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FAITHFUL GROUP ACTIONS AND SCHREIER GRAPHS

Each action of a finitely generated group on a set uniquely defines a labelled directed graph called the Schreier graph of the action. Schreier graphs are used mainly as a tool to establish geometrical and dynamical properties of corresponding group actions. In particilar, they are widely used in order to check amenability of different classed of groups. In the present paper Schreier graphs are utilized to construct new examples of faithful actions of free products of groups. Using Schreier graphs of group actions a sufficient condition for a group action to be faithful is presented. This result is applied to finite automaton actions on spaces of words i.e. actions defined by finite automata over finite alphabets. It is shown how to construct new faithful automaton presentations of groups upon given such a presentation. As an example a new countable series of faithful finite automaton presentations of free products of finite groups is constructed. The obtained results can be regarded as another way to construct new faithful actions of groups as soon as at least one such an action is provided.

Key words and phrases: group action, faithful action, Schreier graph, free product, automaton permutation.

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INTRODUCTION

O. Shreier introduced in [8] graphs to represent cosets by finite index subgroups in free groups. Such kind of graphs were later named after Schreier and they naturally arise in geometric group theory. In particular they were used to produce exotic example of group actions and to establish rare properties of graphs and groups [1,2,4].

In this paper we use Schreier graphs of group actions to give a sufficient condition for a group action to be faithful. This approach gives an alternative way to construct faithful group actions of free products compared to a well-known method based on ping-pong lemma (see e.g. [5,7]). As an application we construct a new countable series of faithful finite automaton presentations of free products of finite groups.

This result generalizes our previous construction from [3] and its proof explores the main theorem from [6].

The paper is organized as follows. In the first section we recall the definition of Schreier graphs and introduce Schreier embedding of group actions. Then we prove the main theorem, which allows to built new faithful group actions upon given one.

In the second section we recall basic definitions about automaton permutations and define a countable series of finite automaton actions of free products of finite groups. In the last section we prove the result about Schreier embeddability of constructed actions and apply the main theorem to obtain faithfulness of them.

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1 SCHREIER GRAPHS

Let G be a group with a finite generation set S, acting on a set M.

Definition 1. *The Schreier graph $\Gamma(G, S, M)$ of the action of the group G on the set M is a directed graph with the set of vertices M and the set of edges $M \times S$, where for every $m \in M$ and $s \in S$ there is an edge (m, s) from m to $s(m)$ and this edge has a label s.*

Definition 2. *The Schreier graphs Γ_1 and Γ_2 of the group G with the generation set S acting on the sets M_1 and M_2 respectively are called isomorphic if they are isomorphic as oriented edge-labeled graphs, i.e., there is a one-to-one function $f : M_1 \to M_2$ such that for arbitrary vertices v_1, v_2 of the graph Γ_1 there is an arrow from v_1 to v_2 with the label $s \in S$ if and only if the graph Γ_2 contains an arrow from $\mathbf{f}(v_1)$ to $\mathbf{f}(v_2)$ with the label s.*

It immediately follows from the definition that for isomorphic Schreier graphs Γ_1 and Γ_2 there exists a path between two vertices v_1 and v_2 in Γ_1 with the labels of the edges $g_1, \ldots, g_n$ if and only if Γ_2 contains a path between $\mathbf{f}(v_1)$ and $\mathbf{f}(v_2)$ with the labels of the edges $g_1, \ldots, g_n$.

It is possible to give a natural sufficient condition for the faithfulness of the group actioni in terms of the Schreier graphs. Namely, let the group G act on the sets M_1 and M_2, that is, the actions ψ_1 and ψ_2 of the group G are given on these sets respectively.

Definition 3. *The action ψ_1 is Schreier-embedded into the action ψ_2 if a group G has a generation set A such that each connected component of the Schreier graph of action ψ_1 of this group with respect to the generation set A is isomorphic to some component of the Schreier graph of the action ψ_2 of this group with respect to the same generation set A.*

We call actions ψ_1, ψ_2 Schreier-equivalent if ψ_1 is Schreier-embedded into ψ_2 and vice versa.

We have the following useful observation.

Theorem 1. *Let ψ_1 and ψ_2 be actions of a group G such that ψ_1 is Shreier-embedded into ψ_2. If the action ψ_1 is faithful then the action ψ_2 is faithful as well.*

Proof. Let ψ_1 and ψ_2 be actions on sets M_1 and M_2 respectively. Denote by A a generating set of G used to construct Schreier-embedding of the action ψ_1 in the action ψ_2.

Assume that the action ψ_2 is not faithful. Then there exists a non-identity element g of the group G that fixes an arbitrary element of the set M_2. Then $g = g_1 \ldots g_n$ for some $g_1, \ldots, g_n \in A$. So paths with the edges labeled $g_1, \ldots, g_n$ in the Schreier graph of the action ψ_2 are cycles.

By the assumption of the theorem, the action ψ_1 of the group G is Schreier-embedded into the action ψ_2 of the same group. Therefore all paths with the labels $g_1, \ldots, g_n$ in the Schreier graph of the action ψ_1 of the group G are cycles as well. This implies that the non-identity element $g = g_1 \ldots g_n$ of the group G fixes arbitrary element from the set M_1. This contradicts with the faithfulness of the action ψ_1. □

2 AUTOMATON ACTIONS OF FREE PRODUCTS

Let an *alphabet* be a finite set X, $|X| > 1$. A sequence $x_1 \ldots x_n$ of elements from the alphabet is the *word* of length n. An *empty word* Λ has a length equal to zero. Denote by X^n the set of words of length n over the alphabet X. Consider X^* and X^ω — words of finite and infinite length respectively. For arbitrary words $u, v \in X^* \bigcup X^\omega$ one can define the product of two words v and u by concatenation $uv \in X^* \bigcup X^\omega$.

Definition 4. *An initial automaton is a tuple $A = \langle X, Q, \varphi, \psi, q_0 \rangle$,*

- *where X is a finite input and output alphabet, $|X| = n$,*

- *Q is a nonempty set, the set of inner states of the automaton A,*

- *φ and ψ are transition and output functions, acting from $Q \times X$ into Q and X, respectively,*

- *$q_0 \in Q$ is an initial state.*

In particular, a *finite* automaton is an automaton with a finite set of states: $|Q| < \infty$. An automaton is called *permutational* if for each state of the automaton the restriction of the output function in this state determines some permutation on the alphabet.

The transformation of the set X^* of all finite words over the alphabet X defined by the finite initial permutational automata form a group $FGA(X)$ with respect to a superposition. Elements of this group are called finite automaton permutations over the alphabet X.

Consider the group G generated by a finite set S of finite automaton permutations over the alphabet X. It acts on the set X^* of all finite words over the alphabet X.

The sets X^n, $n \geq 1$, are invariant under the action of G. Thus the sequence Γ_n of finite Schreier graphs of the action of G over X^n, $n \geq 1$, naturally arise. We call these graphs the Schreier graphs of the action of G on the levels.

Let $G_1, \dots, G_s$ be s ($s \geq 2$) finite groups of orders $p_1, \dots, p_s$ respectively. Without loss of generality suppose $1 < p_1 \leq \ldots \leq p_s$ and denote $n = p_s$. Let us remind the construction from [6] of an embedding of the free product $G_1 * \ldots * G_s$ into the group $FGA(X)$ of finite automaton permutations over the alphabet $X = \{x_1, \dots, x_n\}$.

For every i, $1 \leq i \leq s$ fix a regular action of the group G_i on the first p_i symbols of X and fix remain letters. Denote the letter x_1 by 0 and the word $0 \ldots 0 \in X^s$ of all words of length s by $\overline{0}$. Consider subsets M_i, $1 \leq i \leq s$, in X^s:

$$M_i = \{\underbrace{x \ldots x}_{i} 0 \ldots 0 : x \in X, x \neq 0\}.$$

For each $i, 1 \leq i \leq s$, we define the set $D_i = \bigcup\limits_{j \neq i} M_j^{G_i}$, where

$$M_j^{G_i} = \{\omega^g : \omega \in M_j, g \in G_i\}, 1 \leq i, j \leq s, \quad D_i = \{\overline{x_1}^{hg} : h \in G_j, g \in G_i, h \neq e_j, j \neq i\}.$$

Let φ_{1i} be functions, which assign to each element $g \in G_i$ a map $\varphi_{1i}(g)$ on the set X^∞ of all infinite words over the alphabet X. An infinite word $\omega \in X^\infty$ can be divided into syllables of arbitrary length $k \in \mathbb{N}$:

$$\omega = \omega[k, 1]\omega[k, 2] \ldots$$

For all $g \in G_i$, $u \in X^\infty$ we construct $v_1 = (\varphi_{1i}(g))(u)$ as follows. Let $v_1[s, 1] = u[s, 1]$, and for all $j \geq 2$

$$v_1[s, j] = \begin{cases} (u[s, j])^g, & \text{if } u[s, j-1] \in D_i \\ u[s, j] & \text{otherwise.} \end{cases} \tag{1}$$

Hence we have constructed everywhere defined transformations of the set of infinite words over X.

In [6] it is proved that for each element $g \in G_i$ the transformation $\varphi_{1i}(g)$ is a finite automaton permutation over the alphabet X and the function φ_{1i} is a monomorphism from the group G_i into the group of finite automaton permutations $FGA(X)$. Denoted by $\mathbb{G}_1(G_1, \ldots, G_s)$ a subgroup of $FGA(X)$ generated by the images of these monomorphisms.

Theorem 2 ([6]). *The group $\mathbb{G}_1(G_1, \ldots, G_s)$ splits into the free product as follows:*

$$\mathbb{G}_1(G_1, \ldots, G_s) \simeq G_1 * \ldots * G_s.$$

We proceed to the construction of a class of actions in each of which the action (1) is Schreier-embedded. We define a series of sets of functions $\varphi_{ti}, t \geq 1$ on $G_i, 1 \leq i \leq s$. The function $\varphi_{ti}, t \geq 1$ assigns to each element $g \in G_i$ a finite automaton transformations $\varphi_{ti}(g)$ of the set X^∞ of all infinite words over X. For arbitrary $g \in G_i$, $u \in X^\infty$, we define $v_t = (\varphi_{ti}(g))(u)$ as follows. For arbitrary $1 \leq j < t+1$ we put $v_t[s,j] = u[s,j]$, and for all $j \geq t+1$

$$v_t[s,j] = \begin{cases} (u[s,j])^g, & \text{if } u[s,j-t] \in D_i \\ u[s,j] & \text{otherwise.} \end{cases} \tag{2}$$

It is directly verified that for each $t \geq 1$ and $i, 1 \leq i \leq s$ the function φ_{ti} is a homomorphism on the group G_i. Hence, for each $t \geq 1$ we obtain an action of the free product $G_1 * \ldots * G_s$ by finite state automaton permutations.

Let $\mathbb{G}_t(G_1, \ldots, G_s)$ be a subgroup of the group of finite automaton permutations over X generated by $\varphi_{t1}(G_1), \ldots, \varphi_{ts}(G_s), t \geq 1$. Note that for $t = 1$ we obtain the action given by A. Oliynyk in [6], and for $t = 2$ — by the author in [3].

3 PROPERTIES OF ACTIONS

The Schreier-embeddability of the constructed actions of a free product of finite number of finite groups is proved by the next theorem.

Lemma 1. *The action given by equation* (1) *is Schreier-embedded into each action of the series given by equation* (2).

Proof. To prove the statement of the lemma we will express the first action in terms of the second one. We fix $t > 1$.

We will use representation of an infinite word ω as a product of subwords of length s:

$$\omega = \omega[s,1]\omega[s,2]\omega[s,3]\ldots.$$

Then we construct t infinite words as follows

$$\begin{aligned} \omega_1 &= \omega[s,1]\omega[s,t+1]\omega[s,2t+1]\ldots \\ &\ldots \\ \omega_i &= \omega[s,i]\omega[s,t+i]\omega[s,2t+i]\ldots \\ &\ldots \\ \omega_t &= \omega[s,t]\omega[s,2t]\omega[s,3t]\ldots \end{aligned}$$

In other terms, the representation of infinite word ω_i as a product of subwords of length s consists of those subwords of length s of ω which numbers have the form $tk + i$, $k \geq 0$.

Let $\varphi_{ti}(g)$, $g \in G_i$, defined by (2), acts on infinite words ω. Denote by v_t the word obtained as the result of this action. Denote by $v_{1,i}$, $0 \leq i < t$ infinite words that are the results of the action $\varphi_{1i}(g)$ on ω_i respectively. Comparing the words v_t and $v_{1,i}$, we have the following equations for arbitrary $k \geq 1$:

$$\begin{cases} v_t[s, tk-1] = v_{1,1}[s,k], \\ v_t[s, tk-2] = v_{1,2}[s,k], \\ \dots \\ v_t[s, tk-t+1] = v_{1,t-1}[s,k]. \end{cases} \tag{3}$$

Thus, in order to express a second action in terms of the first one, it is sufficient to decompose the word ω, on which the second mapping acts, on the words ω_i, apply the first transformation to them, and create a new word using equalities (3).

Consider arbitrary connected component of the Schreier graph of the first action. Then fix arbitrary vertex of this component. This vertex correspond to some infinite word ω_1. Let us prove that in the Schreier graph of the second action there is an isomorphic connected component to the selected one. For that purpose we consider the infinite word $\omega_{00,1}$, that for all $k \geq 1$ satisfies the equalities

$$\begin{cases} \omega_{00,1}[s, tk-1] = 00, \\ \dots \\ \omega_{00,1}[s, tk-k+1] = 00, \\ \omega_{00,1}[s, tk] = \omega_1[s,k]. \end{cases} \tag{4}$$

Since $00 \notin D_i, 1 \leq i \leq s$, the $\omega_{00,1}$ blocks whose numbers are not divisible by t will not be changed under the action of the second map. And the blocks which numbers are divisible by t will be changed in the same way as ω_1 under the action of the first map. Thus, the connected component of the Schreier graph of the second action which contains the vertex corresponding to the word $\omega_{00,1}$ will be isomorphic to the connected component of the Schreier graph of the first-action which contains the vertex corresponding to the word ω_1.

Consequently, for arbitrary connected component of the Schreier graph of the first action one can find an isomorphic connected component of the Schreier graph of the second action. That is, the first action is Schreier-embedded into the second one. $\square$

Note that we leave as open a question about Schreier equivalence of these actions.

The main result now can be formulated as follows.

Theorem 3. *Each action of series* (2) *is faithful.*

Proof. Theorem 2 implies that the action (1) is a faithful action of the free product

$$G_1 * \dots * G_s.$$

Theorem 1 implies that the action (1) is Schreier-embedded into each action of the series given by equation (2). Hence by theorem 1 action (2) for all $t \geq 2$ is faithful as well. $\square$

Then we obtain as a corollary the following result.

Corollary 1. *For each $t \geq 1$ the group $\mathbb{G}_t(G_1, \dots, G_s)$ splits into the free product*

$$G_1 * \dots * G_s.$$

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Кожна дія скінченно породженої групи на множині однозначно визначає помічений орієнтований граф, який називається графом Шраєра цієї дії. Графи Шраєра переважно використовуються як інструмент для встановлення геометричних і динамічних властивостей відповідних групових дій. Зокрема, їх вони широко вживані для перевірки аменабельності різноманітних класів груп. В даній статті графи Шраєра вжито для побудови нових прикладів точних дій вільних добутків груп. Використовуючи графи Шраєра дії груп наведено достатню умову того, коли дія групи є точною. Цей результат застосовано до скінченно автоматних дій на просторах слів, тобто до дій, визначених скінченними автоматами над скінченними алфавітами. Показно, як будувати нові точні автоматні зображення груп за умови існування такого зображення. Як приклад, побудовано нову злічену серію точних скінченно автоматних зображень вільних добутків скінченних груп. Отримані результати можна розглядати, як ще один спосіб побудувати нові точні дії груп за умови існування хоча б однієї такої дії.

Ключові слова і фрази: дія групи, точна дія, граф Шраєра, вільний добуток, автоматна підстановка.