{L^*}(f,h) > 0$ *and the other conditions remain the same, then one can easily verify that the conclusion of Theorem 22 remains valid with* $\tau_p^{L^*}(f,h)$ *replaced by* $\overline{\sigma}_p^{L^*}(f,h)$.

Theorem 23. *Let f be a meromorphic function either of finite order or of non-zero lower order such that $\Theta(\infty; f) = \sum_{a\neq\infty} \delta_p(a; f) = 1$ or $\delta(\infty; f) = \sum_{a\neq\infty} \delta(a; f) = 1$ and h be an entire function having regular growth and non zero finite order with $\Theta(\infty; h) = \sum_{a\neq\infty} \delta_p(a; h) = 1$ or $\delta(\infty; h) = \sum_{a\neq\infty} \delta(a; h) = 1$. Also let g be an entire function and $0 < \lambda_p^{L^*}(f, h) \leq \rho_p^{L^*}(f, h) < \infty$, where p is any positive integer. Then*

$$\overline{\lim_{r\to\infty}} \frac{\log T_h^{-1} T_{f\circ g}(r)}{\log T_{P_0[h]}^{-1} T_{P_0[f]}(\exp r^\mu)} \geq \frac{\lambda_p^{L^*}(f, h)}{\rho_p^{L^*}(f, h)},$$

where $0 < \mu < \rho_g \leq \infty$.

Proof. In view of Lemma 10, we obtain for a sequence of values of r tending to infinity that

$$\begin{aligned} &\log T_h^{-1} T_{f\circ g}(r) \geq \log T_h^{-1} T_f(\exp r^\mu), \\ \text{i.e.}\quad &\log T_h^{-1} T_{f\circ g}(r) \geq \left(\lambda_p^{L^*}(f, h) - \varepsilon\right)\left[r^\mu + \exp^{[p-1]} L(\exp r^\mu)\right]. \end{aligned} \tag{12}$$

Also in view of Lemma 5, and for any arbitrary $\varepsilon(> 0)$, it follows for all sufficiently large values of r that

$$\begin{aligned} &\log T_{P_0[h]}^{-1} T_{P_0[f]}(\exp r^\mu) \leq \left(\rho_p^{L^*}(P_0[f], P_0[h]) + \varepsilon\right)\left[r^\mu + \exp^{[p-1]} L(\exp r^\mu)\right], \\ \text{i.e.}\quad &\log T_{P_0[h]}^{-1} T_{P_0[f]}(\exp r^\mu) \leq \left(\rho_p^{L^*}(f, h) + \varepsilon\right)\left[r^\mu + \exp^{[p-1]} L(\exp r^\mu)\right]. \end{aligned} \tag{13}$$

Now from (12) and (13), we get for a sequence of values of r tending to infinity that

$$\frac{\log T_h^{-1} T_{f\circ g}(r)}{\log T_{P_0[h]}^{-1} T_{P_0[f]}(\exp r^\mu)} \geq \frac{\left(\lambda_p^{L^*}(f, h) - \varepsilon\right)\left[r^\mu + \exp^{[p-1]} L(\exp r^\mu)\right]}{\left(\rho_p^{L^*}(f, h) + \varepsilon\right)\left[r^\mu + \exp^{[p-1]} L(\exp r^\mu)\right]}.$$

Since $\varepsilon(> 0)$ is arbitrary, it follows from above that

$$\overline{\lim_{r\to\infty}} \frac{\log T_h^{-1} T_{f\circ g}(r)}{\log T_{P_0[h]}^{-1} T_{P_0[f]}(\exp r^\mu)} \geq \frac{\lambda_p^{L^*}(f, h)}{\rho_p^{L^*}(f, h)}.$$

Thus the theorem follows. □

Theorem 24. *Let f be a meromorphic function, g be an entire function either of finite order or of non-zero lower order such that $\Theta(\infty; g) = \sum_{a\neq\infty} \delta_p(a; g) = 1$ or $\delta(\infty; g) = \sum_{a\neq\infty} \delta(a; g) = 1$ and h be an entire function having regular growth and non zero finite order with $\Theta(\infty; h) = \sum_{a\neq\infty} \delta_p(a; h) = 1$ or $\delta(\infty; h) = \sum_{a\neq\infty} \delta(a; h) = 1$. Let $0 < \lambda_f$ and $0 < \lambda_p^{L^*}(g, h) \leq \rho_p^{L^*}(g, h) < \infty$, where p is any positive integer. Then*

$$\overline{\lim_{r\to\infty}} \frac{\log T_h^{-1} T_{f\circ g}(r)}{\log T_{P_0[h]}^{-1} T_{P_0[g]}(\exp r^\mu)} \geq \frac{\lambda_p^{L^*}(g, h)}{\rho_p^{L^*}(g, h)},$$

where $0 < \mu < \rho_g$.

We omit the proof of the above theorem as it can be carried out in the line of Theorem 23 and with the help of Lemma 11.

In the line of Theorem 23 and Theorem 24 respectively, one can easily prove the following two theorems and therefore their proofs are omitted.

Theorem 25. *Let f be a transcendental meromorphic function of finite order or of non-zero lower order and* $\sum_{a\in\mathbb{C}\cup\{\infty\}} \delta_1(a;f) = 4$ *and h be a transcendental entire function of regular growth having non zero finite order with* $\sum_{a\in\mathbb{C}\cup\{\infty\}} \delta_1(a;h) = 4$. *Also let g be an entire function and* $0 < \lambda_p^{L^*}(f,h) \le \rho_p^{L^*}(f,h) < \infty$, *where p is any positive integer. Then*

$$\overline{\lim_{r\to\infty}} \frac{\log T_h^{-1} T_{f\circ g}(r)}{\log T_{M[h]}^{-1} T_{M[f]}(\exp r^{\mu})} \ge \frac{\lambda_p^{L^*}(f,h)}{\rho_p^{L^*}(f,h)},$$

where $0 < \mu < \rho_g \le \infty$.

Theorem 26. *Let f be a meromorphic function and g be a transcendental entire function of finite order or of non-zero lower order such that* $\sum_{a\in\mathbb{C}\cup\{\infty\}} \delta_1(a;g) = 4$ *and h be a transcendental entire function of regular growth having non zero finite order with* $\sum_{a\in\mathbb{C}\cup\{\infty\}} \delta_1(a;h) = 4$. *Also let* $0 < \lambda_f$ *and* $0 < \lambda_p^{L^*}(g,h) \le \rho_p^{L^*}(g,h) < \infty$, *where p is any positive integer. Then*

$$\overline{\lim_{r\to\infty}} \frac{\log T_h^{-1} T_{f\circ g}(r)}{\log T_{M[h]}^{-1} T_{M[g]}(\exp r^{\mu})} \ge \frac{\lambda_p^{L^*}(g,h)}{\rho_p^{L^*}(g,h)},$$

where $0 < \mu < \rho_g$.

Theorem 27. *Let f be a meromorphic function either of finite order or of non-zero lower order such that* $\Theta(\infty;f) = \sum_{a\neq\infty} \delta_p(a;f) = 1$ *or* $\delta(\infty;f) = \sum_{a\neq\infty} \delta(a;f) = 1$ *and h be an entire function having regular growth and non zero finite order with* $\Theta(\infty;h) = \sum_{a\neq\infty} \delta_p(a;h) = 1$ *or* $\delta(\infty;h) = \sum_{a\neq\infty} \delta(a;h) = 1$. *Also let g be an entire function and* $0 < \lambda_p^{L^*}(f,h) \le \rho_p^{L^*}(f,h) < \infty$, *where p is any positive integer. Then*

$$\underline{\lim_{r\to\infty}} \frac{\log T_h^{-1} T_{f\circ g}(r)}{\log T_{P_0[h]}^{-1} T_{P_0[f]}(\exp r^{\mu})} \le \frac{\rho_p^{L^*}(f,h)}{\lambda_p^{L^*}(f,h)},$$

where $\lambda_g < \mu < \infty$.

Proof. In view of Lemma 12, we obtain for a sequence of values of r tending to infinity that

$$\begin{aligned} &\log T_h^{-1} T_{f\circ g}(r) < \log T_h^{-1} T_f(\exp r^{\mu}), \\ \text{i.e.}\quad &\log T_h^{-1} T_{f\circ g}(r) < \left(\rho_p^{L^*}(f,h) + \varepsilon\right)\left[r^{\mu} + \exp^{[p-1]} L(\exp r^{\mu})\right]. \end{aligned} \tag{14}$$

Also in view of Lemma 5, and for any arbitrary $\varepsilon\,(>0)$, it follows for all sufficiently large values of r that

$$\begin{aligned} &\log T_{P_0[h]}^{-1} T_{P_0[f]}(\exp r^{\mu}) \ge \left(\lambda_p^{L^*}(P_0[f], P_0[h]) - \varepsilon\right)\left[r^{\mu} + \exp^{[p-1]} L(\exp r^{\mu})\right], \\ \text{i.e.}\quad &\log T_{P_0[h]}^{-1} T_{P_0[f]}(\exp r^{\mu}) \ge \left(\lambda_p^{L^*}(f,h) - \varepsilon\right)\left[r^{\mu} + \exp^{[p-1]} L(\exp r^{\mu})\right]. \end{aligned} \tag{15}$$

Now from (14) and (15), we get for a sequence of values of r tending to infinity that

$$\frac{\log T_h^{-1} T_{f \circ g}(r)}{\log T_{P_0[h]}^{-1} T_{P_0[f]}(\exp r^{\mu})} < \frac{\left(\rho_p^{L^*}(f,h)+\varepsilon\right)\left[r^{\mu}+\exp^{[p-1]} L(\exp r^{\mu})\right]}{\left(\lambda_p^{L^*}(f,h)-\varepsilon\right)\left[r^{\mu}+\exp^{[p-1]} L(\exp r^{\mu})\right]}.$$

Since $\varepsilon\,(>0)$ is arbitrary, it follows from above that

$$\varliminf_{r\to\infty} \frac{\log T_h^{-1} T_{f \circ g}(r)}{\log T_{P_0[h]}^{-1} T_{P_0[f]}(\exp r^{\mu})} \leq \frac{\rho_p^{L^*}(f,h)}{\lambda_p^{L^*}(f,h)}.$$

Thus the theorem follows. □

Now we state the following theorem without its proof as it can be carried out in the line of the above theorem and with the help of Lemma 13.

Theorem 28. *Let f be a meromorphic function and g be an entire function either of finite order or of non-zero lower order such that $\Theta(\infty;g) = \sum_{a\neq\infty} \delta_p(a;g) = 1$ or $\delta(\infty;g) = \sum_{a\neq\infty} \delta(a;g) = 1$ and h be an entire function having regular growth and non zero finite order with $\Theta(\infty;h) = \sum_{a\neq\infty} \delta_p(a;h) = 1$ or $\delta(\infty;h) = \sum_{a\neq\infty} \delta(a;h) = 1$. Let $0 < \lambda_f$ and $0 < \lambda_p^{L^*}(g,h) \leq \rho_p^{L^*}(g,h) < \infty$, where p is any positive integer. Then*

$$\varliminf_{r\to\infty} \frac{\log T_h^{-1} T_{f \circ g}(r)}{\log T_{P_0[h]}^{-1} T_{P_0[g]}(\exp r^{\mu})} \leq \frac{\rho_p^{L^*}(g,h)}{\lambda_p^{L^*}(g,h)},$$

where $0 < \lambda_g < \mu < \infty$.

In the line of Theorem 27 and Theorem 28 respectively, one can easily prove the following two theorems and therefore their proofs are omitted.

Theorem 29. *Let f be a transcendental meromorphic function of finite order or of non-zero lower order with $\sum_{a\in\mathbb{C}\cup\{\infty\}} \delta_1(a;f) = 4$ and h be a transcendental entire function of regular growth having non zero finite order with $\sum_{a\in\mathbb{C}\cup\{\infty\}} \delta_1(a;h) = 4$. Also let g be an entire function and $0 < \lambda_p^{L^*}(f,h) \leq \rho_p^{L^*}(f,h) < \infty$, where p is any positive integer. Then*

$$\varliminf_{r\to\infty} \frac{\log T_h^{-1} T_{f \circ g}(r)}{\log T_{M[h]}^{-1} T_{M[f]}(\exp r^{\mu})} \leq \frac{\rho_p^{L^*}(f,h)}{\lambda_p^{L^*}(f,h)},$$

where $\lambda_g < \mu < \infty$.

Theorem 30. *Let f be a meromorphic function and g be a transcendental entire function of finite order or of non-zero lower order such that $\sum_{a\in\mathbb{C}\cup\{\infty\}} \delta_1(a;g) = 4$ and h be a transcendental entire function of regular growth having non zero finite order with $\sum_{a\in\mathbb{C}\cup\{\infty\}} \delta_1(a;h) = 4$. Also let $0 < \lambda_f$ and $0 < \lambda_p^{L^*}(g,h) \leq \rho_p^{L^*}(g,h) < \infty$, where p is any positive integer. Then*

$$\varliminf_{r\to\infty} \frac{\log T_h^{-1} T_{f \circ g}(r)}{\log T_{M[h]}^{-1} T_{M[g]}(\exp r^{\mu})} \leq \frac{\rho_p^{L^*}(g,h)}{\lambda_p^{L^*}(g,h)},$$

where $0 < \lambda_g < \mu < \infty$.

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Бісвас Т. *Прогрес у вивченні аналізу росту диференціальних поліномів і диференціальних мономів в контексті повільно зростаючих функцій* // Карпатські матем. публ. — 2018. — Т.10, №1. — С. 31–57.

Дослідження аналізу росту цілих чи мероморфних функцій, як правило, проводилися через їх характеристичну функцію Неванліни в порівнянні з тими експоненційними функціями. Але якщо потрібно порівняти темпи зростання будь-якої цілої чи мероморфної функції відносно іншої, то потрібно використовувати поняття індикаторів відносного зростання. Область дослідження в цій галузі може бути більш значимою через інтенсивні застосування теорій повільно зростаючих функцій, що фактично означає, що $L(ar) \sim L(r)$ при $r \to \infty$ для кожної додатньої константи a, тобто $\lim\limits_{r\to\infty} \frac{L(ar)}{L(r)} = 1$, де $L \equiv L(r)$ — додатня неперервна функція, яка повільно зростає. Власне, в цій роботі ми отримали деякі результати, що залежать від властивостей відносного зростання композицій цілих і мероморфних функцій, використовуючи ідею відносного ${}_pL^*$-порядку, відносного ${}_pL^*$-типу, відносного ${}_pL^*$-слабкого типу і диференціальних мономів, диференціальних поліномів, породжених одним з коефіцієнтів; ці результати поширюють деякі попередні результати, де ${}_pL^*$ є нічим іншим як слабшим припущенням на L.

Ключові слова і фрази: ціла функція, мероморфна функція, відносний ${}_pL^*$ порядок, відносний ${}_pL^*$ тип, відносний ${}_pL^*$ слабкий тип, ріст, диференціальний моном, диференціальний поліном, функція повільного росту.