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AN EXAMPLE OF A NON-BOREL LOCALLY-CONNECTED FINITE-DIMENSIONAL TOPOLOGICAL GROUP

According to a classical theorem of Gleason and Montgomery, every finite-dimensional locally path-connected topological group is a Lie group. In the paper for every natural number n we construct a locally connected subgroup $G \subset \mathbb{R}^{n+1}$ of dimension n, which is not locally compact. This answers a question posed by S. Maillot on MathOverflow and shows that the local path-connectedness in the result of Gleason and Montgomery can not be weakened to the local connectedness.

Key words and phrases: topological group, Lie group.

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By a classical result of A. Gleason [3] and D. Montgomery [6], every locally path-connected finite-dimensional topological group G is locally compact and hence is a Lie group. Generalizing this result of A. Gleason and D. Montgomery, T. Banakh and L. Zdomskyy [1] proved that a topological group G is a Lie group if G is compactly finite-dimensional and locally continuum-connected. In [5] Sylvain Maillot asked if the locally path-connectedness in the result of A. Gleason and D. Montgomery can be replaced by the local connectedness. In this paper we construct a counterexample to this question of S. Maillot.

We recall that a subset B of a Polish space X is called a *Bernstein set* in X if both B and $X \setminus B$ meet every uncountable closed subset F of X. Bernstein sets in Polish space can be easily constructed by transfinite induction, see [4, 8.24].

Theorem 1. *For every $n \geq 2$ the Euclidean space $\mathbb{R}^n$ contains a dense additive subgroup $G \subset \mathbb{R}^n$ such that*

1) *G is a Bernstein set in $\mathbb{R}^n$;*
2) *G is locally connected;*
3) *G has dimension $\dim(G) = n - 1$;*
4) *G is not Borel and hence not locally compact.*

Proof. Let $(F_\alpha)_{\alpha<\mathfrak{c}}$ be an enumeration of all uncountable closed subsets of $\mathbb{R}^n$ by ordinal $< \mathfrak{c}$. Fix any point $p \in \mathbb{R}^n \setminus \{\mathbf{0}\}$. By transfinite induction, for every ordinal $\alpha < \mathfrak{c}$ we shall choose a point $z_\alpha \in F_\alpha$ such that the subgroup $G_\alpha \subset \mathbb{R}^n$ generated by the set $\{z_\beta\}_{\beta<\alpha}$ does not contain

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the point p. Assume that for some ordinal $\alpha < \mathfrak{c}$ we have chosen points z_β, $\beta < \alpha$, so that the subgroup $G_{<\alpha}$ generated by the set $\{z_\beta\}_{\beta<\alpha}$ does not contain p. Consider the set

$$Z = \left\{ \frac{1}{n}(p - g) : n \in \mathbb{Z} \setminus \{0\},\ g \in G_{<\alpha} \right\}$$

and observe that it has cardinality

$$|Z| \le \omega \cdot |G_{<\alpha}| \le \omega + |\alpha| < \mathfrak{c}.$$

Since the uncountable closed subset F_α of $\mathbb{R}^n$ has cardinality $|F_\alpha| = \mathfrak{c}$ (see [4, 6.5]), there is a point $z_\alpha \in F_\alpha \setminus Z$. For this point we get $p \neq nz_\alpha + g$ for any $n \in \mathbb{Z} \setminus \{0\}$, and $g \in G_{<\alpha}$. Consequently, the subgroup

$$G_\alpha = \{nz_\alpha + g : n \in \mathbb{Z},\ g \in G_{<\alpha}\}$$

generated by the set $\{z_\beta\}_{\beta\le\alpha}$ does not contain the point p. This completes the inductive step.

After completing the inductive construction, consider the subgroup G generated by the set $\{a_\alpha\}_{\alpha<\mathfrak{c}}$ and observe that $p \notin G$ and G meets every uncountable closed subset F of $\mathbb{R}^n$. Moreover, since G meets the closed uncountable set $F - p$, the coset $p + G \subset \mathbb{R}^n \setminus G$ meets F. So, both the subgroup G and its complement $\mathbb{R}^n \setminus G$ meet each uncountable closed subset of $\mathbb{R}^n$, which means that G is a Bernstein set in $\mathbb{R}^n$. The following proposition implies that the group G has properties (2)–(4). □

Proposition 1. *Let $n \ge 2$. Every Bernstein subset B of $\mathbb{R}^n$ has the following properties:*

1) *B is not Borel;*
2) *B is connected and locally connected;*
3) *B has dimension $\dim(B) = n - 1$.*

Proof. 1. By [4, 8.24], the Bernstein set B is not Borel (more precisely, B does not have the Baire property in $\mathbb{R}^n$).

2. To prove that B is connected and locally connected, it suffices to prove that for every open subset $U \subset \mathbb{R}^n$ homeomorphic to $\mathbb{R}^n$ the intersection $U \cap G$ is connected. Assuming the opposite, we could find two non-empty open disjoint sets $U_1, U_2 \subset U$ such that $U \cap B = (U_1 \cap B) \cup (U_2 \cap B)$. Consider the complement $F = U \setminus (U_1 \cup U_2) \subset U \setminus B$ and observe that F is closed in U and hence of type F_σ in $\mathbb{R}^n$. If F is uncountable, then F contains an uncountable closed subset of $\mathbb{R}^n$ and hence meets the set B, which is not the case. So, the closed subset F of U is at most countable and separates the space $U \cong \mathbb{R}^n$, which contradicts Theorem 1.8.14 of [2].

3. Since the subset B has empty interior in $\mathbb{R}^n$, we can apply Theorem 1.8.11 of [2] and conclude that $\dim(B) < n$. On the other hand, Lemma 1.8.16 [2] guarantees that B has dimension $\dim(B) \ge n - 1$ (since B meets every non-trivial compact connected subset of $\mathbb{R}^n$). So, $\dim(B) = n - 1$. □

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Згідно з класичною теоремою Ґлісона-Монтґомері, довільна скінченно-вимірна локально лінійно зв'язна топологічна група є групою Лі. У статті для кожного натурального числа n побудовано локально зв'язну, але не локально компактну адитивну підгрупу $G \subset \mathbb{R}^{n+1}$ топологічного виміру n. Цей приклад дає відповідь на проблему С. Мейло, поставлену на MathOverflow, та показує, що локально лінійну зв'язність у теоремі Ґлісона-Монтґомері не можна послабити до локальної зв'язності.

Ключові слова і фрази: топологічна група, група Лі.