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SOME CLASSES OF DISPERSIBLE DCSL-GRAPHS

A distance compatible set labeling (dcsl) of a connected graph G is an injective set assignment $f : V(G) \rightarrow 2^X$, X being a non empty ground set, such that the corresponding induced function $f^{\oplus} : E(G) \rightarrow 2^X \setminus \{\varphi\}$ given by $f^{\oplus}(uv) = f(u) \oplus f(v)$ satisfies $|f^{\oplus}(uv)| = k^f_{(u,v)} d_G(u,v)$ for every pair of distinct vertices $u, v \in V(G)$, where $d_G(u,v)$ denotes the path distance between u and v and $k^f_{(u,v)}$ is a constant, not necessarily an integer, depending on the pair of vertices u, v chosen. G is distance compatible set labeled (dcsl) graph if it admits a dcsl. A dcsl f of a (p,q)-graph G is dispersive if the constants of proportionality $k^f_{(u,v)}$ with respect to $f, u \neq v, u, v \in V(G)$ are all distinct and G is dispersible if it admits a dispersive dcsl. In this paper, we prove that all paths and graphs with diameter less than or equal to 2 are dispersible.

Key words and phrases: set labeling of graphs, dcsl-graph, dispersible dcsl-graph.

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INTRODUCTION

Acharya B.D. [1] introduced the notion of vertex set valuation as a set analogue of number valuation. For a graph $G = (V, E)$ and a non empty set X, Acharya B.D. defined a set valuation of G as an injective set valued function $f : V(G) \rightarrow 2^X$, and he defined a set-indexer as a set valuation such that the function $f^{\oplus} : E(G) \rightarrow 2^X \setminus \{\varphi\}$ given by $f^{\oplus}(uv) = f(u) \oplus f(v)$ for every $uv \in E(G)$ is also injective, where 2^X is the set of all the subsets of X and $\oplus$ is the binary operation of taking the symmetric difference of subsets of X.

Acharya B.D. and Germina K.A., who has been studying topological set valuation, introduced the particular kind of set valuation for which a metric, especially the cardinality of the symmetric difference, is associated with each pair of vertices in proportion to the distance between them [2]. In otherwords, the question is whether one can determine those graphs $G = (V, E)$ that admit an injective function $f : V \rightarrow 2^X$, X being a non empty ground set such that the cardinality of the symmetric difference $f^{\oplus}(uv)$ is proportional to the usual path distance $d_G(u,v)$ between u and v in G, for each pair of distinct vertices u and v in G. They called f a *distance compatible set labeling* (dcsl) of G, and the ordered pair (G, f), a distance compatible set labeled (dcsl) graph.

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Definition 1 ([2]). *Let $G = (V, E)$ be any connected graph. A distance compatible set labeling (dcsl) of a graph G is an injective set assignment $f : V(G) \to 2^X$, X being a non empty ground set, such that the corresponding induced function $f^{\oplus} : E(G) \to 2^X \setminus \{\varphi\}$ given by $f^{\oplus}(uv) = f(u) \oplus f(v)$ satisfies $|f^{\oplus}(uv)| = k^f_{(u,v)} d_G(u, v)$ for every pair of distinct vertices $u, v \in V(G)$, where $d_G(u, v)$ denotes the path distance between u and v and $k^f_{(u,v)}$ is a constant, not necessarily an integer, depending on the pair of vertices u, v chosen.*

The following universal theorem has been established in [2].

Theorem 1 ([2]). *Every graph admits a dcsl.*

Definition 2 ([3]). *A dcsl f of a (p, q)-graph G is dispersive if the constants of proportionality $k^f_{(u,v)}$ with respect to f, $u \neq v$, $u, v \in V(G)$ are all distinct and G is dispersible if it admits a dispersive dcsl. A dispersive dcsl f of G is (k, r)-arithmetic, if the constants of proportionality with respect to f can be arranged in the arithmetic progression, $k, k + r, k + 2r, \ldots, k + (q - 1)r$ and if G admits such a dcsl then G is a (k, r)-arithmetic dcsl-graph.*

Theorem 2 ([3]). *K_n is dispersible for all $n \geq 1$.*

1 DISPERSIVE DCSL-GRAPH WITH DIAM(G) ≤ 2

Theorem 3. *The star graph $K_{1,n}$ is dispersible for any $n \geq 1$.*

Proof. Let $V(K_{1,n}) = \{v_0, v_1, v_2, \ldots, v_n\}$ with v_0 is the *central vertex*. Let $X = \{1, 2, \ldots, 2^{2n+1}\}$. Define $f : V(K_{1,n}) \to 2^X$ by $f(v_0) = \varphi$ and $f(v_i) = \{1, 2, 3, \ldots, 2^{2i+1}\}, 1 \leq i \leq n$. Clearly $f(v_i) \subset f(v_j)$ and $|f(v_i) \oplus f(v_j)| = 2^{2j+1} - 2^{2i+1}, |f(v_0) \oplus f(v_i)| = 2^{2i+1}$ for $i < j$ and $1 \leq i, j \leq n$. Now, we prove that the constant of proportionality $k^f_{(u,v)}$ are all distinct, for distinct $u, v \in V(K_{1,n})$.

Case 1. For $i \neq j$, if possible

$$\begin{aligned} k^f_{(v_0,v_i)} = k^f_{(v_0,v_j)} &\Rightarrow \frac{|f(v_0) \oplus f(v_i)|}{d(v_0, v_i)} = \frac{|f(v_0) \oplus f(v_j)|}{d(v_0, v_j)} \\ &\Rightarrow \frac{2^{2i+1} - 0}{1} = \frac{2^{2j+1} - 0}{1} \Rightarrow 2^{2i+1} = 2^{2j+1}, \text{ a contradiction.} \end{aligned}$$

Case 2. For i, j, k and $j > k$, if possible

$$\begin{aligned} k^f_{(v_0,v_i)} = k^f_{(v_j,v_k)} &\Rightarrow \frac{|f(v_0) \oplus f(v_i)|}{d(v_0, v_i)} = \frac{|f(v_j) \oplus f(v_k)|}{d(v_j, v_k)} \\ &\Rightarrow \frac{2^{2i+1} - 0}{1} = \frac{2^{2j+1} - 2^{2k+1}}{2} \Rightarrow 2^{2i+1} = 2^{2j} - 2^{2k} \\ &\Rightarrow 2^{2i+1} = 2^{2k}(2^{2j-2k} - 1) \Rightarrow 2^{2i+1-2k} = 2^{2j-2k} - 1 (\text{if } 2i + 1 > 2k). \end{aligned}$$

Here the left hand side is even and right hand side is odd, a contradiction. Also $2i + 1 = 2k$ is not possible and for $2i + 1 < 2k$, a similar contradiction can be derived.

Case 3. Let $v_i, v_j, v_k, v_l, 1 \leq i, j, k, l \leq n$ are four vertices of $K_{1,n}$ with all the four vertices are distinct. We also assume with out loss of generality that $i < j, l < k$ and $i < l$.

$$\begin{aligned}
k^f_{(v_i,v_j)} = k^f_{(v_k,v_l)} &\Rightarrow \frac{|f(v_i) \oplus f(v_j)|}{d(v_i,v_j)} = \frac{|f(v_k) \oplus f(v_l)|}{d(v_k,v_l)} \\
&\Rightarrow \frac{2^{2j+1} - 2^{2i+1}}{2} = \frac{2^{2k+1} - 2^{2l+1}}{2} \Rightarrow 2^{2j} - 2^{2i} = 2^{2k} - 2^{2l} \\
&\Rightarrow 2^{2i}(2^{2j-2i} - 1) = 2^{2l}(2^{2k-2l} - 1) \Rightarrow (2^{2j-2i} - 1) = 2^{2l-2i}(2^{2k-2l} - 1),
\end{aligned}$$

a contradiction that left hand side is odd and right hand side is even. Now if $k^f_{(v_i,v_j)} = k^f_{(v_k,v_l)}$ and any two vertices are same then it is easy to see that the other two vertices are also same. Hence, $k^f_{(u,v)}$ are all distinct for all distinct $u, v \in V(K_{1,n})$, so that $K_{1,n}$ is dispersible dcsl-graph. □

Remark 1. *For $K_{1,n}$, $max\{d(u,v) : u, v \in V(K_{1,n})\} = 2$. The diameter of a connected graph G is defined as $max\{d(u,v) : u, v \in V(G)\}$ and is denoted by $diam(G)$. It can be shown for a graph G with $diam(G) \leq 2$ that it is dispersible dcsl-graph. The result is proved in the following statement.*

Theorem 4. *Any graph G for which $diam(G) \leq 2$, is dispersible dcsl-graph.*

Proof. Let G be a graph with $diam(G) \leq 2$ and $|V(G)| = n$. Choose any 1-1 function $g : V(G) \longrightarrow \{1, 3, 5, 7 \ldots\}$. Consider the function $f : V(G) \longrightarrow 2^{\mathbb{N}}$, where $\mathbb{N} = \{1, 2, 3, \ldots\}$ given by $f(v) = \{1, 2, 3, 4 \ldots, 2^{g(v)}\}$. We prove that f is a dispersive dcsl of G. Rename the vertices of G as $v \in V(G)$ changes to $v_{g(v)}$. We need to prove $k^f_{(v_i,v_j)} \neq k^f_{(v_k,v_l)}$ for all $v_i, v_j, v_k, v_l \in V(G)$. Assume the contrary that,

$$\frac{|f(v_i) \oplus f(v_j)|}{d(v_i,v_j)} = \frac{|f(v_k) \oplus f(v_l)|}{d(v_k,v_l)}.$$

Case 1. $d(v_i, v_j) = d(v_k, v_l) = 1$.
Subcase a. If $v_i = v_k$,

$$2^j - 2^i = 2^l - 2^k \Rightarrow 2^j - 2^i = 2^l - 2^i \Rightarrow 2^j = 2^l \Rightarrow j = l \Rightarrow v_j = v_l.$$

Similarly for $v_j = v_l \Rightarrow v_i = v_k$.
Subcase b. If $v_i = v_l$ and for $j > i > k$,

$$\begin{aligned}
2^j - 2^i = 2^l - 2^k &\Rightarrow 2^j + 2^k = 2^i + 2^l \Rightarrow 2^j + 2^k = 2^{i+1} \\
&\Rightarrow 2^k(2^{j-k} + 1) = 2^{i+1} \Rightarrow 2^{j-k} + 1 = 2^{i+1-k}.
\end{aligned}$$

Left hand side is odd and right hand side is even, a contradiction.
Subcase c. If $v_j = v_k$ and for $l > j > i$,

$$2^j - 2^i = 2^l - 2^k \Rightarrow 2^j + 2^k = 2^i + 2^l \Rightarrow 2^{j+1} = 2^i + 2^l \Rightarrow 2^{j+1-i} = 2^{l-i} + 1.$$

Here left hand side is even and right hand side is odd, a contradiction.
Case 1 implies that if any two vertices are same, either the other two must be same or we arrive

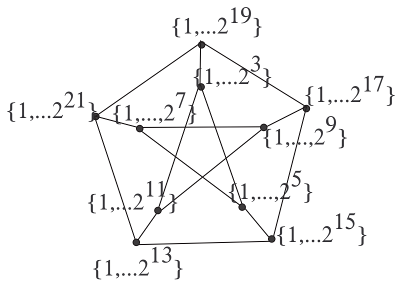

Figure 1: Dispersive dcsl of Peterson graph $[diam(P) = 2]$.

at a contradiction.
Case 2. $d(v_i, v_j) = d(v_k, v_l) = 2$.
Similar arguments of Case 1 implies that if any two vertices are same, either the other two must be same or we arrive at a contradiction.
Case 3. $d(v_i, v_j) = 2$ and $d(v_k, v_l) = 1$.
Subcase a. If $v_j = v_l$, then $v_i \neq v_k$ and for $j > i > k$,

$$2^{j-1} - 2^{i-1} = 2^l - 2^k \Rightarrow 2^{j-1} - 2^{i-1} = 2^j - 2^k \Rightarrow 2^k(2^{j-1-k} - 2^{i-1-k}) = 2^k(2^{j-k} - 1).$$

Left hand side is even and right hand side is odd, a contradiction. A similar contradiction can be obtained when $k > i$.
Subcase b. if $v_j = v_k$ and for $l > j > i$,

$$2^{j-1} - 2^{i-1} = 2^l - 2^k \Rightarrow 2^{j-1-(i-1)} - 1 = 2^{l-(i-1)} - 2^{j-(i-1)}.$$

A contradiction(left hand side is odd and right hand side is even).
Subcase c. If $v_i = v_l$ and for $j > i > k$,

$$2^{j-1} - 2^{i-1} = 2^l - 2^k \Rightarrow 2^{j-1} - 2^{i-1} = 2^i - 2^k \Rightarrow 2^k(2^{j-1-k} - 2^{i-1-k}) = 2^k(2^{i-k} - 1).$$

A contradiction(left hand side is even and right hand side is odd). Case 3 implies that if any two vertices are same, then we arrive at a contradiction.
Case 4. All the four vertices are distinct. if for any i, j, k, l distinct odd natural numbers,

$$2^j - 2^i \neq 2^l - 2^k, 2^{j-1} - 2^{i-1} \neq 2^{l-1} - 2^{k-1}$$

and $2^{j-1} - 2^{i-1} \neq 2^l - 2^k$. So in every case all the four vertices should be distinct, implies $k^f_{(u,v)}$ is distinct for every pair of vertices (u, v) of a connected graph G with $diam(G) \leq 2$. □

Corollary 1. *A graph G with a full degree vertex is dispersive dcsl-graph.*

Proof. Since G has a full degree vertex, $K_{1,n}$ is a spanning subgraph of G. So $diam(G) \leq 2$. □

Corollary 2. *$K_n, K_{m,n}, C_4, C_5$ and Peterson graph are dispersive dcsl-graphs.*

Corollary 3. *Join of two graphs is always dispersive dcsl-graphs.*

Proof. Since $diam(G_1 \vee G_2) \leq 2$ for any two graphs G_1 and G_2, by theorem 5 join of two graph is always dispersible. □

Corollary 4. *The Wheel graph $(K_1 \vee C_n)$ is dispersive dcsl-graph.*

Corollary 5. *A graph G with $\delta(G) > \frac{n}{2}$ is dispersible.*

Proof. Let $u, v \in V(G)$. Since degree of each vertex in G is greater than or equal to $\frac{n}{2}$, both u and v should have a common neighbor. Which in turn implies that $d(u, v) \leq 2$. This is true for any pair of vertices implies the $diam(G) \leq 2$. □

Remark 2. *It is proved in Theorem 4 that all the graphs with diameter less than or equal to two are dispersible. It does not imply that graphs with higher diameter are not dispersible. In fact for every n, we get a dispersible graph with $diam(G) = n$ as shown in the next Theorem 5.*

Theorem 5. *Paths are dispersible dcsl-graphs.*

Proof. Let $P_{n+1} = v_0v_1v_2 \ldots v_{n-1}v_n$ be a path of length n with $n+1$ vertices. Label the vertices with sets which are mutually disjoint and of size in the following way.

$$|f(v_0)| = 0,$$
$$|f(v_1)| = n!,$$
$$|f(v_i)| = i[|f(v_{i-1})| + |f(v_{i-2})|] + n!, \; for \; 2 \leq i \leq n+1.$$

Here the constant $k^f_{(v_0,v_i)}$ is greater than all other constants upto v_{i-1}. Also

$$k^f_{(v_0,v_i)} < k^f_{(v_1,v_i)} < \ldots < k^f_{(v_{i-1},v_i)}$$

for all $2 \leq i \leq n+1$. Since all the constants of proportionality are distinct, this dcsl is a dispersive dcsl. □

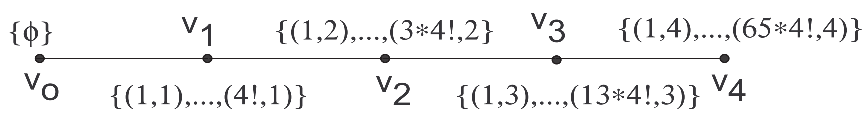

Figure 2: Dispersive dcsl of P_5.

2 CONCLUSION

Much work has been done when the constant of proportionality $k^f_{u,v}$ is a constant for every pair $(u, v) \in V(G) \times V(G)$ of a dcsl-graph G [2, 4, 5]. Here we proved that some classes of graphs are dispersible. But we did not get any graph which is not dispersible. Also dispersive dcsl is not unique for a dispersible graph. So some problems arise automatically.

1. What is the minimum cardinality of ground set X of dispersible graph G, denoted by $\nu(G)$?
2. Trees are dispersible?
3. Every graph admits a dispersive dcsl?
4. Any graph G with $diam(G) \leq 2$ is (k, r)-arithmetic?

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Джінто Дж., Ґерміна К.А., Шаіні П. *Деякі класи розсіюваних dcsl графів* // Карпатські матем. публ. — 2017. — Т.9, №2. — C. 128–133.

Сумісна з відстанями множина міток (dcsl) зв'язаного графа G є ін'єктивним відображенням множин $f : V(G) \to 2^X$, де X — непорожня базова множина така, що відповідна індукована функція $f^{\oplus} : E(G) \to 2^X \setminus \{\varphi\}$ задана як $f^{\oplus}(uv) = f(u) \oplus f(v)$ задовольняє умову $|f^{\oplus}(uv)| = k^f_{(u,v)} d_G(u,v)$ для кожної пари різних вершин $u, v \in V(G)$, де $d_G(u,v)$ позначає довжину шляху між u і v, та $k^f_{(u,v)}$ не обов'язково ціла константа, що залежить від пари обраних вершин u, v. G є графом з сумісною з відстанями множиною міток (dcsl-графом), якщо він дозволяє dcsl. Сумісна з відстанями множина міток f деякого (p,q)-графа G є дисперсною, якщо сталі пропорційності $k^f_{(u,v)}$ відносно $f, u \neq v, u, v \in V(G)$ є різними і G є дисперсним, якщо він доспускає дисперсну dcsl. У цій статті доведено, що всі шляхи і графи з діаметром не більшим 2 є дисперсними.

Ключові слова і фрази: множини міток графів, dcsl-граф, дисперсний dcsl-граф.