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# FINITE HOMOMORPHIC IMAGES OF BEZOUT DUO-DOMAINS

It is proved that for a quasi-duo Bezout ring of stable range 1 the duo-ring condition is equivalent to being an elementary divisor ring. As an application of this result a couple of useful properties are obtained for finite homomorphic images of Bezout duo-domains: they are coherent morphic rings, all injective modules over them are flat, their weak global dimension is either 0 or infinity. Moreover, we introduce the notion of square-free element in noncommutative case and it is shown that they are adequate elements of Bezout duo-domains. In addition, we are going to prove that these elements are elements of almost stable range 1, as well as necessary and sufficient conditions for being square-free element are found in terms of regularity, Jacobson semisimplicity, and boundness of weak global dimension of finite homomorphic images of Bezout duo-domains.

*Key words and phrases:* Bezout ring, duo-domain, distributive ring, stable range 1, square-free element, adequate element, von Neumann regular ring, morphic ring, weak global dimension.

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## INTRODUCTION

All the rings considered in the article are supposed to be associative with nonzero identity element. In [21] it is proved that any right distributive elementary divisor ring satisfies the condition: for any element $a \in R$ one can find an element $b \in R$ such that $RaR = bR = Rb$. Moreover, in the same paper the authors have proved that such a ring has to be a duo-ring if all its zero divisors are in the Jacobson radical. As a consequence we will obtain the following result.

**Theorem 1** ([21]). *Any right distributive elementary divisor domain is a duo-domain.*

On the other hand, in [11] the author has proved that *for any elementary divisor ring, the conditions of being right distributive, left distributive, right quasi-duo, left quasi-duo ring and duo-ring are equivalent*. Also the same author proves in [12] that *a right Bezout ring is right distributive if and only if it is a right quasi-duo ring*, and *a right distributive ring is an arithmetical ring, and if it is a right duo-ring then the reverse inclusion holds*.

Here we need to mention that the quasi-duo rings have been studied in [6, 15], where the reader can find the proofs of their basic properties and their connections to the classes of regular and exchange rings. Furthermore, for the Bezout rings (as well as the arithmetical rings) the quasi-duo conditions have tight connection to the right distributivity of lattice of its right ideals.

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We are going to prove below in this article that the *"duo-ring" condition in Theorem 1 is not only necessary but is also sufficient in the case when R is a right quasi-duo Bezout ring of stable range 1.* The latter means that condition of zero divisors being in Jacobson radical can be omitted.

All mentioned information will be applied to the finite homomorphic images of a Bezout duo-domain $R$ and some corollaries will be obtained for the ring $R/aR$ in the case when $a$ is a square-free element. Actually, we will prove that $a$ is a square-free element of a Bezout duo-domain $R$ if and only if $R/aR$ is a von Neumann regular ring if and only if $R/aR$ has zero Jacobson radical if and only if the weak global dimension of $R/aR$ is finite if and only if $R/aR$ has weak global dimension 0.

Finally, from this fact we conclude that the square-free elements of the Bezout duo-domains are elements of almost stable range 1.

We recall some definitions and facts that we will need below in our proofs. All other notions can be found in [7, 8, 16, 18–20].

Hyman Bass in [1] introduced the notion of stable range that became one of the main K-theory invariants later. This invariant can be used for solving problems of matrix diagonalization over rings [19] and their relations to other classes of rings. Its definition is left-right symmetric due to [14]. Below we will use stable range condition for specific values of $n$, in fact for $n = 1$ and $n = 2$.

**Definition 1.** *We say that a ring $R$ is has the stable range 1 if for any elements $a, b \in R$ the equality $aR + bR = R$ implies that there is some $x \in R$ such that $a + bx$ is an invertible element in $R$.*

*If for any elements $a, b, c$ in a ring $R$ the equality $aR + bR + cR = R$ implies that there are some elements $x, y \in R$ such that $(a + cx)R + (b + cy)R = R$ then we say that the stable range of $R$ is equal to 2.*

*An element $a$ in a ring $R$ is called an almost stable range 1 element if the stable range of $R/aR$ is equal to 1.*

Since in the duo-ring case every von Neumann regular ring is strongly regular, the stable range of $R/aR$ becomes equal to 1 when $R/aR$ is von Neumann regular duo-ring.

Here we gather some results concerning our topic.

**Theorem 2.** *1) A right Bezout ring of stable range 1 is a right Hermite ring [18].*

*2) For any elements $a, b$ in a right Bezout ring $R$ of stable range 1 one can find some elements $x, d \in R$ such that $a + bx = d$ and $aR + bR = dR$ [18].*

*3) Every matrix $A$ over a right Hermite ring $R$ can be reduced to the lower triangular matrix $AU$ via the right multiplication by some invertible matrix $U$ [5].*

*4) If all $2 \times 2$, $2 \times 1$ and $1 \times 2$ matrices over a ring $R$ admit canonical diagonal reduction then $R$ is an elementary divisor ring [5].*

In [7] it is proved that the left morphic rings are the right P-injective. In addition it is useful to mention that a pair $(a, b)$ of elements of a ring $R$ in the previous theorem is called a *left morphic pair* and this fact will be denoted as $a \sim_l b$. Similarly for the right case we use the notation $a \sim_r b$.

## 1 RIGHT QUASI-DUO ELEMENTARY DIVISOR RINGS

Before proving one of the main results we need the following lemma.

**Lemma 1.** *Let $R$ be a Bezout duo-ring of stable range 1. If for any elements $a, b, c \in R$ such that $aR + bR + cR = R$ there are some elements $p, q \in R$ such that $(pa + qb)R + qcR = R$ then $R$ is an elementary divisor ring.*

*Proof.* According to Theorem 2 the Bezout duo-ring R of stable range 1 is a Hermite ring, so it is sufficient to prove the statement for the matrices $A = \begin{pmatrix} a & 0 \\ b & c \end{pmatrix}$, where $aR + bR + cR = R$. By given assumption there are some elements $p, q \in R$ such that $(pa + qb)u + qcv = 1$, for some $u, v \in R$. By Theorem 2 there are some invertible matrices $P = \begin{pmatrix} p & q \\ * & * \end{pmatrix}$, $Q = \begin{pmatrix} u & * \\ v & * \end{pmatrix}$, such that the element $c_{11}$ of the matrix $C = PAQ$ is equal to 1, and then obviously the matrix $C$ (as well as matrix $A$) admits canonical diagonal reduction. Thus $R$ is an elementary divisor ring as was desired. The lemma is proved. □

**Theorem 3.** *Let $R$ be a Bezout quasi-duo ring of stable range 1. Then $R$ is an elementary divisor ring if and only if it is a duo-ring.*

*Proof.* As it was mentioned at the beginning and is proved in [11] being a quasi-duo elementary divisor ring implies the duo-ring condition, so the necessity is proved.

For the proof of the sufficiency suppose that we have any triple $a, b, c \in R$ such that $aR + bR + cR = R$. By Theorem 6 there are some elements $z, h \in R$ such that $b + cz = h$ and $bR + cR = hR$. So, $aR + hR = R$ implies that there exists $q \in R$ such that $a + qh = g \in U(R)$, since $st.r.(R) = 1$. After the substitution we obtain $ag^{-1} + q(b + cz)g^{-1} = 1$ and the rearranging gives $(a + qb)g^{-1} + (qc)(zg^{-1}) = 1$ that means $(a + qb)R + qcR = R$. By Lemma 1 above $R$ is an elementary divisor ring. The theorem is proved. □

**Corollary 1.** *A right distributive Bezout ring of stable range 1 is an elementary divisor ring if and only if it is a duo-ring.*

**Example 1.** *For any Bezout ring of stable range 1 the rings of upper triangular matrices over $R$ satisfy conditions of Theorem 3. The same we have for a ring $R[[x]]$ of a formal power series over any strongly regular ring $R$. However, there are rings that fail the conditions of Theorem 3. The ring of formal power series $R\langle\langle x, y\rangle\rangle$ over a division ring $R$ of two non-commuting variables is a quasi-duo ring, but is not a right duo-ring, therefore cannot be an elementary divisor ring.*

## 2 Finite homomorphic images of Bezout duo-domains

The importance of the duo-ring conditions for the non-commutative Bezout rings was shown in the previous section. Now our goal is to determine what properties of the finite homomorphic images of the commutative Bezout domains are preserved in the duo-situation. Below we give some analogues of the results proved in [8, 13, 17, 19].

**Theorem 4.** *If $R$ is a Bezout duo-domain and $a \in R$ is some its nonzero element then $R/aR$ coincides with its classical ring of quotients: $Q(R/aR) = R/aR$, and $R/aR$ is an almost Baer, P-injective, coherent, reversible morphic IF-ring of weak global dimension equal either to 0 or the infinity, where the left and right morphic pairs coincide.*

*Proof.* For any element $b \in R$ the only possible situations are: either $aR + bR = R$ or $aR + bR = dR$. In the first case one can find some $u, v \in R$ such that $au + bv = 1$, and its image in $\overline{R} = R/aR$ is $\overline{b}\overline{v} = \overline{1}$, and so $\overline{b}$ is right invertible in $\overline{R}$. Since $R$ is a duo-domain, $Ra + Rb = aR + bR = R$ and similarly we obtain that $\overline{b}$ is left invertible as well. Thus such $\overline{b}$ will be invertible. In the case when $aR + bR = dR$ there are some $x, y \in R$ such that $a = dx, b = dy, xR + yR = R$. Then using the fact that $R$ is a duo-domain $bx = dyx = zdx = za \in aR$ for some $z \in R$. Hence $\overline{b}$ in $\overline{R}$ is a left zero divisor. Similarly we can obtain that it is a right zero divisor as well. Thus the localization that leads to the classical ring of quotients coincides with $R/aR$.

Let us show that $\overline{R}$ is a right almost Baer ring, that is we have to show that $r(\overline{b})$ is a right principal ideal for any $\overline{b} \in \overline{R}$. Suppose that $\overline{t} \in r(\overline{b})$, that is $\overline{bt} = \overline{0}$, or it is the same as $bt = as$, for some $s \in R$. Suppose that $bR + aR = hR$. If $hR = R$ then $\overline{b}$ is a unit by property 1 and its right annihilator is a right principal ideal generated by zero. Suppose that $hR \neq R$. Since $R$ is a Bezout domain we can state that $a = hx, b = hy, xR + yR = R$ for some elements $x, y \in R$. Hence the equality $bt = as$ implies $hyt = hxt$, and, after the cancelation, $yt = xs$. Since $x, y$ are coprime, then $x$ has to be a divisor of $t$, that is $t \in xR$ hence $r(\overline{b}) \subseteq \overline{x}\overline{R}$. For any $xr \in xR$ we have that $bxr = hyxr = y_1hxr = y_1ar = ay_2r \in aR$, for some $y_1, y_2 \in R$. The latter means that $\overline{xr} \in r(\overline{b})$ and $\overline{x}\overline{R} \subseteq r(\overline{b})$. At last we have obtained that $\overline{x}\overline{R} = r(\overline{b})$, thus $R/aR$ is a right almost Baer ring. Similarly it is a left almost Baer ring.

Suppose that we have in $\overline{R}$ the inclusion $r(\overline{c}) \subseteq r(\overline{b})$. Let $aR + cR = dR$ and then $a = dx, c = dy$. As $cx = dyx = dxy_1 = ay_1$ for some $y_1 \in R$, as it is a duo-domain. Hence $\overline{cx} = \overline{0}$ and $\overline{x} \in r(\overline{c}) \subseteq r(\overline{b})$, so $\overline{b}\overline{x} = \overline{0}$. The latter means that there is some $k \in R$ such that $bx = ak$. Since $R$ is a duo-domain, there exists $h \in R$ such that $bx = ak = ha = hdx$. After the cancelation we obtain $b = hd \in Rd$. Then $\overline{b} \in \overline{Rd} = \overline{R}\overline{c}$ and $\overline{R}\overline{b} \subseteq \overline{R}\overline{c}$. Thus $R/aR$ is a right P-injective by [8]. Case of a left P-injective case is similar.

Finally, in [2] it is proved that a Bezout ring $R$ is a right and left IF-ring if and only if it is coherent and P-injective. By [3, 10] we know that every IF-ring either has zero weak global dimension or it is infinite.

Suppose that $\overline{x} \in \overline{R} = R/aR$. Then by previous properties we have that there are some $y, z \in R$ such that $l(\overline{x}) = \overline{R}\overline{y} \Rightarrow \overline{x}\overline{R} = rl(\overline{x}) = r(\overline{y})$
$r(\overline{x}) = \overline{x}\overline{R} \Rightarrow \overline{R}\overline{x} = lr(\overline{x}) = r(\overline{z})$. Since $\overline{R}$ is also a duo-ring, $\overline{x}\overline{R} = \overline{R}\overline{x}$ and thus $r(\overline{y}) = l(\overline{z})$.

Let us consider two homomorphisms $f, g : \overline{R} \to \overline{R}$ defined by $f(\overline{r}) = \overline{rx}, g(\overline{r}) = \overline{xr}$. By the First Isomorphism Theorem, $\overline{R}/Ker(f) = \overline{R}/l(\overline{x}) \cong \overline{R}\overline{x}, \overline{R}/Ker(g) = \overline{R}/r(\overline{x}) \cong \overline{x}\overline{R}$. However $\overline{x}\overline{R} = \overline{R}\overline{x}$ and therefore $\overline{R}/l(\overline{x}) \cong \overline{R}/r(\overline{x})$ or $\overline{R}/\overline{z}\overline{R} \cong \overline{R}/\overline{R}\overline{y} = \overline{R}/\overline{y}\overline{R}$. Consider the commutative diagram with exact rows,

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \overline{y}\overline{R} & \longrightarrow & \overline{R} & \longrightarrow & \overline{R}/\overline{y}\overline{R} & \longrightarrow & 0 \\
 & & \big\downarrow j & & \big\downarrow = & & \big\downarrow \cong & & \\
0 & \longrightarrow & \overline{z}\overline{R} & \longrightarrow & \overline{R} & \longrightarrow & \overline{R}/\overline{z}\overline{R} & \longrightarrow & 0
\end{array}$$

where there exists the unique isomorphism $j : \overline{y}\overline{R} \to \overline{z}\overline{R}$ augmenting this diagram by [9]. Thus, we have: $\overline{x}\overline{R} = r(\overline{y}), r(\overline{x}) = \overline{z}\overline{R} \cong \overline{y}\overline{R}, \overline{R}\overline{x} = l(\overline{z}), l(\overline{x}) = \overline{R}\overline{y} = \overline{y}\overline{R} \cong \overline{z}\overline{R} = \overline{R}\overline{z}$. Therefore, by [7] $R/aR$ is the left and right morphic ring. For proving that the left and right morphic pairs coincide we need the following simple fact: if $xR \cong yR$ in a right P-injective ring $R$ then $xR = yR$. Hence we conclude that $\overline{y}\overline{R} \cong \overline{z}\overline{R}$ implies $\overline{y}\overline{R} = \overline{z}\overline{R}$, therefore the left and right morphic pairs coincide. Since the right and left morphic pairs in $\overline{R}$ coincide, for any $\overline{b} \in \overline{R}$ we can find $\overline{c} \in \overline{R}$ such that $l(\overline{b}) = \overline{R}\overline{c} = \overline{c}\overline{R} = r(\overline{b})$. The latter equality means that $R/aR$ is a reversible ring.

As we have proved that $R/aR$ is an almost Baer ring, the only thing we need is to prove that the intersection of any two right (and left) principal ideals is again a right (and left) principal ideal. Consider the ideals $\overline{bR}, \overline{cR}$. Using property 2 we see that there are some $\overline{x}, \overline{y} \in \overline{R}$ such that $\overline{bR} = r(\overline{x}), \overline{cR} = r(\overline{y})$. Then $\overline{rR} \cap \overline{cR} = r(\overline{x}) \cap r(\overline{y}) = r(\overline{xR} + \overline{yR}) = r(\overline{zR}) = \overline{dR}$, where $\overline{z}, \overline{d}$ are some elements in $\overline{R}$. We conclude that $R/aR$ is a right (and similarly left) coherent ring by the definition. The theorem is proved. $\square$

Gatalevych [4] was the first researcher who studied a noncommutative theory of adequate rings and their generalizations. We are also making an effort to deal with this theory.

**Definition 2.** *A nonzero element $a$ in a ring $R$ is said to be right adequate if for any nonzero element $b \in R$ we can find two elements $r, s \in R$ such that the decomposition $a = sr$ satisfies the following properties: $rR + bR = R$ and $s'R + bR \neq R$, where $sR \subseteq s'R \neq R$.*

Similarly, a left adequate element can be defined. In the case of a duo-ring these notions coincide and we simply talk on adequate elements. At first, the examples of adequate elements are the units, irreducible elements, and all square-free elements of a ring. Here is the definition of a square-free element.

**Definition 3.** *A nonzero element $a$ in a ring $R$ is called a square-free element if having any its decomposition $a = xy$, where $x, y \in R$, one can conclude that $xR + yR = R$ and $Rx + Ry = R$.*

It is useful to notice that there are rings without square-free elements, for example such is the ring of all algebraic integers.

**Proposition 1.** *All square-free elements of a Bezout duo-domain are adequate.*

*Proof.* Let $a, b \in R$, where $a$ is a square-free element. Then $aR + bR = dR$, $a = da_0$, $b = db_0$, $a_0R + b_0R = R$ for some elements $d, a_0, b_0 \in R$. Since $a$ is a square free element, $a_0R + dR = R$. The latter equality implies $a_0R + bR = R$ and the decomposition $a = sr$, where $s = d$, $r = a_0$ is the one that was desired. The statement is proved. $\square$

**Theorem 5.** *Let $R$ be a Bezout duo-domain and $a$ be some its nonzero element. The following statements are equivalent:*

*1) $a$ is a square-free element;*
*2) $R/aR$ is a von Neumann regular ring;*
*3) $J(R/aR) = 0$;*
*4)* $\mathrm{w.gl.dim}(R/aR) = 0$*;*
*5)* $\mathrm{w.gl.dim}(R/aR)$ *is finite.*

*Proof.* $1 \Rightarrow 2$. Suppose that $a$ is a square-free element. Let $\overline{y} \in \overline{R} = R/aR$ be an arbitrary element. If $\overline{y}$ is not invertible then by Theorem 4, $\overline{y}$ is a zero divisor, that is $\overline{xy} = 0$ for some element $\overline{x}$ in $\overline{R}$. Then $xy = ak' = ka$, for some $k, k' \in R$. Suppose that $kR + xR = dR = Rd$, and $k = dk_0$, $x = x_0d$, $x_0R + k_0R = R$, for some $x_0, k_0 \in R$. Hence $dx_0y = dk_0a$ and cancellating by $d$ we have $x_0y = k_0a$. Since $x_0$ and $k_0$ are coprime, $k_0$ has to be a divisor of $y$, that is $y = k_0y_0$ for some $y_0 \in R$. Since $R$ is a duo-domain, there is some $x_1 \in R$ such that $x_0k_0y_0 = k_0x_1y_0 = k_0a$ hence $a = x_1y_0$. Since $a$ is a square-free element, we can conclude that $x_1R + y_0R = R$. Then for some elements $p, q, u, v \in R$ we have $x_1u + y_0v = 1$, $x_0p + k_0q = 1$. Multiplying the first equality by $k_0$ we obtain $k_0x_1u + k_0y_0v = k_0$ and $x_0p + (x_0k_0u + yv)q = 1$. The latter equality implies $x_0R + yR = R$. Since $x_0y = k_0a$, we conclude that $\overline{x_0y} = \overline{0}$. As $x_0, y$ are coprime in $R$, this is preserved in $\overline{R}$. Therefore, there are elements $\overline{m}, \overline{n} \in \overline{R}$ such that $\overline{x_0m} + \overline{yn} = \overline{1}$. The ring $R/aR$ is reversible by Theorem 13, therefore $\overline{yx_0} = \overline{0}$. Finally, $\overline{yx_0m} + \overline{y}^2\overline{n} = \overline{y}$ implies $\overline{y}^2\overline{n} = \overline{y}$, thus $R/aR$ is a von Neumann regular ring.

$2 \Rightarrow 3$. The proof is obvious as this is a property of each von Neumann regular ring.

$3 \Rightarrow 1$. Suppose that $a = bc$, where $b$ and $c$ have g.c.d. $d \neq 1$. Then $\overline{x} \in J(R/aR)$ if and only if for any $r, s \in R$ we have $(1 - rxs)R + aR = R$. However the Jacobson radical is zero, thus $x \in aR = Ra$. The equality $bR + cR = dR$ implies $b = db_0$, $c = dc_0$, where $b_0, c_0 \in R$. Suppose that $(1 - b_0dc_0)R + aR = hR$. Then there are some $a', x \in R$ such that $a = ha', 1 - b_0dc_0 = hx$. Hence $hR + (b_0dc_0)R = R$. Since $d(b_0dc_0) = ha'$, the element $b_0dc_0$ has to divide $a'$, namely $a' = b_0dc_0k$, for some $k \in R$. Furthermore, $a = db_0dc_0 = ha' = hb_0dc_0k = hk'(b_0dc_0)$, for some $k' \in R$. Hence $d = hk'$. From the duo-ring condition we know that $Rh + Rb_0dc_0 = R$ and there are $u, v \in R$ such that $uh + vb_0dc_0 = 1$. After the right multiplication by $k'$ we obtain $d = hk' = h(vb_0dc_0k' + uhk') = h(ud + wd) = h(u + w)d$, for some $w \in R$. Thus $d = h(u + w)d$ implies $h(u + w) = 1$ and hence $h$ is invertible. As a result $a - b_0dc_0$ is coprime with $a$, that is $b_0dc_0 = ta = tdb_0dc_0$, for some $t \in R$. Al last we obtain $td = 1$ and $d$ becomes a unit that contradicts with our assumption. Thus, $a$ is a square-free element.

$2 \Leftrightarrow 4$. The necessity is straightforward as this is a property of each von Neumann regular ring, and the sufficiency follows from the observation that $R/aR$ is an IF-ring (by the Theorem 13) of zero weak global dimension [2].

$4 \Leftrightarrow 5$. The necessity is again obvious, while the sufficiency follows from the fact that the weak global dimension of $R/aR$ can be either 0 or infinite. The theorem is proved. $\square$

**Corollary 2.** *The square-free elements of a Bezout duo-domain are the elements of almost stable range 1.*

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У статті доведено, що квазі-дуо кільце Безу стабільного рангу 1 є кільцем елементарних дільників тоді і лише тоді, коли воно є дуо-кільцем. Як застосування цього результату показано, що скінченні гомоморфні образи дуо-областей Безу є когерентними морфійними кільцями слабкої глобальної розмірності рівної 0 або нескінченості, та кожен ін'єктивний модуль є плоский над такими кільцями. Крім того, введене поняття вільного від квадратів елемента у ситуації некомутативного кільця та показано, що такі елементи є адекватними елементами в дуо-областях Безу. Також отримано критерій регулярності скінченних гомоморфних образів дуо-областей Безу в термінах вільних від квадратів елементів, виродженості радикалу Джекобсона та скінченності слабкої глобальної розмірності.

*Ключові слова і фрази:* кільце Безу, подвійна область визначення, дистрибутивне кільце, стабільність рангу 1, вільно квадратований елемент, адекватний елемент, регулярне кільце фон Неймана, морфічне кільце, слабка глобальна вимірність.

Сорокин А.С. *Конечные гомоморфные образы дуо-областей Безу* // Карпатские матем. публ. — 2014. — Т.6, №2. — С. 360–366.

В статье доказано, что квази-дуо кольцо Безу стабильного ранга 1 является кольцом элементарных делителей тогда и только тогда, когда оно является дуо-кольцом. Как применение этого результата показано, что конечные гомоморфные образы дуо-областей Безу являются когерентными морфическими кольцами слабой глобальной размерности равной 0 или бесконечности, и каждый иньективный модуль будет плоским над такими кольцами. Кроме того, введено понятие свободного от квадратов элемента в ситуации некоммутативного кольца, и показано, что такие элементы являются адекватными элементами в дуо-областях Безу. Также получен критерий регулярности конечных гомоморфных образов дуо-областей Безу в терминах свободных от квадратов элементов, вырожденности радикала Джекобсона и конечности слабой глобальной размерности.

*Ключевые слова и фразы:* кольцо Безу, двойственная область определения, дистрибутивное кольцо, стабильность ранга 1, свободно квадратируемый элемент, адекватный элемент, регулярное кольцо фон Неймана, морфическое кольцо, слабое глобальное измерение.