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HAJJEJ Z., BALEGH M.

UNIFORM BOUNDARY CONTROLLABILITY OF A DISCRETE 1-D SCHRÖDINGER EQUATION

In this paper we study the controllability of a finite dimensional system obtained by discretizing in space and time the linear 1-D Schrödinger equation with a boundary control. As for other problems, we can expect that the uniform controllability does not hold in general due to high frequency spurious modes. Based on a uniform boundary observability estimate for filtered solutions of the corresponding conservative discrete system, we show the uniform controllability of the projection of the solutions over the space generated by the remaining eigenmodes.

Key words and phrases: observability, controllability, filtering.

Faculty of Sciences of Monastir, University of Monastir, Avenue of Environment, 5019, Monastir, Tunisia
E-mail: hajjej.zayd@gmail.com (Hajjej Z.), mohamedbalegh@gmail.com (Balegh M.)

INTRODUCTION

Let us consider the 1-D Schrödinger equation

$$\begin{cases} u_t(x,t) + iu_{xx}(x,t) = 0, & 0 < x < 1, \ \ 0 < t < T, \\ u(0,t) = u(1,t) = 0, & 0 < t < T, \\ u(x,0) = u^0(x), & 0 < x < 1, \end{cases} \tag{1}$$

where $u^0 \in H_0^1(0,1)$. It is well known that the energy

$$E(t) = \frac{1}{2}\int_0^1 |u_x(x,t)|^2\, dx \tag{2}$$

of the solutions is conserved in time. Applying Fourier series techniques one can prove a boundary observability inequality showing that, for every $T > 0$, there exists $C = C(T) > 0$ such that

$$E(0) \leq C \int_0^T |u_x(1,t)|^2\, dt \tag{3}$$

for every solution of (1).

As a consequence of this observability inequality and the HUM method [10], the following boundary controllability property may be proved.

For all $T > 0$ and $y^0 \in H^{-1}(0,1)$ there exists a control $v \in L^2(0,T)$ such that the solution of

$$\begin{cases} y_t(x,t) + iy_{xx}(x,t) = 0, & 0 < x < 1, \ \ 0 < t < T, \\ y(0,t) = 0, \ \ y(1,t) = v, & 0 < t < T, \\ y(x,0) = y^0(x), & 0 < x < 1, \end{cases} \tag{4}$$

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satisfies $y(T) = 0$.

This article aims at studying the observability and the controllability properties for space-discrete and fully discrete approximations schemes of (1) and (4).

In the last years many works have dealt with the numerical approximations of the control problem of the wave equation using the HUM approach [1, 4, 11]. It is by now well-known that discretization processes may create high frequency spurious solutions which might lead to non-uniform observability properties. The conclusion was that the controllability property is not uniform as the discretization parameter h goes to zero and, consequently, the control of the discrete model do not converge to the control of the continuous model. Some remedies are then necessary to restore the convergence of the discrete control to the continuous one. We can mention the Tychonoff regularization [6], a mixed finite element method [1], or a filtering technique [7]. In the context of fully discrete conservative equations, we refer to [3], which deals with very general approximation schemes for conservative linear systems. For space semi-discrete approximations of Shrödinger equation, we mention the work [2] which study interior observability and controllability properties, based on spectral estimates. Let us also mention that the time semi-discrete Schrödinger equation has been studied in [13]. Our article seems to be the first one that deals with fully discrete Schrödinger equation in details providing an uniform result of boundary controllability.

The outline of this paper is as follows.

The second section briefly recalls some controllability results for the Schrödinger equation. In section 3, we study the space discrete observability and controllability properties. Section 4 is devoted to prove observability and controllability problems of fully discrete approximation schemes of (1) and (4).

1 THE CONTINUOUS PROBLEM

In this section, we recall briefly the controllability property of the Schrödinger system (4) (see [10, 14] for more details).

Theorem 1. *For all $T > 0$ and $(y^0) \in H^{-1}(0,1)$ there exists a control $v \in L^2(0,T)$ such that the solution of (4) satisfies $y(T) = 0$.*

Multiplying in (4) by $\bar{u}$, integrating by parts in $(0,1) \times (0,T)$ and using the equations (1) that u satisfies we deduce that

$$i\int_0^T v\bar{u}_x(1)dt + \int_0^1 y^0\bar{u}^0dx = \int_0^1 y(T)\bar{u}(T)dx.$$

Taking imaginary parts in the last equality, we deduce that

$$\operatorname{Re}\int_0^T v\bar{u}_x(1)dt + \operatorname{Im}\int_0^1 y^0\bar{u}^0dx = 0.$$

Here and in the sequel Re, Im and $\bar{u}$ stand respectively for the real part, the imaginary part of a complex number and the conjugate of u.

The control of minimal L^2-norm can be obtained by minimizing functional $J : H_0^1(0,1) \to \mathbb{R}$ defined as follows:

$$J(u^0) = \frac{1}{2}\int_0^T |u_x(1,t)|^2dt - \operatorname{Im}\int_0^1 y^0\bar{u}^0dx. \tag{5}$$

The functional J is continuous and convex. Moreover, J is coercive because of the observability inequality (3). Then, the following result holds.

Theorem 2. *Given any $T > 0$ and $y^0 \in H^{-1}(0,1)$ the functional J has an unique minimizer $\hat{u}^0 \in H_0^1(0,1)$. If $\hat{u}$ is the corresponding solution of (1) with initial data $\hat{u}^0$ then $v(t) = -\hat{u}_x(1,t)$ is the control of (4) with minimal L^2-norm.*

As said in the introduction, the controllability property is equivalent to the observability inequality (3).

Let us finally remark that the solution of (1) admits the Fourier expansion

$$u(x,t) = \sum_{k>0} c_k e^{ik^2\pi^2 t} \sin(k\pi x),$$

with suitable Fourier coefficients depending on the initial data u^0.

2 Space semi-discretizations

In this section, we consider the space semi-discrete version of the continuous observability and controllability problems. Let N be a nonnegative integer. Set $h = \frac{1}{N+1}$ and consider the subdivision of $(0,1)$ given by

$$0 = x_0 < x_1 = h < \cdots < x_j = jh < \cdots < x_{N+1} = 1,$$

i.e., $x_j = jh$ for all $j = 0, \ldots, N+1$. Consider the following finite difference approximation of (4):

$$\begin{cases} y_j'(t) + i\frac{y_{j+1}(t)-2y_j(t)+y_{j-1}(t)}{h^2} = 0, & 0 < t < T,\ j = 1, \ldots, N, \\ y_0(t) = 0,\ \ y_{N+1}(t) = v_h(t), & 0 < t < T, \\ y_j(0) = y_j^0, & j = 1, \ldots, N. \end{cases} \tag{6}$$

As in the context of the continuous Schrödinger equation above, we consider the uncontrolled system

$$\begin{cases} u_j'(t) + i\frac{u_{j+1}(t)-2u_j(t)+u_{j-1}(t)}{h^2} = 0, & 0 < t < T,\ j = 1, \ldots, N, \\ u_0(t) = u_{N+1}(t) = 0, & 0 < t < T, \\ u_j(0) = u_j^0, & j = 1, \ldots, N. \end{cases} \tag{7}$$

The energy of system (7) is given by

$$E_h(t) = \frac{h}{2}\sum_{j=0}^{N} \left| \frac{u_{j+1}(t) - u_j(t)}{h} \right|^2,$$

which is a discretization of the continuous energy $E(t)$. It is easy to see that the energy E_h is conserved along time for the solutions of (7), i.e.

$$E_h(t) = E_h(0) \qquad \text{for all} \quad 0 < t < T.$$

We observe that the system (7) can be rewritten in the following simplified form

$$u_h'(t) - iA_h u_h(t) = 0,\ 0 < t < T,\ u_h(0) = u_h^0, \tag{8}$$

where u_h stands for the column vector $(u_1, \ldots, u_N)^T$, A_h denotes the matrix

$$A_h = \frac{1}{h^2}\begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}$$

entering in the finite difference discretization of the Laplacian with Dirichlet boundary conditions. We consider the eigenvalue problem associated with (7):

$$\begin{cases} i\frac{\Phi_{j+1}-2\Phi_j+\Phi_{j-1}}{h^2} = \beta_j \Phi_j, \quad j = 1, \ldots, N, \\ \Phi_0 = \Phi_{N+1} = 0. \end{cases} \tag{9}$$

Let us denote by $\beta_{1,h}, \ldots, \beta_{N,h}$ the N eigenvalues of (9). These eigenvalues can be computed explicitly [8]. We have

$$\beta_{k,h} = -i\lambda_{k,h} = -i\frac{4}{h^2}\sin^2\left(\frac{\pi hk}{2}\right), \quad k = 1, \ldots, N.$$

The eigenfunction $\Phi^{k,h} = (\Phi_1^{k,h}, \ldots, \Phi_N^{k,h})$ associated to the eigenvalue $\beta_{k,h}$ can also be computed explicitly:

$$\Phi_j^{k,h} = \sin(j\pi hk), \quad j = 1, \ldots, N.$$

Solutions of (7) admit a Fourier development on the basis of eigenvectors of system (9). More precisely, every solution $u_h = (u_j)_j$ of (7) can be written as

$$u_h(t) = \sum_{k=1}^{N} a_k e^{i\lambda_{k,h}t}\Phi^{k,h},$$

for suitable coefficients $a_k \in \mathbb{C}$, $k = 1, \ldots, N$, that can be computed explicitly in terms of the initial data.

2.1 Uniform observability of (7)

The main goal of this subsection is to analyze the following discrete version of (3):

$$E_h(0) \leq C(T,h)\int_0^T \left|\frac{u_N(t)}{h}\right|^2 dt, \tag{10}$$

where $C(T,h)$ is independent of the solution of (7).

The observability inequality (10) is said to be uniform, if the constants $C(T,h)$ are bounded uniformly in h, as $h \to 0$. However, the following result asserts that this is false.

Theorem 3. *Let u is a solution of (7). For any $T > 0$ we have*

$$\sup_u \left[\frac{E_h(0)}{\int_0^T \left|\frac{u_N(t)}{h}\right|^2 dt}\right] \to \infty \quad as \quad h \to 0.$$

Before getting into the proof of Theorem 3, let us recall the following property of the eigenvectors of (9) proved in [7].

Lemma 1. *For any eigenvector Φ with eigenvalue β of (9), the following identity holds:*

$$h\sum_{j=0}^{N}\left|\frac{\Phi_{j+1}-\Phi_j}{h}\right|^2 = \frac{2}{4-i\beta h^2}\left|\frac{\Phi_N}{h}\right|^2 = \frac{2}{4-\lambda h^2}\left|\frac{\Phi_N}{h}\right|^2.$$

Proof of Theorem 3. For $h > 0$, consider the particular solution of (7)

$$u_h(t) = e^{i\lambda_{N,h}t}\Phi^{N,h}.$$

For this solution we have

$$E_h(0) = h\sum_{j=0}^{N} \left| \frac{\Phi_{j+1}^{N,h} - \Phi_j^{N,h}}{h} \right|^2 = \frac{2}{4 - \lambda_{N,h}h^2} \left| \frac{\Phi_N^{N,h}}{h} \right|^2 .$$

On the other hand,

$$\int_0^T \left| \frac{u_N(t)}{h} \right|^2 dt = T \left| \frac{\Phi_N^{N,h}}{h} \right|^2 .$$

Note that

$$4 - \lambda_{N,h}h^2 = 4 - 4\sin^2\left(\frac{\pi}{2} - \frac{\pi h}{2}\right) = 4 - 4\cos^2\left(\frac{\pi h}{2}\right) \to 0 \quad \text{as} \quad h \to 0.$$

Thus, the result is established. □

To overcome this obstacle, we rule out the high frequency spurious modes. We define

$$C_s = \text{span}\{\Phi^{k,h} \ \text{ such that } \ \lambda_{k,h} \leq s\}.$$

In order to obtain a positive counterpart to Theorem 3, we have to introduce suitable subclasses of solutions of (7) generated by eigenvectors of (9) associated with eigenvalues such that $\lambda h^2 \leq \gamma$. For a given $\gamma \in (0, 4)$, we take solutions of (7) in C_{γ/h^2}.

We are ready to prove the following uniform boundary observability of the discrete Schrödinger equation.

Theorem 4. *Let* $0 < \gamma < 4$. *For all* $T > 0$ *there exist* $C = C(T, \gamma) > 0$ *such that*

$$E_h(0) \leq C \int_0^T \left| \frac{u_N(t)}{h} \right|^2$$

for every solution u_h *of (8) with* $u_h^0 \in C_{\gamma/h^2}$.

Sketch of the proof. In the range of eigenvalues $\lambda h^2 \leq \gamma$, according to the identity of Lemma 1, it follows that

$$h\sum_{j=0}^{N} \left| \frac{\Phi_{j+1} - \Phi_j}{h} \right|^2 \leq \frac{2}{4-\gamma} \left| \frac{\Phi_N}{h} \right|^2 \tag{11}$$

for any eigenvector $\Phi = (\Phi_1, \dots, \ \Phi_N)$ associated to an eigenvalue β such that $i\beta h^2 \leq \gamma$, or equivalent $\lambda h^2 \leq \gamma$.

Let us now consider a solution u_h of (7) in the class C_{γ/h^2}. It can be written as

$$u_h(t) = \sum_{\lambda_{k,h}h^2 \leq \gamma} a_k e^{i\lambda_{k,h}t} \Phi^{k,h}.$$

As was proved in [9], roughly speaking, the asymptotic gap tends to infinity as $k \to \infty$, uniformly on the parameter h. Then applying Lemma 2.3 [9] and using (11) we deduce that for $T > 0$,

$$C(T, \gamma) \int_0^T \left| \frac{u_N(t)}{h} \right|^2 \geq \sum_{\lambda_{k,h}h^2 \leq \gamma} |a_k|^2 h \sum_{j=0}^{N} \left| \frac{\Phi_{j+1}^{k,h} - \Phi_j^{k,h}}{h} \right|^2 .$$

Moreover,

$$E_h(0) = \frac{1}{2} \sum_{\lambda_{k,h}h^2 \leq \gamma} |a_k|^2 h \sum_{j=0}^{N} \left| \frac{\Phi_{j+1}^{k,h} - \Phi_j^{k,h}}{h} \right|^2 .$$

Therefore, we obtain the desired inequality. □

2.2 Uniform controllability of (6)

In this subsection we apply the observability result obtained above to analyze the controllability properties of the semi-discrete system (6).

For every $s \in \mathbb{R}$, introduce the finite dimensional Hilbert spaces

$$H_h^s = span\{\Phi^{1,h}, \ldots, \Phi^{N,h}\}$$

endowed with the norm

$$\|f_h\|^2_{H_h^s} = \sum_{k=1}^{N} \lambda^s_{k,h} |d_k|^2, \text{ whenever } f_h = \sum_{k=1}^{N} d_k \Phi^{k,h},$$

where $\lambda_{k,h} = \frac{4}{h^2} \sin^2(\frac{k\pi h}{2})$.

Let $0 < \gamma < 4$ and $T > 0$. The partial controllability problem of system (6) in the space H_h^{-1} consists in finding a control $v_h \in L^2(0,T)$ such that the solution $y_h = (y_j)_j$ of (6) satisfies

$$\Pi_\gamma(y_h(T)) = 0, \tag{12}$$

where Π_γ is the orthogonal projection over C_{γ/h^2}.

Multiplying (6) by $\bar{u}_j$, adding in j and integrating in time, we get

$$\text{Im } h \sum_{j=0}^{N} y_j^0 \bar{u}_j^0 - \text{Re} \int_0^T v_h(t) \frac{\bar{u}_N(t)}{h} dt = 0.$$

We obtain the following characterization of the partial controllability property of system (6).

Lemma 2. *Let $T > 0$ and $0 < \gamma < 4$. Problem (6) is partially controllable in H_h^{-1} if for every $y_h^0 \in H_h^{-1}$ there exists a control v_h such that*

$$Im\, h \sum_{j=0}^{N} y_j^0 \bar{u}_j^0 = Re \int_0^T v_h(t) \frac{\bar{u}_N(t)}{h} dt,$$

for any initial data $u_h^0 \in C_{\gamma/h^2}$.

The following uniform partial controllability property holds in the space C_{γ/h^2}.

Theorem 5. *For all $T > 0$ and $0 < \gamma < 4$, the problem (6) is partially controllable in H_h^{-1} for all $0 < h < 1$. Moreover, we have:*

(a) the corresponding controls v_h in the semi-discrete system (6) satisfying (12) are bounded in $L^2(0,T)$;

(b) the controls v_h converge as $h \to 0$ to a control $v \in L^2(0,T)$ of the minimal $L^2(0,T)$-norm of the system (4) such that $y(T) = 0$.

The proof of this theorem is similar to that in [9], also it can be done as the proof in subsection 4.2.

3 FULLY DISCRETE APPROXIMATIONS

Let $M, N \in \mathbb{N}$. We set $h = \frac{1}{N+1}$ and $\Delta t = \frac{T}{M+1}$ and introduce the nets

$$\begin{aligned} &0 = x_0 < x_1 = h < \cdots < x_j = jh < \cdots < x_{N+1} = 1, \\ &0 = t_0 < t_1 = \Delta t < \cdots < t_k = k\Delta t < \cdots < t_{M+1} = 1. \end{aligned} \tag{13}$$

We consider the following Crank-Nicolson discretization of (4)

$$\begin{cases} \frac{y_j^{n+1}-y_j^n}{\Delta t} + i\frac{y_{j+1}^{n+1}+y_{j-1}^{n+1}-2y_j^{n+1}}{2h^2} + i\frac{y_{j+1}^{n}+y_{j-1}^{n}-2y_j^{n}}{2h^2} = 0, & j = 1, \dots, N,\ n = 1, \dots, M, \\ y_0^n = 0,\ \frac{y_{N+1}^{n+1}+y_{N+1}^n}{2} = v_h^n, & n = 1, \dots, M, \\ y_j^0 = y_{0j}, & j = 1, \dots, N. \end{cases} \tag{14}$$

We shall denote by $\tilde{y}^n = (y_1^n, \dots, y_N^n)$ the solution at the time step n. We consider also the system

$$\begin{cases} \frac{u_j^{n+1}-u_j^n}{\Delta t} + i\frac{u_{j+1}^{n+1}+u_{j-1}^{n+1}-2u_j^{n+1}}{2h^2} + i\frac{u_{j+1}^{n}+u_{j-1}^{n}-2u_j^{n}}{2h^2} = 0, & j = 1, \dots, N,\ n = 1, \dots, M, \\ u_0^n = u_{N+1}^n = 0, & n = 1, \dots, M, \\ u_j^0 = u_{0j}, & j = 1, \dots, N. \end{cases} \tag{15}$$

Simple formal calculations give

$$\tilde{u}^{n+1} = (I - \frac{\Delta t}{2} i A_h)^{-1}(I + \frac{\Delta t}{2} i A_h)\tilde{u}^n = e^{i\alpha_{k,h}\Delta t}\tilde{u}^n,$$

where $\tilde{u}^n = (u_1^n, \dots, u_N^n)$ is the solution at the time step n and $e^{i\alpha_{k,h}\Delta t} = \frac{1+\frac{\Delta t}{2} i\lambda_{k,h}}{1-\frac{\Delta t}{2} i\lambda_{k,h}}$. Writing

$$\tilde{u}^0 = \sum_{k=1}^{N} a_k \tilde{\Phi}_k,$$

then the solution $\tilde{u}^n$ is given by

$$\tilde{u}^n = \sum_{k=1}^{N} a_k e^{i\alpha_{k,h} n\Delta t}\tilde{\Phi}_k, \tag{16}$$

with $a_k \in \mathbb{C},\ \tilde{\Phi}_k = (\Phi_1^{k,h}, \dots, \Phi_N^{k,h}) = (\sin(k\pi h), \dots, \sin(Nk\pi h))$ and

$$\alpha_{k,h} = \frac{2}{\Delta t}\arctan\Big(\frac{\lambda_{k,h}\Delta t}{2}\Big).$$

The energy of (15) is

$$E^n = \frac{h}{2}\sum_{j=0}^{N}\left|\frac{u_{j+1}^n - u_j^n}{h}\right|^2,$$

which is a discretization of the continuous energy E in (2), and it is conserved in all the time steps: $E^n = E^0, n = 0, \dots, M$, for the solutions of (15).

3.1 Uniform observability of (15)

In this subsection, our goal is to prove the uniform observability inequality of system (15). We have the following theorem.

Theorem 6. *Let* $0 < \gamma < 4$. *Assume that*

$$\frac{h^2}{\Delta t} \leq \tau, \tag{17}$$

where τ *is a positive constant. Then for any* $0 < \delta < \frac{\gamma}{\tau}$, *there exists* T_δ *such that for any* $T > T_\delta$ *there exists* $C_{T,\delta,\gamma}$ *such that the observability inequality*

$$E^0 \leq C_{T,\delta,\gamma} \Delta t \sum_{n=0}^{M} \left| \frac{u_N^{n+1} + u_N^n}{2h} \right|^2 \tag{18}$$

holds for every solution of (15) with initial data in the class $C_{\delta/\Delta t}$ *for all* h *and* Δt *small enough satisfying (17).*

The proof of this Theorem will essentially rely on the following Theorem proved in [5].

Theorem 7. *Let* $I = \mathbb{N}$ *or* $\mathbb{Z}$ *and* $(\mu_j)_{j\in\mathbb{N}}$ *be an increasing sequence of real numbers such that, for some* $\theta > 0$,

$$\inf_{j\in I} |\mu_{j+1} - \mu_j| \geq \theta. \tag{19}$$

Let f *be a smooth function satisfies the assumptions:* $f \in C^\infty$ *and satisfies* $f(0) = 0$, $f'(0) = 1$; f *is odd;* $f : [-R, R] \to [-\pi, \pi]$, *where* $R \in \mathbb{R}_+^* \cup \{+\infty\}$; $\inf\{f'(\alpha) | |\alpha| \leq \delta\} > 0$, *where* $\delta \in (0, R)$. *Then for all time*

$$T > \frac{2\pi}{\theta \inf\limits_{|\alpha|\leq\delta} f'(\alpha)}$$

there exist two positive constants C *and* $\tau_0 > 0$ *such that for all* $\tau \in (0, \tau_0)$, *for all* $(a_j)_{j\in I} \in l^2(I)$ *vanishing for* $j \in I$ *such that* $|\mu_j|\tau \geq \delta$,

$$\frac{1}{C}\sum_{j\in I} |a_j|^2 \leq \tau \sum_{k\tau\in(0,T)} \left| \sum_{j\in I} a_j e^{if(\mu_j\tau)k} \right|^2 \leq C \sum_{j\in I} |a_j|^2.$$

Proof of Theorem 6. The energy of solutions (15) is

$$E^0 = \frac{h}{2} \sum_{j=0}^{N} \left| \frac{u_{j+1}^0 - u_j^0}{h} \right|^2 = \frac{h}{2} \sum_{\lambda_{k,h} \leq \frac{\delta}{\Delta t}} |a_k|^2 \lambda_{k,h} \sum_{j=0}^{N} \left| \Phi_j^{k,h} \right|^2,$$

where we used

$$\sum_{j=0}^{N} \left| \frac{\Phi_{j+1}^{k,h} - \Phi_j^{k,h}}{h} \right|^2 = \lambda_{k,h} \sum_{j=0}^{N} \left| \Phi_j^{k,h} \right|^2.$$

Normalizing the eigenvector $\Phi^{k,h}$, i.e. $h \sum_{j=0}^{N} \left| \Phi_j^{k,h} \right|^2 = 1$, we get

$$E^0 = \frac{2}{h^2} \sum_{\lambda_{k,h} \leq \frac{\delta}{\Delta t}} |a_k|^2 \sin^2\left(\frac{k\pi h}{2}\right) = \frac{2}{h^2} \sum_{\lambda_{k,h} \leq \frac{\delta}{\Delta t}} |a_k|^2 \frac{\sin^2(k\pi h)}{4\cos^2(\frac{k\pi h}{2})} = \frac{1}{2} \sum_{\lambda_{k,h} \leq \frac{\delta}{\Delta t}} |b_k|^2 \frac{4 + \lambda_{k,h}^2 \Delta t^2}{4\cos^2(\frac{k\pi h}{2})},$$

where

$$b_k = (-1)^k a_k (1 + e^{i\alpha_{k,h}\Delta t}) \frac{\sin(k\pi h)}{2h}.$$

Here we used the fact that

$$|1+e^{i\alpha_{k,h}\Delta t}|^2 = 4\cos^2\left(\frac{\alpha_{k,h}\Delta t}{2}\right) = \frac{16}{4+\lambda_{k,h}^2\Delta t^2}.$$

In virtue of (17), we have $C_{\frac{\delta}{\Delta t}} \subset C_{\frac{\gamma}{h^2}}$ and then we get

$$\frac{1}{4\cos^2(\frac{\alpha_{k,h}\Delta t}{2})} \leq \frac{1}{4-\gamma} \text{ and } 4+\lambda_{k,h}^2\Delta t^2 \leq 4+\delta^2.$$

On the other hand, we have

$$\begin{aligned}
\Delta t\sum_{n=0}^{M}\left|\frac{u_N^{n+1}+u_N^n}{2h}\right|^2 &= \Delta t\sum_{n=0}^{M}\left|\sum_{\lambda_{k,h}\leq\frac{\delta}{\Delta t}} a_k e^{i\alpha_{k,h}n\Delta t}(1+e^{i\alpha_{k,h}\Delta t})\frac{\varphi_N^{|k|,h}}{2h}\right| \\
&= \Delta t\sum_{n=0}^{M}\left|\sum_{\lambda_{k,h}\leq\frac{\delta}{\Delta t}} b_k e^{i\alpha_{k,h}n\Delta t}\right|^2 = \Delta t\sum_{n=0}^{M}\left|\sum_{\lambda_{k,h}\leq\frac{\delta}{\Delta t}} b_k e^{if(\lambda_{k,h}\Delta t)n}\right|^2,
\end{aligned}$$

where $f(t) = 2\arctan(\frac{t}{2})$. It is clear that the function f satisfies the assumptions of Theorem 7. Besides, it was proved in [9] that for all $\varepsilon \in (0,1)$, we have

$$\lambda_{k+1,h} - \lambda_{k,h} \geq 3\pi^2 - \varepsilon.$$

Consequently (19) is verified with $\theta = 3\pi^2 - \varepsilon$. Applying Theorem 7, we obtain

$$E^0 \leq \frac{C(4+\delta^2)}{4-\gamma}\Delta t\sum_{n=0}^{M}\left|\frac{u_N^{n+1}+u_N^n}{2h}\right|^2,$$

for all $T > T_\delta = \frac{\pi(4+\delta^2)}{2\theta}$. □

3.2 Uniform controllability of (14)

In this part, we present the following uniform partial controllability result for system (14) and the convergence result for the controls.

The partial controllability problem for system (14) in the space H_h^{-1} consists of finding a control $(v_h^n)_{0,1,\ldots,M}$ such that for all initial data $\tilde{y}^0 \in H_h^{-1}$ the solution $\tilde{y}^n$ of (14) satisfies

$$P_\delta \tilde{y}^{M+1} = 0,$$

where δ is the same in Theorem 6 and P_δ is the orthogonal projection over $C_{\delta/\Delta t}$.

The main result of this paper reads as follows.

Theorem 8. *Let T, γ, τ and δ be given as in Theorem 6. Then for every Δt and h small enough and every $y^0 \in H^{-1}(0,1)$, the system (14) is partially controllable on H_h^{-1} with controls v_h^n. Moreover, we have:*

i) the controls of minimal norm are uniformly bounded with respect to Δt;

ii) the controls v_h^n converge to a control v of the minimal L^2-norm of the controllable system (4).

Proof. For any given $T > T_\delta$, choose γ, τ and δ as in Theorem 6 to guarantee the uniform observability (18). Multiplying the first equation in (14) by a solution $\frac{\bar{u}_j^{n+1}+\bar{u}_j^n}{2}$ of (15), adding in j and n and taking the imaginary parts, we get

$$\text{Re } \Delta t \sum_{n=0}^{M} v_h^n \frac{\bar{u}_N^{n+1} + \bar{u}_N^n}{2h} - \text{Im } h \sum_{j=0}^{N} y_j^0 \bar{u}_j^0 = 0. \tag{20}$$

Let $\tilde{u}^n \in C_{\delta/\Delta t}$ be the solution of (15) with initial data $\tilde{u}^0$ and define the functional $J_{h,\Delta t} : \mathbb{R}^n \to \mathbb{R}$ by

$$J_{h,\Delta t}(\tilde{u}^0) = \frac{\Delta t}{2} \sum_{n=0}^{M} \left| \frac{u_N^{n+1} + u_N^n}{2h} \right|^2 - \text{Im } h \sum_{j=0}^{N} y_j^0 \bar{u}_j^0.$$

For $\tilde{u}^n \in C_{\delta/\Delta t}$ we have

$$\left| \text{Im } h \sum_{j=0}^{N} y_j^0 \bar{u}_j^0 \right| \leq \left| (P_\delta \tilde{y}^0, \bar{\tilde{u}}^0)_{\mathbb{R}^N} \right| \leq \|P_\delta \tilde{y}^0\|_{H_h^{-1}} \|\tilde{u}^0\|_{H_h^1}. \tag{21}$$

The functional $J_{h,\Delta t}$ is continuous and convex. Moreover, in view of the observability inequality (18), it is clear that $J_{h,\Delta t}$ is coercive. Thus, there exists unique minimizer $\hat{u}^0$ of $J_{h,\Delta t}$,

$$J_{h,\Delta t}(\hat{u}^0) = \min_{\tilde{u}^0 \in C_{\delta/\Delta t}} J_{h,\Delta t}(\tilde{u}^0).$$

Let $\hat{u}^n \in C_{\delta/\Delta t}$ be the solution of the system (15) with initial data $\hat{u}^0$. The $\hat{u}^0$ satisfies the Euler-Lagrange equation. Calculating The Gateaux derivative of $J_{h,\Delta t}$ in $\hat{u}^0$, we get

$$0 = \lim_{t \to 0} \frac{J_{h,\Delta t}(\hat{u}^0 + t\tilde{u}^0) - J_{h,\Delta t}(\hat{u}^0)}{t} = \text{Re } \Delta t \sum_{n=0}^{M} \frac{\hat{u}_N^{n+1} + \hat{u}_N^n}{2h} \frac{\bar{u}_N^{n+1} + \bar{u}_N^n}{2h} - \text{Im } h \sum_{j=0}^{N} y_j^0 \bar{u}_j^0.$$

Therefore, according to (20) we choose the control function v_h^n in system (14) as follows

$$v_h^n = \frac{\hat{u}_N^{n+1} + \hat{u}_N^n}{2h}, \quad n = 0, \dots, M.$$

We now check the uniform boundedness of the controls v_h^n. We have

$$J_{h,\Delta t}(\hat{u}^0) \leq J_{h,\Delta t}(0) = 0,$$

and by (21), we get

$$\frac{\Delta t}{2} \sum_{n=0}^{M} \left| \frac{\hat{u}_N^{n+1} + \hat{u}_N^n}{2h} \right|^2 \leq \|P_\delta \tilde{y}^0\|_{H_h^{-1}} \|\tilde{u}^0\|_{H_h^1}.$$

Applying the observability inequality (18) we obtain

$$\Delta t \sum_{n=0}^{M} \left| \frac{\hat{u}_N^{n+1} + \hat{u}_N^n}{2h} \right|^2 \leq 2\sqrt{\frac{2C(4+\delta^2)}{4-\gamma}} \|P_\delta \tilde{y}^0\|_{H_h^{-1}} \left(\Delta t \sum_{n=0}^{M} \left| \frac{\hat{u}_N^{n+1} + \hat{u}_N^n}{2h} \right|^2 \right)^{\frac{1}{2}},$$

where we used

$$E^0 = \frac{1}{2} \|\tilde{u}^0\|_{H_h^1}.$$

Consequently, the controls $v_h^n = \frac{\hat{u}_N^{n+1} + \hat{u}_N^n}{2h}$ satisfy

$$\left(\Delta t \sum_{n=0}^{M} |v_h^n|^2 \right)^{\frac{1}{2}} \leq C(T, \delta, \gamma) \|P_\delta \tilde{y}^0\|_{H_h^{-1}}.$$

Therefore, the controls are uniformly bounded with respect to Δt.

Let us now give some details for the proof of the convergence result. Indeed the proof is standard and one may use the method developed in [12]. Note that with the notations (16), the controls (v_h^n) are of the form

$$\frac{1}{2h}\sum_{\lambda_{k,h}\leq\delta/\Delta t} m_k e^{i\alpha_{k,h}n\Delta t}(1+e^{i\alpha_{k,h}\Delta t})\sin(k\pi Nh),$$

where $(m_k)_k$ are the Fourier coefficients of the solution $\hat{u}^n \in C_{\delta/\Delta t}$ of (15), with initial data $\hat{u}^0$ being the minimizer of the functional $J_{h,\Delta t}$.

We define the continuous extension of the discrete controls by

$$v_h(t) = \frac{1}{2h}\sum_{\lambda_{k,h}\leq\delta/\Delta t} m_k e^{i\alpha_{k,h}t}(1+e^{i\alpha_{k,h}\Delta t})\sin(k\pi Nh).$$

Now, from the boundedness of (v_h^n), we see that, extracting subsequences, for some $v \in L^2(0,T)$ and $\hat{u}^0 \in H_0^1(0,1)$, $v_h \to v$ weakly in $L^2(0,T)$, $\hat{u}_h^0 \to \hat{u}^0$ weakly in $H_0^1(0,1)$, as $\Delta t \to 0$.

Moreover, one can show by standard arguments, that

$$v = -\hat{u}_x(1,t),$$

where $\hat{u}$ is the solution of (1) with initial data $\hat{u}^0 \in H_0^1(0,1)$, the unique minimizer of the functional J given in (5). Letting $\Delta t \to 0$ and $\Delta x \to 0$ in (20), we get

$$\mathrm{Re}\int_0^T v\bar{u}_x(1)dt + \mathrm{Im}\int_0^1 y^0\bar{u}^0 dx = 0,$$

and this later condition implies that the solution of system (4) with control v given as above satisfies $y(T) = 0$.

On the other hand, taking into account the convergence of the linear term of the discrete functional $J_{h,\Delta t}$ to the linear term of the discrete continuous functional J, and the structure of J and $J_{h,\Delta t}$, we deduce that

$$\int_0^T |v_h|^2 dt \to \int_0^T |v|^2 dt \quad \text{as } \Delta \to 0.$$

This combined with the weak convergence ensure the strong convergence desired. □

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У статті досліджується керованість системи скінченної розмірності, яка отримана в результаті дискретизації в просторі та часі лінійного 1-D рівняння Шредінгера з граничним контролем. Як і для інших задач, можна очікувати, що рівномірна керованість не виконується у загальному випадку у зв'язку з високою частотою появи некоректних моделей. Базуючись на рівномірній граничній спостережуваній оцінці для фільтрованих розв'язків відповідної консервативної дискретної системи, показано рівномірну керованість проекції розв'язків на простір, породжений рештою власних форм.

Ключові слова і фрази: спостережуваність, контрольованість, фільтрування.