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SIDDIQI M.D.[1], HASEEB A.[1], AHMAD M.[2]

SKEW SEMI-INVARIANT SUBMANIFOLDS OF GENERALIZED QUASI-SASAKIAN MANIFOLDS

In the present paper, we study a new class of submanifolds of a generalized Quasi-Sasakian manifold, called skew semi-invariant submanifold. We obtain integrability conditions of the distributions on a skew semi-invariant submanifold and also find the condition for a skew semi-invariant submanifold of a generalized Quasi-Sasakian manifold to be mixed totally geodesic. Also it is shown that a skew semi-invariant submanifold of a generalized Quasi-Sasakian manifold will be anti-invariant if and only if $A_\xi = 0$; and the submanifold will be skew semi-invariant submanifold if $\nabla w = 0$. The equivalence relations for the skew semi-invariant submanifold of a generalized Quasi-Sasakian manifold are given. Furthermore, we have proved that a skew semi-invariant $\xi^\perp$-submanifold of a normal almost contact metric manifold and a generalized Quasi-Sasakian manifold with non-trivial invariant distribution is CR-manifold. An example of dimension 5 is given to show that a skew semi-invariant $\xi^\perp$ submanifold is a CR-structure on the manifold.

Key words and phrases: skew semi-invariant submanifold, generalized quasi-Sasakian manifold, integrability conditions of the distributions, CR-structure.

[1] Jazan University, Al Maarefah Rd, P.O. Box 114, Jazan, Kingdom of Saudi Arabia
[2] Integral University, Kursi Road, 226026, Lucknow, India
E-mail: msiddiqi@jazanu.edu.sa (Siddiqi M.D.), haseeb@jazanu.edu.sa (Haseeb A.), mobinahmad@rediffmail.com (Ahmad M.)

INTRODUCTION

The theory of submanifolds in spaces endowed with additional structure is very interesting topic in the field of differential geometry [5]. The theory of CR-submanifolds has been introduced by A. Bejancu for almost contact geometry [1] and also for almost complex geometry [2], after that several papers have been appeared in this field. M. Barros et al. [5], B. Y. Chen [6, 7], A. Bejancu and N. Papaghuic [3], V. Mangione [10] and N. Papaghiuc [11] have studied semi-invariant submanifolds in Sasakian manifolds and the study was also extended to other ambient spaces. Moreover, some related topics were studied by V. V. Goldberg and R. Rosca [16–20]. In 2012, C. Calin et al. [8] have studied the semi-invariant $\xi^\perp$-submanifold of a generalized quasi-Sasakian manifold. Later on, A. Bejancu defined and studied a semi-invariant submanifold of a locally product manifold [4]. Recently, L. Ximin and F. M. Shao [12] have discussed a new class of submanifolds of a locally product manifold, that is, known as skew semi-invariant submanifolds. The purpose of the present work is to investigate some interseting results on the skew semi-invariant submanifolds of a generalized Quasi-Sasakian manifold.

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1 PRELIMINARIES

Let $\bar{M}$ be a real $(2n+1)$-dimensional smooth manifold equipped with an almost contact metric structure (φ, ξ, η, g), where φ is $(1,1)$-tensor field, ξ is a vector field, η is a 1-form and g is a Riemannian metric such that [1]

$$\varphi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \ \varphi(\xi) = 0, \ \eta \circ \varphi = 0, \tag{1}$$

$$g(\varphi X, Y) = -g(X, \varphi Y), \quad g(X, \xi) = \eta(X), \quad g(\xi, \xi) = 1 \tag{2}$$

for all X, Y on space M. The almost contact manifold $\bar{M}(\varphi, \xi, \eta)$ is said to be normal, if

$$N_\varphi(X, Y) + 2d\eta(X, Y)\xi = 0$$

for all $X, Y \in (TM)$, where

$$N_\varphi(X, Y) = [\varphi X, \varphi Y] + \varphi^2[X, Y] - \varphi[\varphi X, Y] - \varphi[X, \varphi Y]$$

is the Nijenhuis tensor field corresponding to the tensor field φ. The fundamental 2-form Φ on $\bar{M}$ is defined by

$$\Phi(X, Y) = g(X, \varphi Y). \tag{3}$$

S. S. Eum [9], considered the hypersurface of an almost contact metric manifold $\bar{M}$ whose structure tensor field satisfy the following relation:

$$(\bar{\nabla}_X \varphi)Y = g(\bar{\nabla}_{\varphi X}\xi, Y)\xi - \eta(Y)\bar{\nabla}_{\varphi X}\xi, \tag{4}$$

where $\bar{\nabla}$ is the Levi-Civita connection of metric tensor g. For the sake of simplicity we say that a manifold $\bar{M}$ with an almost contact metric structure satisfying (4) is a generalized Quasi-Sasakian manifold. We define a $(1,1)$-tensor field F by

$$FX = -\bar{\nabla}_X \xi.$$

Now, we assume that $\bar{M}$ is a generalized Quasi-Sasakian manifold and M is an m-dimensional submanifold isometrically immersed in $\bar{M}$. Denote by g the induced metric on M and by ∇ its Levi-Civita connection. For $p \in M$ and the tangent vector $X_p \in T_pM$, we can write

$$FX_p = PX_p + QX_p, \tag{5}$$

where $PX_p \in T_pM$ and $QX_p \in T^\perp{}_pM$. For any two vectors $X_p, Y_p \in T_pM$, we have $g(FX_p, Y_p) = g(PX_p, Y_p)$, which implies that $g(PX_p, Y_p) = g(X_p, PY_p)$. Therefore P and P^2 are all symmetric operators on the tangent space T_pM. If $\alpha(p)$ is the eigen value of P^2 at $p \in M$, since P^2 is a composition of an isometry and a projection, then $\alpha(p) \in [0,1]$.

For each $p \in M$, we set

$$D_p^\alpha = \mathcal{K}er(P^2 - \alpha(p)I),$$

where I is an identity transformation on T_pM and $\alpha(p)$ an eigenvalue of P^2 at $p \in M$. Obviously, we have $D_p^0 = \mathcal{K}er P$ and $D_p^1 = \mathcal{K}er Q$, where D_p^1 is the maximal F-invariant subspace of T_pM and D_p^0 is the maximal F-anti invariant subspace of T_pM. If $\alpha_1(p), \ldots, \alpha_k(p)$ are all eigenvalues of P^2 at p, then T_pM can be decomposed as the direct sum of the mutually orthogonal eigenspaces, *i.e.*,

$$T_pM = D_p^{\alpha_1} \oplus \cdots \oplus D_p^{\alpha_k}.$$

Definition 1 ([12]). *A submanifold M of a generalized quasi-Sasakian manifold $\bar{M}$ is said to be skew semi-invariant submanifold, if there exists an integer k and the functions $\alpha_1, \cdots, \alpha_k$ defined on M with values in $(0,1)$ such that*

(1) *each $\alpha_1(x), \cdots, \alpha_k(x)$ are distinct eigenvalues of P^2 at each $p \in M$ with*

$$T_pM = D_p^1 \oplus D_p^0 \oplus D_p^{\alpha_1} \oplus \cdots \oplus D_p^{\alpha_k};$$

(2) *the dimensions of $D_p^0, D_p^1, D_p^{\alpha_1}, \cdots, D_p^{\alpha_k}$ are independent of $p \in M$.*

Remark 1. *(i) From the second case of Definition 1, we can also define P-invariant mutually orthogonal distributions*

$$D^\alpha = \bigcup_{p \in M} D_p^\alpha, \quad \alpha \in \{0, \alpha_1, \cdots, \alpha_k, 1\}$$

on M and $TM = D^1 \oplus D^0 \oplus D^{\alpha_1} \oplus \cdots \oplus D^{\alpha_k}$ are differentiable (see [7]).

(ii) If $k = 0$ in Definition 1, then it follows that P is a structure of type $f(3, -1)$ on M [13] and $\dim(D_p^1) = rank(P_p)$, $\dim(D_p^0)$ are independent of $p \in M$ [14].

(iii) If $k = 0$, (1) implies (2), then M is called a semi-invariant $\xi^\perp$-submanifold.

(iv) If $k = 0$ and $D_p^1 = \{0\}$ (resp., $D_p^0 = \{0\}$), then M is called an anti-invariant (resp., invariant) $\xi^\perp$-submanifold.

(v) If $D_p^1 = \{0\} = D_p^0$, $k = 1$ and $\alpha_1^2(x)$ is constant, then M may be said to be a θ-slant submanifold with slant angle $\cos\theta = \alpha_1$.

Example 1. *We consider the Euclidean space R^9 and denote its points by $y = (y^i)$. Let $(e_j), j = 1, \ldots, 9$ be the natural basis defined by $e^j = \partial/\partial y^j$. We define a vector field ξ and a 1-form η by $\xi = \partial/\partial y^9$ and $\eta = dy^9$ respectively and φ is $(1,1)$ tensor field defined by*

$$\begin{aligned}
&\varphi e_1 = e_2, \ \ \varphi e_2 = e_1, \ \ \varphi e_3 = e_8, \ \ \varphi e_8 = e_3, \\
&\varphi e_4 = cost(y)e_5 - sint(y)e_6, \varphi e_5 = cost(y)e_4 + sint(y)e_7, \\
&\varphi e_6 = -sint(y)e_4 + cost(y)e_7, \varphi e_7 = sint(y)e_5 + cost(y)e_6, \varphi e_9 = 0,
\end{aligned}$$

where $t : R^9 \to (0, \pi/2)$ is a smooth function. Then it is easy to verify that R^9 is an almost contact metric manifold with almost contact structure (φ, ξ, η, g) with associated metric g given by $g(e_i, e_j) = \delta_{ij}$. The submanifold

$$M = \left\{ (y^1, \ldots, y^9) \in R^9 | y^6, y^7, y^8, y^9 = 0 \right\}$$

of R^9 is a skew semi-invariant submanifold with

$$D^1 = Span\{e_1, e_2\}, \ \ D^0 = Span\{e_3\}, \ \ D^\alpha = Span\{e_4, e_5\},$$

where for $x \in M$ one has $\alpha(y) = cost(y)$.

Denote the induced connection in M by ∇, then the Gauss and Weingarten eqautions are given respectively by

$$\bar{\nabla}_X Y = \nabla_X Y + h(X,Y), \quad \bar{\nabla}_X N = -A_N X + \nabla^{\perp}_X N, \quad X,Y \in TM; N \in T^{\perp}M, \tag{6}$$

where $\bar{\nabla}$, ∇ and $\nabla^{\perp}$ are the Riemannian, induced Riemannian and induced normal connections in M, $\bar{M}$ and the normal bundle $T^{\perp}M$ of $\bar{M}$ *respectively* and h is the second fundamental form related to A by the equation

$$g(h(X,Y),N) = g(A_N X, Y). \tag{7}$$

Let M be a submanifold of a generalized Quasi-Sasakian manifold $\bar{M}$ for $X,Y \in TM$, $N \in T^{\perp}M$. By using

$$\varphi X = tX + wX, \quad tX \in TM,\ wX \in T^{\perp}M, \tag{8}$$

$$\varphi N = BN + CN, \quad BN \in TM,\ CN \in T^{\perp}M, \tag{9}$$

we have

$$(\bar{\nabla}_X \varphi)Y = ((\nabla_X t)Y - A_{wY}X - Bh(X,Y)) + ((\nabla_X w)Y + h(X,tY) - Ch(X,Y)), \tag{10}$$

$$(\bar{\nabla}_X \varphi)N = ((\nabla_X B)N - A_{BN}X + BA_N X)) + ((\nabla_X B)N + h(X,BN) + wA_N X)),$$

where

$$(\nabla_X t)Y = \nabla_X tY - t\nabla_X Y, \quad (\nabla_X w)Y = \nabla^{\perp}_X wY - w\nabla_X Y,$$

$$(\nabla_X B)N = \nabla_X BN - t\nabla^{\perp}_X N, \quad (\nabla_X C)N = \nabla^{\perp}_X CN - C\nabla^{\perp}_X N.$$

Comparing the tangential and normal components in (10), we obtain

$$t\nabla_X Y = \nabla_X tY - Bh(X,Y) - A_{wY}X, \tag{11}$$

$$\nabla_X tY = h(X,tY) + \nabla^{\perp}_X wY - Ch(X,Y). \tag{12}$$

From (11) and (12) we have

$$t[X,Y] = \nabla_X tY - \nabla_Y tY + A_{wX}Y - A_{wY}X, \tag{13}$$

$$w[X,Y] = h(X,tY) - h(tX,Y) + \nabla^{\perp}_Y wX - \nabla^{\perp}_X wY. \tag{14}$$

Thus from (11), (12), (13) and (14), we have the following lemmas.

Lemma 1 ([8]). *Let M be a skew semi-invariant submanifold of a generalized quasi-Sasakian manifold $\bar{M}$. Then, we have*

$$(\nabla_X t)Y = A_{wY}X + Bh(X,Y), \quad (\nabla_X w)Y = Ch(X,Y) - h(X,tY) + g(FX,\varphi Y)\xi \tag{15}$$

for all $X,Y \in TM$.

Proof. The Lemma follows from (10)–(11) by taking into the consideration decomposition of $TM^{\perp}$. □

Lemma 2 ([8]). *Let M be a skew semi-invariant submanifold of a generalized quasi-Sasakian manifold $\bar{M}$. Then we have for any $N \in TM^{\perp}$*

1) $BN \in D^0$,

2) $CN \in D^1$.

Lemma 3 ([8]). *Let M be a skew semi-invariant submanifold of a generalized quasi-Sasakian manifold $\bar{M}$, then the distribution D^0 is integrable if and only if*

$$A_{wZ}W = A_{wW}Z, \quad \text{for all } X, Y \in D^0. \tag{16}$$

The following results give necessary and sufficient conditions for the integrability of the distributions D^0 and D^1.

Theorem 1. *Let M be a skew semi-invariant submanifold of a generalized quasi-Sasakian manifold $\bar{M}$. Then the distribution D^0 is integrable.*

Proof. Let $Z, W \in D^0$, then from (8), (15) and (16), we deduce that

$$t[Z, W] = A_{wZ}W - A_{wW}Z = 0.$$

Hence the conclusion. □

Theorem 2. *Let M be a skew semi-invariant submanifold of a generalized quasi-Sasakian manifold $\bar{M}$, then the distribution D^1 is integrable if and only if*

$$h(tX, Y) - h(X, tY) = (L_\xi g)(X, \varphi Y)\xi \quad \text{for all } X, Y \in D^1. \tag{17}$$

Proof. The statement yields from (15)

$$w([tX, Y]) = h(X, tY) - h(tX, Y) + (L_\xi g)(X, \varphi Y)\xi \quad \text{for all } X, Y \in D^1. \tag{18}$$

□

Proposition 1. *If M is a skew semi-invariant submanifold of a generalized quasi-Sasakian manifold $\bar{M}$, then the following relations take place:*

$$-A_\xi X = t^2 X, \tag{19}$$

$$\nabla^\perp_X \xi = w^2 X, \tag{20}$$

$$\eta(h(X, Y)) = g(X, t^2 Y), \tag{21}$$

$$\eta(H) = -\frac{1}{n}\, trace(t^2)$$

for any $X, Y \in TM$, where H is the mean curvature vector.

Proof. Form equation (18), it follows that $\bar{\nabla}_X \xi = \varphi^2 X = -X + \eta(X)\xi$.

Using (19), (8) and $\eta(X) = 0$ in (6), we get

$$-A_\xi X + \nabla^\perp_X \xi = t^2 X + w^2 X. \tag{22}$$

Equating tangential and normal part of (22), we get (19) and (20), *respectively*. From (2), (7) and (15) it follows that

$$\eta(h(X, Y)) = g(h(X, Y), \xi) = g(A_\xi X, Y) = -g(t^2 X, Y),$$

which gives (21). If $\{e_1, e_2, \dots, e_n\}$, $n = dimM$ is a local orthonormal frame field, then from (17) we get

$$\eta(H) = \frac{1}{n}\eta\left(\sum_{i=1}^{n} h(e_i, e_i)\right) = -\frac{1}{n}\left(\sum_{i=1}^{n} g(P^2 e_i, e_i)\right).$$

Therefore (16) holds. □

From (16), we have the following.

Corollary 1. *Let M be skew semi-invariant submanifold of a generalized quasi-Sasakian manifold $\bar{M}$. If $trace(t^2) \neq 0$, then M can not be minimal.*

In view of (16), we have the following theorem.

Theorem 3 ([8])**.** *Let M be a skew semi-invariant submanifold of a generalized quasi-Sasakian manifold $\bar{M}$. Then M is anti-invariant if and only if $A_\xi = 0$.*

Let D^1 and D^2 be two distributions defined on a manifold $\bar{M}$. We say that D^1 is parallel to D^2 for all $X \in D^2$ and $Y \in D^1$, we have

$$\nabla_X Y \in D^1.$$

If D^1 is parallel then for $X \in TM$ and $Y \in D^1$, we have $\nabla_X Y \in D^1$. It is easy to verify that D^1 is parallel if and only if the orthogonal complementary distribution of D^1 is also parallel.

Let M be a skew semi-invariant submanifold of $\bar{M}$. A distribution D is said to be totally geodesic, if $h(X,Y) = 0$ for all $X, Y \in D$. The distributions D^1 and D^2 are said to be D^1-D^2-mixed totally geodesic, if $h(X,Y) = 0$ for all $X \in D^1$ and $Y \in D^2$.

Proposition 2. *Let M be a skew semi-invariant submanifold of generalized quasi-Sasakian manifold $\bar{M}$. For any distribution D^α, if*

$$A_N tX = tA_N X \text{ for all } X \in D^\alpha,\ N \in T^\perp M,$$

then M is D^α-D^β-mixed totally geodesic, where $\alpha \neq \beta$.

Proof. From the assumption, we have

$$t^2 A_N X - \alpha A_N X = 0.$$

This implies that $A_N X \in D^\alpha$. So for all $Y \in D^\beta$, $N \in T^\perp M$, $\alpha \neq \beta$, we have

$$g(A_N X, Y) = g(h(X,Y), N) = 0.$$

Therefore $h(X,Y) = 0$. Hence M is D^α-D^β-mixed totally geodesic. □

Now from (5), (8) and (9), we find

$$CwX_p = -wtX_p, \quad wBN = N - C^2 N \tag{23}$$

for all $X_p \in T_pM$, $N \in T_p^\perp M$. Furthermore for $X_p \in D_p^{\alpha_i}$, $\alpha \in \{\alpha_1, \ldots, \alpha_k\}$, we have

$$C^2 wX_p = \alpha_i wX_p.$$

Also, if $X_p \in D_p^0$, then it is clear that $t^2 wX_p = 0$. Thus if X_p is an eigenvector of t^2 corresponding to the eigenvalue $\alpha(p) \neq 1$, then wX_p is an eigenvector of C^2 with the same eigenvalue $\alpha(p)$. Thus, (23) implies that $\alpha(p)$ is an eigenvalue of B^2 if and only if $\gamma(p) = 1 - \alpha(p)$ is an eigenvalue of wt. Since wB and f^2 are symmetric operators on the normal bundle $T^\perp M$, then their eigenspaces are orthogonal. The dimension of the eigenspace of wB corresponding to the eigenvalue $1 - \alpha(p)$ is equal to the dimension of D_p^α if $\alpha(p) \neq 1$. Consequently, we have the following lemma.

Lemma 4. *A submanifold M is a skew semi-invariant submanifold of generalized quasi-Sasakian manifold $\bar{M}$ if and only if the eigenvalues of wB are constant and the eigenspaces of wB have constant dimension.*

2 Skew semi-invariant submanifold

Theorem 4. *Let M be a submanifold of a generalized quasi-Sasakian manifold $\bar{M}$. If $\nabla t = 0$, then M is a skew semi-invariant submanifold. Furthermore each of the t-invariant distributions D^0, D^1 and D^{α_i}, $1 \leq i \leq k$ are parallel.*

Proof. For a fix $p \in M$ any $Y_p \in D^{\alpha_{ip}}$ and $X \in TM$. Let Y be the parallel translation of Y_p along with the integral curve of X. Since $(\nabla_X t)Y = 0$ and from (11) we have

$$\nabla_X(t^2 - \alpha(p)Y) = t^2\nabla_X Y - \alpha(p)\nabla_X Y = 0.$$

Since $(t^2 Y - \alpha(p)Y) = 0$ at p, it is identical to 0 on $\bar{M}$. Thus the eigenvalues of t^2 are constant. Moreover, parallel translation of T_pM along any curve is an isometry which preserves each D^α. Thus the dimension of D^α is constant and $\bar{M}$ is a skew semi-invariant submanifold.

Now if Y is any vector field in D^α, then we have $t^2 Y = \alpha Y$ (α constant), *i.e*, $t^2\nabla_X Y = \alpha\nabla_X Y$ which implies that D^α is parallel. □

Now, we see the vanishing of ∇w. For $X, Y \in TM$ if $(\nabla_X w)Y = 0$, then (21) yields

$$Ch(X, Y) = h(X, tY) - g(FX, \varphi Y)\xi. \tag{24}$$

In particular if $Y \in D^\alpha$, then (24) implies

$$C^2 h(X, Y) = \alpha h(X, Y) - \alpha g(FX, \varphi Y)\xi.$$

Consequently we have the following proposition.

Proposition 3. *Let M be a skew semi-invariant submanifold of a generalized quasi-Sasakian manifold $\bar{M}$, if $\nabla w = 0$, then M is D^α-D^β -mixed totally geodesic for all $\alpha \neq \beta$. Moreover, if $X \in D^\alpha$ then either $h(X, X) = 0$ or $h(X, X)$ is an eigenvector of C^2 with eigenvalue α.*

Lemma 5. *Let M be a submanifold of a generalized quasi-Sasakian manifold $\bar{M}$, then $\nabla w = 0$ if and only if $\nabla_X BN = B\nabla^\perp N$ for all $X \in TM$ and $N \in T^\perp M$.*

Theorem 5. *Let M be a submanifold of a generalized quasi-Sasakian manifold $\bar{M}$. If $\nabla w = 0$, then M is a skew semi-invariant submanifold.*

Proof. If $TM = D^1$, then we are done. Otherwise, we may find a point $p \in M$ and a vector $X_p \in D^\alpha_p$, $\alpha \neq 1$. Set $N_p = wX_p$, then N_p is an eigenvector of wB with eigenvalue $\mu(p) = 1 - \alpha(p)$. Now, let $Y \in TM$ and N be the translation of N_p in the normal bundle $T^\perp M$ along with integral curve of Y, we have

$$\nabla^\perp_Y(wBN - \mu(p)N) = \nabla^\perp_Y wBN - \mu(p)\nabla^\perp_Y N = w(\nabla_Y BN) - \mu(p)\nabla^\perp_Y N.$$

In view of Lemma 5,

$$\nabla^\perp_Y(wBN - \mu(p)N) = \nabla^\perp_Y wBN - \mu(p)\nabla^\perp_Y N = 0.$$

Since $wBN - \mu(p)N = 0$ at p and $tBN - \mu(p)N = 0$ on M. It follows from Lemma 4 that M is a skew semi-invariant submanifold. □

Theorem 6. *Let M be a skew semi-invariant submanifold of a generalized quasi-Sasakian manifold $\bar{M}$, then the following relations are equivalent.*

1. $(\nabla_X w)Y - (\nabla_Y w)X = 0$ for all $X, Y \in D^\alpha$.
2. $h(tX, Y) = h(X, tY)$ for all $X, Y \in D^\alpha$.
3. $w[X, Y] = \nabla^{\perp}_X wY - \nabla^{\perp}_Y wX$ for all $X, Y \in D^\alpha$.
4. *$A_N tY - tA_N Y$ is perpendicular to D^α for all $Y \in D^\alpha$ and $N \in T^{\perp}N$.*

Proof. The proof is trivial, hence we omit it. □

3 CR-STRUCTURE

Let $\bar{M}$ be a differentiable manifold and $T^c\bar{M}$ be the complexified tangent bundle to $\bar{M}$. A CR-structure on M is complex subbundle H of $T^c\bar{M}$ such that $H \cap \bar{H} = \{0\}$ and H is involutive [15]. A manifold endowed with a CR-structure is called a CR-manifold. It is known that a differentiable manifold $\bar{M}$ admits a CR-structure [1] if and only if there is a differentiable distribution $\bar{D}$ and a $(1, 1)$ tensor field P on M such that for all $X, Y \in \bar{D}$

$$P^2X = -X, [P, P](X, Y) \equiv [PX, PY] - [X, Y] - P[PX, Y] - P[X, PY] = 0, [PX, PY] - [X, Y] \in \bar{D}.$$

Definition 2. *A differentiable manifold $\bar{M}$ is said to admit a CR-structure if there is a differentiable distribution $\bar{D}$ and a $(1, 1)$ tensor field P on $\bar{M}$ such that for all $X, Y \in \bar{D}$*

$$P^2X = X, [P, P](X, Y) \equiv [PX, PY] + [X, Y] - P[PX, Y] - P[X, PY] = 0, [PX, PY] = [X, Y] \in D.$$

A manifold equipped with a CR-structure is called a CR-manifold.

Lemma 6. *An almost contact metric structure (φ, ξ, η, g) is normal if the Nijenhuis tensor $[\varphi, \varphi]$ of φ satisfies [3]*

$$[\varphi, \varphi] + 2d\eta \otimes \xi = 0. \tag{25}$$

Now, we prove the following theorem.

Theorem 7. *If M is a skew semi-invariant $\xi^{\perp}$-submanifold of a normal almost contact metric manifold $\bar{M}$ with non-trivial invariant distribution, then $\bar{M}$ possesses a CR-structure.*

Proof. Since M is normal for $X, Y \in \bar{D}^{\perp}$, we get $P^2X = -X$ and in view of (25), we have

$$0 = [P, P](X, Y) - Q([X, PY] + [PX, Y])$$

from which it follows that $Q([PX, Y] + [X, PY]) = 0$, that is, $[PX, Y] + [X, PY] \in \bar{D}^1$. Thus

$$[PX, PY] + [X, Y] = P([PX, Y] + [X, PY]) \in \bar{D}^1$$

and hence $(\bar{D}^1, P)$ is a CR-structure on M. □

Theorem 8. *A skew semi-invariant $\xi^{\perp}$-submanifold of a generalized quasi-Sasakian manifold with non-trivial invariant distribution is a CR-manifold.*

Proof. Since every generalized quasi-Sasakian manifold is normal (see [8], Theorem 7), the proof is obvious. □

From Theorem 7, it is obvious that normality of $\bar{M}$ is a sufficient condition for a skew semi-invariant submanifold with nontrivial invariant distribution to carry a CR-structure. However, this is not neccessary, and now we give an example of skew semi-invariant submanifold.

Example 2. *We consider the Euclidean space R^5 and denote its points by $x = (x^i)$. Let $(e_j), j = 1, \dots, 5$ be the natural basis defined by $e^j = \partial/\partial x^j$. We define a vector field ξ and a 1-form η by $\xi = \partial/\partial x^5$ and $\eta = dx^5$ respectively. For each $x \in R^5$, and g the canonical metric defined by $g(e_i, e_j) = \delta_{i,j}$, $i, j = 1, \dots, 5$, the set E_j defined by*

$$E_1 = e_1,\ \ E_2 = cos(x^1)e_2 + sin(x^1)e_3,\ \ E_3 = -sin(x^1)e_2 + cos(x^1)e_3,\ \ E_4 = e_4,\ \ E_5 = e_5$$

forms an orthonormal basis. As the point x varies in R^5, the above set of equations defines 5 vector fields also denoted by (E_j) and φ is $(1,1)$ tensor field defined by

$$\varphi(E_1) = E_2,\ \ \varphi(E_2) = E_1,\ \ \varphi(E_3) = E_4,\ \ \varphi(E_4) = E_3\ \ \varphi(E_5) = 0.$$

Then (φ, ξ, η, g) defines an almost contact metric structure on R^5. Since

$$[\varphi, \varphi](E_1, E_4) + 2d\eta(E_1, E_4)\xi = E_1 \neq 0,$$

then, the almost contact structure is not normal. The submanifold

$$M = \left\{x \in R^5 : x^4, x^5 = 0\right\}$$

is a skew semi-invariant submanifold of R^5 with $D^1 = Span\{E_1, E_2\}$ and $D^0 = Span\{E_3\}$ such that (D^1, φ) is a CR-structure on M. Moreover, D^1 is not integrable because $D^0 = E_3$.

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Сіддікі М.Д., Хасіб А., Ахмад М. *Антинапівінваріантні підмноговиди узагальнених квазі-Сасакянових многовидів* // Карпатські матем. публ. — 2017. — Т.9, №2. — С. 188–197.

У цій роботі ми вивчаємо новий клас підмноговидів узагальнених квазі-Сасакянових многвидів, що називаються антинапівінваріантними підмноговидами. Нами отримано умови інтегровності розподілів на антинапівінваріантному підмноговиді, а також знайдемо умову того, що антинапівінваріантний підмноговид узагальненого квазі-Сасакянового многовиду є змішаним цілком геодезичним. Також показано, що антинапівінваріантний підмноговид узагальненого квазі-Сасакянового многовиду буде антиінваріантним тоді і тільки тоді, якщо $A_(\xi) = 0$; і підмноговид буде антинапівінваріантним підмноговидом, якщо $\nabla w = 0$. Отримано співвідношення еквівалентності для антинапівінваріантного підмноговиду узагальненого квазі-Сасакянового многовиду. Більше того, ми довели, що антинапівінваріантний $\xi^{\perp}$-підмноговид нормального майже контактного метричного многовиду та узагальненого квазі-Сасакянового многовиду з нетривіальним інваріантним розподілом є CR-многовидом. Наведено приклад розмірності 5 для того, щоб показати, що антинапівінваріантний $\xi^{\perp}$-підмноговид є CR-структурою на многовиді.

Ключові слова і фрази: антинапівінваріантний многовид, узагальнений квазі-Сасакяновий многовид, умови інтегровності розподілів, CR-структура.