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BOUNDARY PROBLEM FOR THE SINGULAR HEAT EQUATION

The scheme for solving of a mixed problem with general boundary conditions is proposed for a heat equation

$$a(x)\frac{\partial T}{\partial \tau} = \frac{\partial}{\partial x}\left(\lambda(x)\frac{\partial T}{\partial x}\right)$$

with coefficient $a(x)$ that is the generalized derivative of a function of bounded variation, $\lambda(x) > 0$, $\lambda^{-1}(x)$ is a bounded and measurable function. The boundary conditions have the form

$$\begin{cases} p_{11}T(0,\tau) + p_{12}T_x^{[1]}(0,\tau) + q_{11}T(l,\tau) + q_{12}T_x^{[1]}(l,\tau) = \psi_1(\tau), \\ p_{21}T(0,\tau) + p_{22}T_x^{[1]}(0,\tau) + q_{21}T(l,\tau) + q_{22}T_x^{[1]}(l,\tau) = \psi_2(\tau), \end{cases}$$

where by $T_x^{[1]}(x,\tau)$ we denote the quasiderivative $\lambda(x)\frac{\partial T}{\partial x}$. A solution of this problem seek by the reduction method in the form of sum of two functions $T(x,\tau) = u(x,\tau) + v(x,\tau)$. This method allows to reduce solving of proposed problem to solving of two problems: a quasistationary boundary problem with initial and boundary conditions for the search of the function $u(x,\tau)$ and a mixed problem with zero boundary conditions for some inhomogeneous equation with an unknown function $v(x,\tau)$. The first of these problems is solved through the introduction of the quasiderivative. Fourier method and expansions in eigenfunctions of some boundary value problem for the second-order quasidifferential equation $(\lambda(x)X'(x))' + \omega a(x)X(x) = 0$ are used for solving of the second problem. The function $v(x,\tau)$ is represented as a series in eigenfunctions of this boundary value problem. The results can be used in the investigation process of heat transfer in a multilayer plate.

Key words and phrases: boundary problem, quasiderivative, eigenfunctions, Fourier method.

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INTRODUCTION

Boundary problems for differential equations of heat conduction with smooth coefficients were studied quite comprehensively in the literature (e.g., see [5]). However, during the modeling of heat transfer processes, the boundary problems with piecewise continuous coefficients or coefficients that have generalized derivatives of discontinuous functions are often appeared. Such problems have already begun to be studied in the works [3, 4].

The present paper deals with solving of a boundary problem for a heat equation with a coefficient that is the generalized derivative of a function of bounded variation. A reduction method [5] is used for solving of this problem. This method allows to reduce solving of this problem to solving of two problems: a quasistationary boundary problem with initial and boundary conditions and a mixed problem with zero boundary conditions for some inhomogeneous equation. Fourier method and expansions in eigenfunctions of some boundary value

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problem for the second-order quasidifferential equation are used for solving of the second of these problems.

Quasidifferential equations are equations that contain terms of the form $(p(x)y^{(m)})^{(n)}$. These equations cannot be reduced to conventional differential equations by n-fold differentiation if the coefficient $p(x)$ is not sufficiently smooth. The introduction of quasiderivatives is used for their research [2].

1 Formulation of the problem

Consider the next boundary value problem for a differential heat equation. It is necessary to find a solution $T(x, \tau)$ of the equation

$$a(x)\frac{\partial T}{\partial \tau} = \frac{\partial}{\partial x}\left(\lambda(x)\frac{\partial T}{\partial x}\right) \tag{1}$$

with boundary conditions

$$\begin{cases} p_{11}T(0,\tau) + p_{12}T_x^{[1]}(0,\tau) + q_{11}T(l,\tau) + q_{12}T_x^{[1]}(l,\tau) = \psi_1(\tau), \\ p_{21}T(0,\tau) + p_{22}T_x^{[1]}(0,\tau) + q_{21}T(l,\tau) + q_{22}T_x^{[1]}(l,\tau) = \psi_2(\tau) \end{cases} \tag{2}$$

and initial condition

$$T(x,0) = \varphi(x), \tag{3}$$

where $a(x) = b'(x)$, $b(x)$ is a right continuous nondecreasing real function of bounded variation on the interval $[0,l]$, $\lambda(x) > 0$, $\lambda^{-1}(x)$ is a bounded and measurable function on the interval $[0,l]$, $\varphi(x)$ is a continuous function on the interval $[0,l]$, $\psi_1(\tau)$ and $\psi_2(\tau)$ are continuously differentiable functions for $\tau \geqslant 0$, p_{ij}, q_{ij} $(i,j = 1,2)$ are real numbers. By $T_x^{[1]}(x,\tau)$ we denote the quasiderivative $\lambda(x)\frac{\partial T}{\partial x}$. The prime in the formula $a(x) = b'(x)$ stands for the generalized differentiation, and hence the function $a(x)$ is a measure, i.e., a zero-order distribution on the space of continuous compactly supported functions [1].

A solution of problem (1)–(3) seek by the reduction method in the form of sum of two functions

$$T(x,\tau) = u(x,\tau) + v(x,\tau). \tag{4}$$

Any of functions u or v can be chosen by a special way, then another one will be determined uniquely.

2 Quasistationary boundary problem for $u(x,\tau)$

We define $u(x,\tau)$ as the solution of the boundary problem

$$\frac{\partial}{\partial x}\left(\lambda(x)\frac{\partial u}{\partial x}\right) = 0, \tag{5}$$

$$\begin{cases} p_{11}u(0,\tau) + p_{12}u_x^{[1]}(0,\tau) + q_{11}u(l,\tau) + q_{12}u_x^{[1]}(l,\tau) = \psi_1(\tau), \\ p_{21}u(0,\tau) + p_{22}u_x^{[1]}(0,\tau) + q_{21}u(l,\tau) + q_{22}u_x^{[1]}(l,\tau) = \psi_2(\tau), \end{cases} \tag{6}$$

which is derived from problem (1)–(3) if τ is a parameter. Here the quasiderivative $u_x^{[1]}(x,\tau) \overset{df}{=} \lambda(x)\frac{\partial u}{\partial x}$, then $\frac{\partial u}{\partial x} = \frac{u^{[1]}}{\lambda(x)}$. With the help of the vector $\bar{u} = (u, u^{[1]})^T$ equation (5) is reduced to the system

$$\begin{pmatrix} u \\ u^{[1]} \end{pmatrix}' = \begin{pmatrix} 0 & \frac{1}{\lambda(x)} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} u \\ u^{[1]} \end{pmatrix}. \tag{7}$$

Boundary conditions (6) are also represented in the vector form

$$P \cdot \bar{u}(0,\tau) + Q \cdot \bar{u}(l,\tau) = \bar{\Gamma}(\tau), \tag{8}$$

where

$$P = \begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix}, \quad Q = \begin{pmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{pmatrix}, \quad \bar{\Gamma}(\tau) = \begin{pmatrix} \psi_1(\tau) \\ \psi_2(\tau) \end{pmatrix}.$$

By direct verification one can make sure such that the Cauchy matrix $B(x,s)$ of system (7) has the form

$$B(x,s) = \begin{pmatrix} 1 & \sigma(x,s) \\ 0 & 1 \end{pmatrix}, \quad \sigma(x,s) = \int_s^x \frac{dt}{\lambda(t)}.$$

Then $\bar{u}(x,\tau) = B(x,0)\bar{u}_0$, where $\bar{u}_0 = \bar{u}(0,\tau)$. We shall determine $\bar{u}_0$. From boundary conditions (8) we obtain $P \cdot \bar{u}_0 + Q \cdot B(l,0) \cdot \bar{u}_0 = \bar{\Gamma}$ whence $\bar{u}_0 = (P + Q \cdot B(l,0))^{-1} \cdot \bar{\Gamma}$. Therefore,

$$\bar{u}(x,\tau) = B(x,0) \cdot (P + Q \cdot B(l,0))^{-1} \cdot \bar{\Gamma}(\tau). \tag{9}$$

3 Mixed problem for $v(x,\tau)$

We substitute $u(x,\tau)$ and $v(x,\tau)$ into equation (1)

$$a(x)\left(\frac{\partial u}{\partial \tau} + \frac{\partial v}{\partial \tau}\right) = \frac{\partial}{\partial x}\left(\lambda(x)\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial x}\right)\right).$$

In consequence of (5) we have the equation

$$a(x)\frac{\partial v}{\partial \tau} = \frac{\partial}{\partial x}\left(\lambda(x)\frac{\partial v}{\partial x}\right) - a(x)\frac{\partial u}{\partial \tau}. \tag{10}$$

According to formula (9) the derivative $\frac{\partial u}{\partial \tau}$ is a continuous function of the variable x on $[0,l]$ and so the last term in equation (10) is correct.

By taking into account formula (4), we define the boundary conditions for v from conditions (2)

$$\begin{aligned} p_{11}u(0,\tau) &+ p_{12}u_x^{[1]}(0,\tau) + q_{11}u(l,\tau) + q_{12}u_x^{[1]}(l,\tau) \\ &+ p_{11}v(0,\tau) + p_{12}v_x^{[1]}(0,\tau) + q_{11}v(l,\tau) + q_{12}v_x^{[1]}(l,\tau) = \psi_1(\tau), \\ p_{21}u(0,\tau) &+ p_{22}u_x^{[1]}(0,\tau) + q_{21}u(l,\tau) + q_{22}u_x^{[1]}(l,\tau) \\ &+ p_{21}v(0,\tau) + p_{22}v_x^{[1]}(0,\tau) + q_{21}v(l,\tau) + q_{22}v_x^{[1]}(l,\tau) = \psi_2(\tau). \end{aligned}$$

By virtue of (6), we obtain

$$\begin{cases} p_{11}v(0,\tau) + p_{12}v_x^{[1]}(0,\tau) + q_{11}v(l,\tau) + q_{12}v_x^{[1]}(l,\tau) = 0, \\ p_{21}v(0,\tau) + p_{22}v_x^{[1]}(0,\tau) + q_{21}v(l,\tau) + q_{22}v_x^{[1]}(l,\tau) = 0. \end{cases} \tag{11}$$

The initial condition is determined similarly

$$v(x,0) = \varphi(x) - u(x,0) \overset{df}{=} \tilde{\varphi}(x). \tag{12}$$

4 FOURIER METHOD AND EIGENVALUE PROBLEM

We search for non-trivial solutions of the homogeneous differential equation

$$a(x)\frac{\partial v}{\partial \tau} = \frac{\partial}{\partial x}\left(\lambda(x)\frac{\partial v}{\partial x}\right)$$

with boundary conditions (11) in the form

$$v(x,\tau) = e^{-\omega\tau}X(x), \tag{13}$$

where ω is a parameter, and $X(x)$ is a function. Then

$$-\omega a(x)e^{-\omega\tau}X(x) = (\lambda(x)X'(x))'e^{-\omega\tau}$$

whence we get the quasidifferential equation

$$(\lambda(x)X'(x))' + \omega a(x)X(x) = 0. \tag{14}$$

Substituting formula (13) in boundary conditions (11), we obtain

$$\begin{cases} p_{11}X(0) + p_{12}X^{[1]}(0) + q_{11}X(l) + q_{12}X^{[1]}(l) = 0, \\ p_{21}X(0) + p_{22}X^{[1]}(0) + q_{21}X(l) + q_{22}X^{[1]}(l) = 0. \end{cases} \tag{15}$$

We denote by ω_k the eigenvalues of boundary problem (14), (15). Let $X_k(\omega_k, x)$ be the corresponding eigenfunctions, $k = 1, 2, \ldots, \infty$.

By [6], all eigenvalues ω_k of boundary problem (14), (15) are real, there are a countable number of them, and their set has not a finite limit point. The eigenfunctions $X_k(\omega_k, x)$ that are corresponded to the different eigenvalues are orthogonal in the sense

$$\int_0^l X_m(\omega_m, x)X_n(\omega_n, x)db(x) = 0, \quad \omega_m \neq \omega_n.$$

5 METHOD OF THE EIGENFUNCTIONS

We seek $v(x,\tau)$ in the form of the series

$$v(x,\tau) = \sum_{k=1}^{\infty} t_k(\tau)X_k(\omega_k, x), \tag{16}$$

where $X_k(\omega_k, x)$ are the eigenfunctions of boundary problem (14), (15). We substitute formula (16) into equation (10)

$$a(x)\frac{\partial}{\partial \tau}\left(\sum_{k=1}^{\infty} t_k(\tau)X_k\right) = \frac{\partial}{\partial x}\left(\lambda(x)\frac{\partial}{\partial x}\left(\sum_{k=1}^{\infty} t_k(\tau)X_k\right)\right) - a(x)\frac{\partial u}{\partial \tau}$$

where, under the assumption of uniform convergence of series (16) and series derived from it by differentiation by x or τ, we have

$$a(x)\sum_{k=1}^{\infty} t_k'(\tau)X_k = \sum_{k=1}^{\infty} t_k(\tau)\big(\lambda(x)X_k'\big)' - a(x)\frac{\partial u}{\partial \tau}.$$

As a result of equation (14) there is equality $\left(\lambda(x)X_k'\right)' = -\omega_k a(x)X_k$, then

$$a(x)\sum_{k=1}^{\infty} t_k'(\tau)X_k = -\sum_{k=1}^{\infty} t_k(\tau)\omega_k a(x)X_k - a(x)\frac{\partial u}{\partial \tau}.$$

Therefore,

$$\sum_{k=1}^{\infty}\left(t_k'(\tau)+\omega_k t_k(\tau)\right)X_k = -\frac{\partial u}{\partial \tau}. \tag{17}$$

We expand the known function $\frac{\partial u}{\partial \tau}$ in a series in the eigenfunctions of boundary problem (14), (15):

$$\frac{\partial u}{\partial \tau} = \sum_{k=1}^{\infty} d_k(\tau)X_k(\omega_k, x), \tag{18}$$

where

$$d_k(\tau) = \frac{1}{\|X_k\|}\int_0^l \frac{\partial u}{\partial \tau}X_k(\omega_k, x)db(x), \quad \|X_k\| = \int_0^l X_k^2(\omega_k, x)db(x).$$

By substituting formula (18) into (17), we obtain

$$t_k'(\tau)+\omega_k t_k(\tau) = -d_k(\tau), \quad k = 1, 2, \ldots, \infty. \tag{19}$$

Since formulas (12) and (16), we have

$$v(x,0) = \sum_{k=1}^{\infty} t_k(0)X_k(\omega_k, x) \equiv \tilde{\varphi}(x).$$

We expand the function $\tilde{\varphi}(x)$ in a series in the eigenfunctions

$$\tilde{\varphi}(x) = \sum_{k=1}^{\infty}\varphi_k X_k(\omega_k, x), \quad \varphi_k = \frac{1}{\|X_k\|}\int_0^l \tilde{\varphi}(x)X_k(\omega_k, x)db(x).$$

Consequently,

$$t_k(0) = \varphi_k, \quad k = 1, 2, \ldots, \infty. \tag{20}$$

Then for all positive integer k we have Cauchy problems (19), (20) for ordinary differential equations.

General solutions of linear inhomogeneous equations (19) acquire the formulas

$$t_k(\tau) = \left(C_k - \int_0^\tau d_k(s)e^{\omega_k s}ds\right)e^{-\omega_k\tau},$$

where C_k are arbitrary constants. Therefore, by using initial conditions (20), we find for each positive integer k the solution of the corresponding Cauchy problem

$$t_k(\tau) = \varphi_k e^{-\omega_k\tau} - \int_0^\tau d_k(s)e^{\omega_k(s-\tau)}ds.$$

Then, by virtue of formula (16), we obtain

$$v(x,\tau) = \sum_{k=1}^{\infty}\left(\varphi_k e^{-\omega_k\tau} - \int_0^\tau d_k(s)e^{\omega_k(s-\tau)}ds\right)X_k(\omega_k, x).$$

Thus, by using the reduction method, Fourier method and the expansion in a series in eigenfunctions, we built the solution of the boundary problem for the heat equation with a distribution. The results can be used in the investigation of the process of heat transfer in a multilayer plate.

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Махней О. В. *Крайова задача для сингулярного рівняння теплопровідності* // Карпатські матем. публ. — 2017. — Т.9, №1. — С. 86–91.

Запропоновано схему розв'язування мішаної задачі за загальних крайових умов для рівняння теплопровідності

$$a(x)\frac{\partial T}{\partial \tau} = \frac{\partial}{\partial x}\left(\lambda(x)\frac{\partial T}{\partial x}\right)$$

з коефіцієнтом $a(x)$, який є узагальненою похідною функції обмеженої варіації, $\lambda(x) > 0$, $\lambda^{-1}(x)$ – обмежена і вимірна функція. Крайові умови мають вигляд

$$\begin{cases} p_{11}T(0,\tau) + p_{12}T_x^{[1]}(0,\tau) + q_{11}T(l,\tau) + q_{12}T_x^{[1]}(l,\tau) = \psi_1(\tau), \\ p_{21}T(0,\tau) + p_{22}T_x^{[1]}(0,\tau) + q_{21}T(l,\tau) + q_{22}T_x^{[1]}(l,\tau) = \psi_2(\tau), \end{cases}$$

де через $T_x^{[1]}(x,\tau)$ позначено квазіпохідну $\lambda(x)\frac{\partial T}{\partial x}$. Розв'язок цієї задачі шукається методом редукції у вигляді суми двох функцій $T(x,\tau) = u(x,\tau) + v(x,\tau)$. Цей метод дає змогу звести розв'язування поставленої задачі до розв'язування двох задач: крайової квазістаціонарної задачі з початковими і крайовими умовами для відшукання функції $u(x,\tau)$ і мішаної задачі з нульовими крайовими умовами для деякого неоднорідного рівняння з невідомою функцією $v(x,\tau)$. Перша з цих задач розв'язується з допомогою введення квазіпохідної. Для розв'язування другої задачі застосовується метод Фур'є і розвинення за власними функціями деякої крайової задачі для квазідиференціального рівняння другого порядку $(\lambda(x)X'(x))' + \omega a(x)X(x) = 0$. Функція $v(x,\tau)$ подається у вигляді ряду за власними функціями цієї крайової задачі. Отримані результати можна використовувати для дослідження процесу теплопередачі в багатошаровій плиті.

Ключові слова і фрази: крайова задача, квазіпохідна, власні функції, метод Фур'є.