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A CLASS OF JULIA EXCEPTIONAL FUNCTIONS

The class of p-loxodromic functions (meromorphic functions, satisfying the condition $f(qz) = pf(z)$ for some $q \in \mathbb{C}\backslash\{0\}$ and all $z \in \mathbb{C}\backslash\{0\}$) is studied. Each p-loxodromic function is Julia exceptional. The representation of these functions as well as their zero and pole distribution are investigated.

Key words and phrases: p-loxodromic function, the Schottky-Klein prime function, Julia exceptionality.

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INTRODUCTION

Denote $\mathbb{C}^* = \mathbb{C}\backslash\{0\}$, and let $q, p \in \mathbb{C}^*$, $|q| < 1$.

Definition 1. *A meromorphic in $\mathbb{C}^*$ function f is said to be p-loxodromic of multiplicator q if for every $z \in \mathbb{C}^*$*

$$f(qz) = pf(z). \tag{1}$$

Let $\mathcal{L}_{qp}$ denotes the class of p-loxodromic functions of multiplicator q.

The case $p = 1$ has been studied earlier in the works of O. Rausenberger [9], G. Valiron [11] and Y. Hellegouarch [5]. In his work [3, p. 133] which A. Ostrowski [8] called "besonders schöne und überraschende" G. Julia gave an example of a meromorphic in the punctured plane $\mathbb{C}^*$ function satisfying (1) with $p = 1$ for some non-zero q, $|q| \neq 1$, and all $z \in \mathbb{C}^*$. He noted that the family $\{f_n(z)\}$, $f_n(z) = f(q^n z)$ is normal [7] in $\mathbb{C}^*$ because $f_n(z) = f(z)$ for all $z \in \mathbb{C}^*$.

If $p = 1$ the function f is called loxodromic. Loxodromic functions of multiplicator q form a field, which is denoted by $\mathcal{L}_q$. The set $\mathcal{L}_{qp}$ forms an Abelian group with respect to addition.

It is obvious that a ratio of two functions from $\mathcal{L}_{qp}$ is a loxodromic function, and the derivative of the loxodromic function is p-loxodromic with $p = \frac{1}{q}$.

Remark 1. *Every $f \equiv const$ belongs to $\mathcal{L}_q$, but the unique constant function belonging to $\mathcal{L}_{qp}$ is $f \equiv 0$.*

If $f \in \mathcal{L}_{qp}$ and a is a zero of f, then aq^n, $n \in \mathbb{Z}$, are as well. That is, in the case of non-positive q the zeros of f lay on a logarithmic spiral. Let $a = |a|e^{i\alpha}$, $q = |q|e^{i\gamma}$. Then the logarithmic spiral in polar coordinates (r, φ) takes the form

$$\log r - \log|a| = k(\varphi - \alpha),$$

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where $k = \frac{\log|q|}{\gamma}$. The same concerns the poles of f. The image of a logarithmic spiral on the Riemann sphere by the stereographic projection intersects each meridian at the same angle and is called loxodromic curve ($\lambda o \xi o \zeta$ - oblique, $\delta \rho o \mu o \zeta$ - way). That is why we call (following G. Valiron) the function from $\mathcal{L}_q$ loxodromic.

Remark 2. *If $f \in \mathcal{L}_q$ and z is its a-point, $a \in \mathbb{C} \cup \{\infty\}$, then $q^n z, n \in \mathbb{Z}$, are its a-points too. In the case, $f \in \mathcal{L}_{qp}$, the previous considerations are valid only for the zeros and the poles of f.*

It is easy to verify, that $\mathcal{L}_{qp}$ forms the linear spaces over the fields $\mathbb{C}$ and $\mathcal{L}_q$. Also it is clear that $\mathcal{L}_{qp}$ has the following properties.

Proposition. *The linear space $\mathcal{L}_{qp}$ has the following properties.*

1. *The map $D : f(z) \mapsto zf'(z)$ maps $\mathcal{L}_{qp}$ to $\mathcal{L}_{qp}$.*
2. *The map $D_l : f(z) \mapsto z\frac{f'(z)}{f(z)}$ maps $\mathcal{L}_{qp}$ to $\mathcal{L}_q$.*
3. *$f(z) \in \mathcal{L}_{qp} \Rightarrow f(\frac{1}{z}) \in \mathcal{L}_{q\frac{1}{p}}$.*

Let us give nontrivial example of p-loxodromic function of multiplicator q. Put

$$h(z) = \prod_{n=1}^{\infty}(1 - q^n z), \quad 0 < |q| < 1.$$

Definition 2. *The function*

$$P(z) = (1-z)h(z)h\left(\frac{1}{z}\right) = (1-z)\prod_{n=1}^{\infty}(1-q^n z)(1-\frac{q^n}{z})$$

is called the Schottky-Klein prime function.

This function is holomorphic in $\mathbb{C}^*$ with zero sequence $\{q^n\}$, $n \in \mathbb{Z}$. It was introduced by Schottky [10] and Klein [6] for the study of conformal mappings of doubly-connected domains, see also [2].

It is easy to obtain the following property of P

$$P(qz) = -\frac{1}{z}P(z). \tag{2}$$

Example 1. *Consider the function*

$$f(z) = \frac{P\left(\frac{z}{p}\right)}{P(z)}.$$

Using (2)*, it is easy to show that $f \in \mathcal{L}_{qp}$.*

1 THE NUMBERS OF ZEROS AND POLES OF p-LOXODROMIC FUNCTIONS IN AN ANNULUS

Let $A_q(R) = \{z \in \mathbb{C} : |q|R < |z| \leq R\}$, $R > 0$ and $A_q = A_q(1)$.

Theorem 1. *Let $f \in \mathcal{L}_{qp}$ and the boundary of $A_q(R)$ contains neither zeros nor poles of f. Then f has equal numbers of zeros and poles (counted according to their multiplicities) in every $A_q(R)$.*

Proof. Let $\Gamma_1 = \{z \in \mathbb{C} : |z| = |q|R\}$ and $\Gamma_2 = \{z \in \mathbb{C} : |z| = R\}$ denote the circles bounding $A_q(R)$. Let $n(f)$ be the number of poles of f in $A_q(R)$.

By the argument principle, we have

$$n\left(\frac{1}{f}\right) - n(f) = \frac{1}{2i\pi}\left(\int\limits_{\Gamma_2^+} \frac{f'(z)}{f(z)}dz - \int\limits_{\Gamma_1^+} \frac{f'(\xi)}{f(\xi)}d\xi\right). \tag{3}$$

Setting $\xi = qz$ in the second integral of (3), we obtain

$$n\left(\frac{1}{f}\right) - n(f) = \frac{1}{2i\pi}\int\limits_{\Gamma_2^+} \left(\frac{f'(z)}{f(z)} - q\frac{f'(qz)}{f(qz)}\right) dz. \tag{4}$$

Since $f \in \mathcal{L}_{qp}$, the relation (1) implies

$$f'(qz) = \frac{p}{q}f'(z). \tag{5}$$

Putting (1) and (5) in (4), we obtain the conclusion of the theorem. □

Remark 3. *Every non-constant loxodromic function of multiplicator q has at least two poles (and two zeros) in every annulus $A_q(R)$ [5]. As we see from Example 1, the p-loxodromic function f has the unique pole $z = 1$ in A_q. This is an essential difference between loxodromic and p-loxodromic functions with $p \neq 1$.*

2 REPRESENTATION OF p-LOXODROMIC FUNCTIONS

The representation of loxodromic functions from $\mathcal{L}_q$ was given in [11], [5]. The following theorem gives the representation of a function from $\mathcal{L}_{qp}$.

Let $a_1, ..., a_m$ and $b_1, ..., b_m$ be the zeros and the poles of $f \in \mathcal{L}_{qp}$ in A_q respectively. Denote

$$\lambda = \frac{a_1 \cdot ... \cdot a_m}{b_1 \cdot ... \cdot b_m}. \tag{6}$$

Theorem 2. *The non-identical zero meromorphic in $\mathbb{C}^*$ function f belongs to $\mathcal{L}_{qp}$, $p \neq 1$, if and only if there exists $\nu \in \mathbb{Z}$ such that $\lambda = \frac{p}{q^\nu}$ and f has the form*

$$f(z) = cz^\nu \frac{P\left(\frac{z}{a_1}\right) \cdot ... \cdot P\left(\frac{z}{a_m}\right)}{P\left(\frac{z}{b_1}\right) \cdot ... \cdot P\left(\frac{z}{b_m}\right)}, \tag{7}$$

where c is a constant.

Proof. Firstly, denote

$$M(z) = \frac{P\left(\frac{z}{a_1}\right) \cdot ... \cdot P\left(\frac{z}{a_m}\right)}{P\left(\frac{z}{b_1}\right) \cdot ... \cdot P\left(\frac{z}{b_m}\right)}$$

and consider the function

$$g(z) = \frac{f(z)}{M(z)}.$$

Since the functions f and M have the same zeros and poles, it follows that their ratio g is holomorphic in $\mathbb{C}^*$ function. Let $g(z) = \sum\limits_{n=-\infty}^{+\infty} c_n z^n$ be the Laurant expansion of g in $\mathbb{C}^*$. Using relation (1) and the equality (2), we obtain

$$\lambda g(qz) = pg(z). \tag{8}$$

According to (8), we obtain

$$\lambda \sum_{n=-\infty}^{+\infty} c_n q^n z^n = p \sum_{n=-\infty}^{+\infty} c_n z^n$$

for any $z \in \mathbb{C}^*$. This implies $\lambda c_n q^n = pc_n$ or $c_n(\lambda q^n - p) = 0$. Then there exists at least one $c_\nu \neq 0, \nu \in \mathbb{Z}$, such that

$$c_\nu(\lambda q^\nu - p) = 0. \tag{9}$$

Hence, the relation (9) implies $q^\nu = \dfrac{p}{\lambda}$. We see also that $c_n = 0$ if $n \neq \nu$, so we have $g(z) = c_\nu z^\nu$. Thus, we can conclude

$$f(z) = g(z)M(z) = cz^\nu M(z),$$

where c is a constant.

Secondly, we have $f(z) = cz^\nu M(z)$, $\nu \in \mathbb{Z}$. Show that it belongs to $\mathcal{L}_{qp}$. Thus, $f(qz) = cq^\nu z^\nu M(qz)$. Indeed, using (2), we obtain

$$f(qz) = cq^\nu z^\nu \lambda M(z) = pf(z).$$

This completes the proof.

□

Corollary 1. *Assume $f \in \mathcal{L}_{qp}$, if f is holomorphic in $\mathbb{C}^*$, then $f(z) \equiv 0$ or there exists $k \in \mathbb{Z}\backslash\{0\}$ such that $p = q^k$ and $f(z) = cz^k$, where c is a constant. Conversely, a holomorphic in $\mathbb{C}^*$ function of the form $f(z) = cz^k$, where $k \in \mathbb{Z}\backslash\{0\}$, c is a constant, belongs to $\mathcal{L}_{qp}$.*

3 ZERO AND POLE DISTRIBUTION

Let $\{a_j\}$, $\{b_j\}$, $j \in \mathbb{Z}$ be a couple of sequences in $\mathbb{C}^*$, $p \neq 1$. Put

$$\mu(r) = [\log r / \log |q|] - 1.$$

Note that $\mu(r) = 0$ if $|q| \leq r < 1$. Denote

$$\mathfrak{M}_\nu(r) = \frac{1}{|p|^{\mu(r)}} \times \begin{cases} r^\nu \dfrac{\prod\limits_{1<|a_j|\leq r} \frac{r}{|a_j|}}{\prod\limits_{1<|b_j|\leq r} \frac{r}{|b_j|}}, & r > 1; \\[2ex] r^\nu \dfrac{\prod\limits_{r<|a_j|\leq 1} \frac{|a_j|}{r}}{\prod\limits_{r<|b_j|\leq 1} \frac{|b_j|}{r}}, & 0 < r \leq 1. \end{cases}$$

Theorem 3. *The zero sequence $\{a_j\}$ and the pole sequence $\{b_j\}$ of a non-identical zero meromorphic p-loxodromic function of multiplicator q satisfy the following conditions:*

(i) *the number of a_j and b_j in every annulus of the form $\{z : r < |z| < 2r\}$, $r > 0$ is bounded by an absolute constant;*

(ii) *the difference between the numbers of a_j and b_k in every annulus $\{z : r_1 < |z| < r_2\}$, $0 < r_1 < r_2 < +\infty$ is bounded by an absolute constant;*

(iii) *there exists $C_1 > 0$ such that $\left|\frac{a_j}{b_k} - 1\right| > C_1$ for every $j, k \in \mathbb{Z}$;*

(iv) *the function $\mathfrak{M}_\nu(r)$, where $\nu \in \mathbb{Z}$ such that $\lambda = \frac{p}{q^\nu}$, and λ is given by* (6)*, is bounded for $r > 0$.*

Proof. Let f be a p-loxodromic of multiplicator q function. If f is holomorphic then by Corollary 1 there exists $k \in \mathbb{Z}\backslash\{0\}$ such that $f(z) = cz^k$, and c is a constant. Hence, f has no zeros in $\mathbb{C}^*$. So there is nothing to prove.

Let f be meromorphic. Then by Remark 2 and Theorem 1 it has infinitely many zeros and poles.

(i) First we remark that there exists a unique $n_0 \in \mathbb{Z}_+$ such that $\dfrac{1}{|q|^{n_0}} \leq 2 < \dfrac{1}{|q|^{n_0+1}}$. This n_0 is equal to $\left[\dfrac{\log 2}{\log \frac{1}{|q|}}\right]$.

Since every annulus $\{z : \dfrac{r}{|q|^k} < |z| \leq \dfrac{r}{|q|^{k+1}}\}$, where $k \in \mathbb{Z}$, contains the same number of zeros of f, say m, and

$$(r, 2r] = \left(\bigcup_{k=0}^{n_0-1}\left(\frac{r}{|q|^k}, \frac{r}{|q|^{k+1}}\right]\right) \cup \left(\frac{r}{|q|^{n_0}}, 2r\right]$$

it follows that the annulus $\{z : r < |z| \leq 2r\}$ contains at least $n_0 m$ and less than $(n_0+1)m$ zeros of f. The same is true about the poles of f.

(ii) Similarly as in (i) we can find unique $n_1, n_2 \in \mathbb{Z}$ such that

$$|q|^{n_1+1} < r_1 \leq |q|^{n_1} < |q|^{n_2} < r_2 \leq |q|^{n_2-1}.$$

Hence

$$(r_1, r_2) = (r_1, |q|^{n_1}] \cup \left(\bigcup_{k=n_1}^{n_2-1}(|q|^k, |q|^{k+1}]\right) \cup (|q|^{n_2}, r_2).$$

Every annulus of the form $\{z : |q|^{k+1} < |z| \leq |q|^k\}$, where $k \in \mathbb{Z}$, contains equal amount of zeros and poles of f counted according to their multiplicities (we have denoted this number by m). Therefore the difference between the numbers of zeros and poles of f in the annulus $\{z : r_1 < |z| < r_2\}$ is no greater than $2m$ because of the choice of n_1, n_2.

(iii) Let $a_1, a_2, ..., a_m$ and $b_1, b_2, ..., b_m$ be the zeros and the poles of f in $\{z : |q| < |z| \leq 1\}$ respectively. Then all the zeros of f have the form $\alpha_{\mu,k} = a_k q^\mu$, where $\mu \in \mathbb{Z}$, $k = 1, 2, ..., m$.

The same is true about the poles of f, namely $\beta_{\nu,k} = b_k q^\nu$, where $\nu \in \mathbb{Z}$, $k = 1,2,...,m$. So, $\frac{\alpha_{\mu,j}}{\beta_{\nu,k}} = \frac{a_j}{b_k} q^l$, where $l \in \mathbb{Z}$.

It is necessary to show that there exists $C > 0$ such that the inequality

$$\left| \frac{a_j}{b_k} q^l - 1 \right| > C$$

holds for all $j,k \in \{1,2,...,m\}$, and $l \in \mathbb{Z}$.

Suppose that for any $\varepsilon > 0$ there exist $j,k \in \{1,2,...,m\}$, and $l \in \mathbb{Z}$ such that

$$\left| \frac{a_j}{b_k} q^l - 1 \right| \leq \varepsilon. \tag{10}$$

Without loss of generality we can assume that $|l| \leq 2$. Indeed, taking into account where a_j, b_k belong to, we have

$$\left| \frac{a_j}{b_k} q^l \right| \leq \frac{1}{|q|} |q|^l \leq |q|, \qquad l \geq 2.$$

Similarly,

$$\left| \frac{a_j}{b_k} q^l \right| \geq |q| |q|^l \geq \frac{1}{|q|}, \qquad l \leq -2.$$

So, for all $j,k \in \{1,2,...,m\}$, and $l \geq 2$

$$\left| \frac{a_j}{b_k} q^l - 1 \right| \geq 1 - |q|,$$

and for $l \leq -2$

$$\left| \frac{a_j}{b_k} q^l - 1 \right| \geq \frac{1}{|q|} - 1.$$

Let now $|l| < 2$. Choose

$$\varepsilon = \frac{1}{2} \min\{|a_j q^l - b_k| : j,k \in \{1,2,...,m\}, -1 \leq l \leq 1\}.$$

Then (10) implies

$$|a_j q^l - b_k| \leq \varepsilon |b_k| \leq \varepsilon\,.$$

That is

$$|a_j q^l - b_k| \leq \frac{1}{2} \min\{|a_j q^l - b_k| : j,k \in \{1,2,...,m\}, -1 \leq l \leq 1\}$$

which gives a contradiction.

(iv) We remind that f has representation (7). It can be rewritten as follows

$$f(z) = c z^\nu \prod_{k=1}^{m} \frac{\prod\limits_{n=0}^{+\infty} \left(1 - \dfrac{q^n z}{a_k}\right) \prod\limits_{n=1}^{+\infty} \left(1 - \dfrac{q^n a_k}{z}\right)}{\prod\limits_{n=0}^{+\infty} \left(1 - \dfrac{q^n z}{b_k}\right) \prod\limits_{n=1}^{+\infty} \left(1 - \dfrac{q^n b_k}{z}\right)}, \qquad z \in \mathbb{C}^*. \tag{11}$$

Clearly, we can assume $c \neq 0$. Consider the integral means $I(r) = \frac{1}{2\pi} \int\limits_0^{2\pi} \log |f(re^{i\theta})| \, d\theta$, $r > 0$.

Let $z = re^{i\theta}$. We have for $r > 1$ [4, p. 8]

$$\frac{1}{2\pi}\int_0^{2\pi} \log\left|1 - \frac{z}{a_j}\right| d\theta = \log^+ \frac{r}{|a_j|},$$

and, if $|a_j| \le 1$

$$\frac{1}{2\pi}\int_0^{2\pi} \log\left|1 - \frac{a_j}{z}\right| d\theta = 0\,.$$

The same is true for b_j.

Since for every $k \in \{1, 2, ..., m\}$ we have $|c_k q^{-n}| > 1$ for $n \in \mathbb{N}$, and $|c_k q^n| \le 1$ for $n \in \mathbb{N} \cup \{0\}$, where c_k is a zero or pole of f, then (11) implies

$$I(r) = \nu \log r + \sum_{|a_j|>1} \log^+ \frac{r}{|a_j|} - \sum_{|b_j|>1} \log^+ \frac{r}{|b_j|} + \log|c|, \quad r > 1\,.$$

Similarly, for $0 < r \le 1$ we obtain

$$I(r) = \nu \log r + \sum_{|a_j|\le 1} \log^+ \frac{|a_j|}{r} - \sum_{|b_j|\le 1} \log^+ \frac{|b_j|}{r} + \log|c|\,.$$

Hence,

$$\mathfrak{M}_\nu(r) = \frac{1}{|p|^{\mu(r)}}\frac{1}{|c|}\exp I(r) = \frac{1}{|c|}\exp\{I(r) - \mu(r)\log|p|\}, \quad r > 0\,.$$

Since $I(r)$ is convex with respect to $\log r$ and consequently continuous, $I(r)$ is bounded on $[|q|, 1]$. It follows from the definition of a p-loxodromic function of multiplicator q that

$$I(|q|^k r) = I(r) + k\log|p|$$

for every $k \in \mathbb{Z}$. On the other hand

$$\mu(|q|^k r) = \left[\frac{k\log|q| + \log r}{\log|q|}\right] - 1 = k, \qquad |q| \le r < 1.$$

That is

$$\mathfrak{M}_\nu(|q|^k r) = \mathfrak{M}_\nu(r), \qquad |q| \le r < 1$$

for all $k \in \mathbb{Z}$. Then we conclude that $\mathfrak{M}_\nu(r)$ remains bounded for all $r > 0$ which completes the proof.

□

4 JULIA EXCEPTIONALITY

Definition 3. *Let $f_n, n \in \mathbb{N}$, be meromorphic functions in a domain G. A sequence $\{f_n(z)\}$ is said to be uniformly convergent to $f(z)$ on G in the Carathéodory-Landau sense [1] if for any point $z_0 \in G$ there exists a disk $K(z_0)$ centered at this point such that $K(z_0) \subset G$ and*

$$(\forall\varepsilon > 0)(\exists n_0 \in \mathbb{N})(\forall n > n_0)(\forall z \in K(z_0)) : |f_n(z) - f(z)| < \varepsilon,$$

whenever $f(z_0) \ne \infty$, or

$$\left|\frac{1}{f_n(z)} - \frac{1}{f(z)}\right| < \varepsilon,$$

whenever $f(z_0) = \infty$.

Note that this convergence is equivalent to the convergence in the spherical metric.

Definition 4. *A family $\mathcal{F}$ of meromorphic in $\mathbb{C}^*$ functions is said to be normal if every sequence $\{f_n\} \subseteq \mathcal{F}$ contains a subsequence which converges uniformly in the Carathéodory-Landau sense.*

Definition 5. *A meromorphic in $\mathbb{C}^*$ function f is called Julia exceptional (see [7]) if for some q, $0 < |q| < 1$, the family $\{f_n(z)\}$, $n \in \mathbb{Z}$, where $f_n(z) = f(q^n z)$, is normal in $\mathbb{C}^*$.*

In $\mathbb{C}$ there are few simple examples of Julia exceptional functions. But in $\mathbb{C}^*$ we have the following.

Let $f \in \mathcal{L}_{qp}$. We have

$$f_n(z) = f(q^n z) = p^n f(z)$$

for every $z \in \mathbb{C}^*$.

If $|p| > 1$, then a limiting function of the family $\{f_n(z)\}$, $n \in \mathbb{Z}$, is ∞. Otherwise, if $|p| < 1$, then a limiting function is 0. If $|p| = 1$, that is $p = e^{i\alpha}$, we have $f_n(z) = e^{in\alpha} f(z)$. Hence, the set of limit functions depends on α. If $\alpha = \frac{\pi m}{k}$, where $m \in \mathbb{Z}$, $k \in \mathbb{N}$, the number of limiting functions is less than or equals to $2k$. Otherwise, if $\alpha = \pi r$, where $r \in \mathbb{R} \backslash \mathbb{Q}$, the number of limiting functions is infinite.

Example 2. *Let $f \in \mathcal{L}_q^\alpha$ with $\alpha = \frac{\pi}{4}$. Then*

$$f_n(z) = f(q^n z) = p^n f(z) = e^{in\frac{\pi}{4}} f(z).$$

Thus, we obtain eight limiting functions

$$\pm f,\ \pm i f,\ \left(\frac{\sqrt{2}}{2} \pm i\frac{\sqrt{2}}{2}\right) f,\ \left(-\frac{\sqrt{2}}{2} \pm i\frac{\sqrt{2}}{2}\right) f.$$

Hence, f is Julia exceptional in $\mathbb{C}^*$.

These results can be summarized as follows.

Theorem 4. *Each function $f \in \mathcal{L}_{qp}$ is Julia exceptional in $\mathbb{C}^*$.*

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Досліджується клас p-локсодромних функцій (мероморфних функцій, що задовольняють умову $f(qz) = pf(z)$ при деяких $q \in \mathbb{C}\setminus\{0\}$ для всіх $z \in \mathbb{C} \setminus \{0\}$). Доведено, що кожна p-локсодромна функція є Жюліа винятковою. Подано зображення таких функцій та описано розподіл їх нулів та полюсів.

Ключові слова і фрази: p-локсодромна функція, первинна функція Шотткі-Кляйна, Жюліа винятковість.