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CONTINUOUS BLOCK-SYMMETRIC POLYNOMIALS OF DEGREE AT MOST TWO ON THE SPACE $(L_\infty)^2$

We introduce block-symmetric polynomials on $(L_\infty)^2$ and prove that every continuous block-symmetric polynomial of degree at most two on $(L_\infty)^2$ can be uniquely represented by some "elementary" block-symmetric polynomials.

Key words and phrases: block-symmetric polynomial, symmetric function on L_∞.

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INTRODUCTION

Firstly symmetric functions of infinite number of variables were studied by Nemirovski and Semenov in [5]. Authors considered functions on ℓ_p and L_p spaces. Some of their results were generalized by González, Gonzalo and Jaramillo [2] to real separable rearrangement-invariant function spaces. In [3] Kravtsiv and Zagorodnyuk considered block-symmetric polynomials on ℓ_1-sum of copies of Banach space. In the joint paper of the author with Galindo and Zagorodnyuk [1] the algebra of symmetric analytic functions of bounded type on the complex space L_∞ is studied in detail and its spectrum is described.

A map $P : X \to \mathbb{C}$, where X is a complex Banach space, is called an n-homogeneous polynomial if there exists an n-linear symmetric form $A_P : X^n \to \mathbb{C}$, such that $P(x) = A_P(x, \overset{n}{\ldots}, x)$ for every $x \in X$. Here "symmetric" means that

$$A_P(x_{\tau(1)}, \ldots, x_{\tau(n)}) = A_P(x_1, \ldots, x_n)$$

for every permutation $\tau : \{1, \ldots, n\} \to \{1, \ldots, n\}$. Note that A_P is called the symmetric n-linear form *associated* with P. It is known (see e.g. [4], Theorem 1.10) that A_P can be recovered from P by means of the so-called Polarization Formula:

$$A_P(x_1, \ldots, x_n) = \frac{1}{n!2^n} \sum_{\varepsilon_1,\ldots,\varepsilon_n=\pm 1} \varepsilon_1 \ldots \varepsilon_n P(\varepsilon_1 x_1 + \ldots + \varepsilon_n x_n). \tag{1}$$

In the case $n = 2$ formula (1) can be written as

$$A_P(x_1, x_2) = \frac{1}{4}\Big(P(x_1 + x_2) - P(x_1 - x_2)\Big). \tag{2}$$

It is also convenient to define 0-homogeneous polynomials as constant mappings.

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A mapping $P : X \to \mathbb{C}$ is called a polynomial of degree at most m if it can be represented as

$$P = P_0 + P_1 + \ldots + P_m,$$

where P_j is a j-homogeneous polynomial for $j = 0, \ldots, m$.

Let L_∞ be the *complex* Banach space of all Lebesgue measurable essentially bounded complex-valued functions x on $[0,1]$ with norm

$$\|x\|_\infty = \operatorname{ess\,sup}_{t\in[0,1]}|x(t)|.$$

Let Ξ be the set of all measurable bijections of $[0,1]$ that preserve the measure. A function $F : L_\infty \to \mathbb{C}$ is called Ξ-*symmetric* (or just *symmetric* when the context is clear) if for every $x \in L_\infty$ and for every $\sigma \in \Xi$

$$F(x \circ \sigma) = F(x).$$

The functions $R_n : L_\infty \to \mathbb{C}$ defined by

$$R_n(x) = \int_0^1 x^n(t)\,dt$$

for every $n \in \mathbb{N} \cup \{0\}$ are called the *elementary symmetric polynomials*. In [1] it is shown that for each continuous Ξ-symmetric polynomial $P : L_\infty \to \mathbb{C}$ of degree at most m there is a unique finitely many variables polynomial q such that

$$P(x) = q(R_0(x), \ldots, R_m(x))$$

for every $x \in L_\infty$.

Let $(L_\infty)^2$ be the Cartesian square of the space L_∞, endowed with norm $\|(x,y)\| = \max\{\|x\|_\infty, \|y\|_\infty\}$. Clearly, $(L_\infty)^2$ is a complex Banach space. A function $F : (L_\infty)^2 \to \mathbb{C}$ we call *block-symmetric* if for every $(x,y) \in (L_\infty)^2$ and for every $\sigma \in \Xi$

$$F((x \circ \sigma, y \circ \sigma)) = F((x,y)).$$

We restrict our attention to continuous block-symmetric polynomials of degree at most two on $(L_\infty)^2$. In Section 1 we prove that every such a polynomial can be uniquely represented as an algebraic combination of the polynomials

$$R_0((x,y)) = 1, \quad R_{10}((x,y)) = R_1(x), \quad R_{01}((x,y)) = R_1(y),$$
$$R_{20}((x,y)) = R_2(x), \quad R_{11}((x,y)) = \int_0^1 x(t)y(t)\,dt, \quad R_{02}((x,y)) = R_2(y),$$

which we call *the elementary block-symmetric polynomials of degree at most two.*

1 THE MAIN RESULT

By $\mathbf{1}_E$ we denote the characteristic function of a set $E \subset [0,1]$. We also define functions $\mathbf{1} = \mathbf{1}_{[0,1]}$ and $\mathbf{r} = \mathbf{1}_{[0,\frac{1}{2}]} - \mathbf{1}_{[\frac{1}{2},1]}$.

Theorem 1. *Every continuous block-symmetric polynomial* $P = P_0 + P_1 + P_2$*, where* P_j *is a* j*-homogeneous polynomial for* $j = 0, 1, 2$*, can be represented as*

$$P = a_0 R_{00} + a_{10} R_{10} + a_{01} R_{01} + a_{20} R_{20} + a_{11} R_{11} + a_{02} R_{02} + a_{1010} R_{10}^2 + a_{1001} R_{10} R_{01} + a_{0101} R_{01}^2,$$

where

$$\begin{aligned}
&a_0 = P_0, \quad a_{10} = P_1((\mathbf{1}, 0)), \quad a_{01} = P_1((0, \mathbf{1})),\\
&a_{20} = P_2((\mathbf{r}, 0)), \quad a_{11} = A_{P_2}((\mathbf{r}, 0), (0, \mathbf{r})), \quad a_{02} = P_2((0, \mathbf{r})),\\
&a_{1010} = P_2((\mathbf{1}, 0)) - P_2((\mathbf{r}, 0)), \quad a_{1001} = A_{P_2}((\mathbf{1}, 0), (0, \mathbf{1})) - A_{P_2}((\mathbf{r}, 0), (0, \mathbf{r})),\\
&a_{0101} = P_2((0, \mathbf{1})) - P_2((0, \mathbf{r})).
\end{aligned}$$

Here we denote by A_{P_2} the symmetric bilinear form, associated with P_2.
Proof. It can be easily checked that

$$\begin{aligned}
&P_0((x, y)) = P((0, 0)), \quad P_1((x, y)) = \frac{1}{2}\Big(P((x, y)) - P((-x, -y))\Big),\\
&P_2((x, y)) = P((x, y)) - P_1((x, y)) - P_0((x, y))
\end{aligned}$$

for every $(x, y) \in (L_\infty)^2$. This implies that P_0, P_1 and P_2 are continuous and block-symmetric.

By the linearity of P_1

$$P_1((x, y)) = P_1((x, 0) + (0, y)) = P_1((x, 0)) + P_1((0, y)).$$

Let $f_1(x) = P_1((x, 0))$ for $x \in L_\infty$. Clearly, f_1 is a continuous linear Ξ-symmetric functional on L_∞. It is known (see [1, 6]) that every such a functional f can be represented as

$$f(x) = f(\mathbf{1}) R_1(x). \tag{3}$$

Therefore $f_1(x) = f_1(\mathbf{1}) R_1(x)$, i. e. $P_1((x, 0)) = P_1((\mathbf{1}, 0)) R_1(x)$. Analogously, $P_1((0, y)) = P_1((0, \mathbf{1})) R_1(y)$. Thus

$$P_1((x, y)) = P_1((\mathbf{1}, 0)) R_1(x) + P_1((0, \mathbf{1})) R_1(y) = a_{10} R_{10}((x, y)) + a_{01} R_{01}((x, y)).$$

Since A_{P_2} is bilinear and symmetric, it follows that

$$P_2((x, y)) = A_{P_2}((x, 0), (x, 0)) + 2A_{P_2}((x, 0), (0, y)) + A_{P_2}((0, y), (0, y)).$$

We define following bilinear forms:

$$\begin{aligned}
&B_I(x_1, x_2) = A_{P_2}((x_1, 0), (x_2, 0)), \quad B_{II}(x_1, x_2) = A_{P_2}((x_1, 0), (0, x_2)),\\
&B_{III}(x_1, x_2) = A_{P_2}((0, x_1), (0, x_2)),
\end{aligned} \tag{4}$$

where $x_1, x_2 \in L_\infty$. Note that B_I and B_{III} are symmetric. By the formula (2)

$$A_{P_2}((x_1, y_1), (x_2, y_2)) = \frac{1}{4}\Big(P_2((x_1 + x_2, y_1 + y_2)) - P_2((x_1 - x_2, y_1 - y_2))\Big).$$

Therefore by the symmetry of P_2

$$A_{P_2}((x_1 \circ \sigma, y_1 \circ \sigma), (x_2 \circ \sigma, y_2 \circ \sigma)) = A_{P_2}((x_1, y_1), (x_2, y_2)) \tag{5}$$

for every $\sigma \in \Xi$ and $(x_1, y_1), (x_2, y_2) \in (L_\infty)^2$. By (5) we have that

$$B_j(x_1 \circ \sigma, x_2 \circ \sigma) = B_j(x_1, x_2), \tag{6}$$

for every $j \in \{I, II, III\}$, $x_1, x_2 \in L_\infty$ and $\sigma \in \Xi$.

Let Q_I be the restriction of B_I to the diagonal. By the continuity of B_I and by (6) we have that Q_I is a continuous 2-homogeneous Ξ-symmetric polynomial. It is known (see [1]) that every continuous 2-homogeneous Ξ-symmetric polynomial Q on L_∞ can be represented as

$$Q = \alpha R_1^2 + \beta R_2. \tag{7}$$

It can be easily checked that $\alpha = Q(\mathbf{1}) - Q(\mathbf{r})$ and $\beta = Q(\mathbf{r})$. Note that

$$Q_I(x) = A_{P_2}((x,0),(x,0)) = P_2((x,0)).$$

Thus

$$\begin{aligned} A_{P_2}((x,0),(x,0)) &= \Big(P_2((\mathbf{1},0)) - P_2((\mathbf{r},0))\Big) R_1^2(x) + P_2((\mathbf{r},0)) R_2(x) \\ &= a_{1010} R_{10}^2((x,y)) + a_{20} R_{20}((x,y)). \end{aligned}$$

Analogously

$$A_{P_2}((0,y),(0,y)) = a_{0101} R_{10}^2((x,y)) + a_{02} R_{20}((x,y)).$$

The bilinear form B_{II} can be represented as the sum of the symmetric and the antisymmetric forms

$$B_{II}^s(x_1, x_2) = \frac{1}{2}\Big(B_{II}(x_1, x_2) + B_{II}(x_2, x_1)\Big)$$

and

$$B_{II}^a(x_1, x_2) = \frac{1}{2}\Big(B_{II}(x_1, x_2) - B_{II}(x_2, x_1)\Big)$$

respectively. Let us prove that $B_{II}^a(x_1, x_2) = 0$ for every $x_1, x_2 \in L_\infty$.

Lemma 1. $B_{II}^a(\mathbf{1}_{[0,\frac{1}{2}]}, \mathbf{1}_{[\frac{1}{2},1]}) = 0.$

Proof. Let $\sigma(t) = 1 - t$. By (6) $B_{II}^a(\mathbf{1}_{[0,\frac{1}{2}]}, \mathbf{1}_{[\frac{1}{2},1]}) = B_{II}^a(\mathbf{1}_{[0,\frac{1}{2}]} \circ \sigma, \mathbf{1}_{[\frac{1}{2},1]} \circ \sigma) = B_{II}^a(\mathbf{1}_{[\frac{1}{2},1]}, \mathbf{1}_{[0,\frac{1}{2}]})$. On the other hand, since B_{II}^a is antisymmetric, it follows that

$$B_{II}^a(\mathbf{1}_{[0,\frac{1}{2}]}, \mathbf{1}_{[\frac{1}{2},1]}) = -B_{II}^a(\mathbf{1}_{[\frac{1}{2},1]}, \mathbf{1}_{[0,\frac{1}{2}]}).$$

Therefore $B_{II}^a(\mathbf{1}_{[0,\frac{1}{2}]}, \mathbf{1}_{[\frac{1}{2},1]}) = 0.$ □

Lemma 2. $B_{II}^a(\mathbf{1}_E, \mathbf{1}_F) = 0$ *for every measurable sets* $E \subset [0, \frac{1}{2}]$ *and* $F \subset [\frac{1}{2}, 1]$.

Proof. For every $x \in L_\infty$ we define $\widehat{x} \in L_\infty$ by

$$\widehat{x}(t) = \begin{cases} x(2t), & \text{if } t \in [0, \frac{1}{2}], \\ 0, & \text{if } t \in (\frac{1}{2}, 1]. \end{cases}$$

Let $z \in L_\infty$ be such that its restriction to $[0, \frac{1}{2})$ is constant. Let $f_z(x) = B_{II}^a(\widehat{x}, z)$, where $x \in L_\infty$. Clearly, f_z is a continuous linear functional on L_∞. Let us prove that f_z is Ξ-symmetric. For every $\sigma \in \Xi$ let

$$\widetilde{\sigma}(t) = \begin{cases} \frac{1}{2}\sigma(2t), & \text{if } t \in [0, \frac{1}{2}], \\ t, & \text{if } t \in (\frac{1}{2}, 1]. \end{cases}$$

Clearly, $\widetilde{\sigma} \in \Xi$ and $z \circ \widetilde{\sigma} = z$. It can be checked that $\widehat{x \circ \sigma} = \widehat{x} \circ \widetilde{\sigma}$. Therefore by (6)

$$f_z(x \circ \sigma) = B^a_{II}(\widehat{x \circ \sigma}, z) = B^a_{II}(\widehat{x} \circ \widetilde{\sigma}, z \circ \widetilde{\sigma}) = B^a_{II}(\widehat{x}, z) = f_z(x).$$

Thus f_z is Ξ-symmetric. By (3) $f_z(x) = f_z(\mathbf{1})R_1(x)$, i. e. $B^a_{II}(\widehat{x}, z) = B^a_{II}(\widehat{\mathbf{1}}, z)R_1(x)$. Since $\widehat{\mathbf{1}} = \mathbf{1}_{[0,\frac{1}{2}]}$, $\widehat{\mathbf{1}_{2E}} = \mathbf{1}_E$ and $R_1(\mathbf{1}_{2E}) = 2\mu(E)$, where $2E = \{2t : \ t \in E\}$, it follows that

$$B^a_{II}(\mathbf{1}_E, z) = B^a_{II}(\mathbf{1}_{[0,\frac{1}{2}]}, z)2\mu(E).$$

Analogously it can be proven that $B^a_{II}(u, \mathbf{1}_F) = B^a_{II}(u, \mathbf{1}_{[\frac{1}{2},1]})2\mu(F)$, where $u \in L_\infty$ such that its restriction to $(\frac{1}{2}, 1]$ is constant. Therefore

$$B^a_{II}(\mathbf{1}_E, \mathbf{1}_F) = B^a_{II}(\mathbf{1}_{[0,\frac{1}{2}]}, \mathbf{1}_F)2\mu(E) = B^a_{II}(\mathbf{1}_{[0,\frac{1}{2}]}, \mathbf{1}_{[\frac{1}{2},1]})4\mu(E)\mu(F) = 0$$

by Lemma 1. $\square$

Lemma 3. *$B^a_{II}(\mathbf{1}_E, \mathbf{1}_F) = 0$ for disjoint measurable sets $E, F \subset [0, 1]$ such that $\mu(E) \leq \frac{1}{2}$ and $\mu(F) \leq \frac{1}{2}$.*

Proof. By [1, Proposition 1.2] there exists $\sigma_{E,F} \in \Xi$ such that $\mathbf{1}_E = \mathbf{1}_{[0,a]} \circ \sigma_{E,F}$ and $\mathbf{1}_F = \mathbf{1}_{[a,a+b]} \circ \sigma_{E,F}$, where $a = \mu(E)$ and $b = \mu(F)$. Let

$$\sigma_1(t) = \begin{cases} t - a + \frac{1}{2}, & \text{if } t \in [a, a + b], \\ t - \frac{1}{2} + a, & \text{if } t \in [\frac{1}{2}, \frac{1}{2} + b], \\ t, & \text{otherwise.} \end{cases}$$

Clearly, $\sigma_1 \in \Xi$, $\mathbf{1}_{[0,a]} = \mathbf{1}_{[0,a]} \circ \sigma_1$ and $\mathbf{1}_{[a,a+b]} = \mathbf{1}_{[\frac{1}{2},\frac{1}{2}+b]} \circ \sigma_1$. Therefore $\mathbf{1}_E = \mathbf{1}_{[0,a]} \circ \sigma_1 \circ \sigma_{E,F}$ and $\mathbf{1}_F = \mathbf{1}_{[\frac{1}{2},\frac{1}{2}+b]} \circ \sigma_1 \circ \sigma_{E,F}$. By (6) and by Lemma 2

$$B^a_{II}(\mathbf{1}_E, \mathbf{1}_F) = B^a_{II}(\mathbf{1}_{[0,a]} \circ \sigma_1 \circ \sigma_{E,F}, \mathbf{1}_{[\frac{1}{2},\frac{1}{2}+b]} \circ \sigma_1 \circ \sigma_{E,F}) = B^a_{II}(\mathbf{1}_{[0,a]}, \mathbf{1}_{[\frac{1}{2},\frac{1}{2}+b]}) = 0.$$

$\square$

Lemma 4. *$B^a_{II}(\mathbf{1}_E, \mathbf{1}_F) = 0$ for every disjoint measurable sets $E, F \subset [0, 1]$.*

Proof. If $\mu(E) = \mu(F)$, then $\mu(E)$ and $\mu(F)$ cannot be greater than $\frac{1}{2}$ and $B^a_{II}(\mathbf{1}_E, \mathbf{1}_F) = 0$ by Lemma 3. Note that $B^a_{II}(\mathbf{1}_E, \mathbf{1}_F) = 0$ if $\mu(E) = 0$ or $\mu(F) = 0$. Let $\mu(E) > \mu(F) > 0$. Let $N = \left\lfloor \frac{\mu(E)}{\mu(F)} \right\rfloor$. We can choose disjoint measurable subsets $E_1, \ldots, E_N \subset E$ such that $\mu(E_1) = \ldots = \mu(E_N) = \mu(F)$. Set $E_0 = E \setminus \cup_{j=1}^N E_j$. Then

$$B^a_{II}(\mathbf{1}_E, \mathbf{1}_F) = \sum_{j=0}^N B^a_{II}(\mathbf{1}_{E_j}, \mathbf{1}_F) = B^a_{II}(\mathbf{1}_{E_0}, \mathbf{1}_F).$$

Since $\mu(E_0) < \mu(F) < \frac{1}{2}$, it follows that $B^a_{II}(\mathbf{1}_{E_0}, \mathbf{1}_F) = 0$ by Lemma 3. $\square$

Lemma 5. *$B^a_{II}(\mathbf{1}_E, \mathbf{1}_F) = 0$ for every measurable sets $E, F \subset [0, 1]$.*

Proof. Note that $E = (E \setminus F) \sqcup (E \cap F)$ and $F = (F \setminus E) \sqcup (E \cap F)$. Therefore

$$B^a_{II}(\mathbf{1}_E, \mathbf{1}_F) = B^a_{II}(\mathbf{1}_{E\setminus F}, \mathbf{1}_{F\setminus E}) + B^a_{II}(\mathbf{1}_{E\setminus F}, \mathbf{1}_{E\cap F}) + B^a_{II}(\mathbf{1}_{E\cap F}, \mathbf{1}_{F\setminus E}) + B^a_{II}(\mathbf{1}_{E\cap F}, \mathbf{1}_{E\cap F}) = 0$$

by Lemma 4 and by the antisymmetry of B^a_{II}. $\square$

Proof of the Theorem 1 (continuation). For the simple measurable functions $x_1, x_2 \in L_\infty$ we have $B^a_{II}(x_1, x_2) = 0$ by the bilinearity of B^a_{II}. Since the set of simple measurable functions is dense in L_∞, the continuity of B^a_{II} leads to $B^a_{II}(x_1, x_2) = 0$ for every $x_1, x_2 \in L_\infty$. Thus $B_{II} = B^s_{II}$, i. e. B_{II} is symmetric. Let Q_{II} be the restriction of B_{II} to the diagonal. Q_{II} is a continuous 2-homogeneous Ξ-symmetric polynomial. Therefore by (7) $Q_{II}(x) = (Q_{II}(\mathbf{1}) - Q_{II}(\mathbf{r}))R_1^2(x) + Q_{II}(\mathbf{r})R_2(x)$.

By (2) $B_{II}(x, y) = \frac{1}{4}\Big(Q_{II}(x + y) - Q_{II}(x - y)\Big)$. Since

$$B_{II}(x, y) = A_{P_2}((x, 0), (0, y)), \quad Q_{II}(\mathbf{1}) = A_{P_2}((\mathbf{1}, 0), (0, \mathbf{1})), \quad Q_{II}(\mathbf{r}) = A_{P_2}((\mathbf{r}, 0), (0, \mathbf{r})),$$

$$R_1^2(x + y) - R_1^2(x - y) = 4R_1(x)R_1(y), \quad R_2(x + y) - R_2(x - y) = 4\int_0^1 x(t)y(t)\, dt,$$

it follows that

$$\begin{aligned} A_{P_2}((x, 0), (0, y)) &= (A_{P_2}((\mathbf{1}, 0), (0, \mathbf{1})) - A_{P_2}((\mathbf{r}, 0), (0, \mathbf{r})))\, R_1(x)R_1(y) \\ &+ A_{P_2}((\mathbf{r}, 0), (0, \mathbf{r})) \int_0^1 x(t)y(t)\, dt = a_{1001}R_{10}((x, y))R_{01}((x, y)) + a_{11}R_{11}((x, y)). \end{aligned}$$

□

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Введено поняття блочно-симетричного полінома на просторі $(L_\infty)^2$ і показано, що кожен неперервний блочно-симетричний поліном степеня щонайбільше два на просторі $(L_\infty)^2$ можна єдиним чином виразити через деякі “елементарні” блочно-симетричні поліноми.

Ключові слова і фрази: блочно-симетричний поліном, симетрична функція на L_∞.