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## PROBLEM WITH TWO-POINT CONDITIONS FOR PARABOLIC EQUATION OF SECOND ORDER ON TIME

The correctness of the problem with two-point conditions on time variable and Dirichlet-type conditions on spatial coordinates for the linear parabolic equations are established. The metric theorem about estimate from below of small denominators of the problem is proved.

*Key words and phrases:* parabolic equations, two-point problem, Fourier series, small denominators, Hausdorff measure.

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### INTRODUCTION

The problems with two-point and multipoint conditions with respect to the time variable for partial differential equations were studied in many scientific papers (see, for example [2–5, 7–11] and the references there). In particular, the correctness of multipoint problems for evolution equations in unbounded domain was investigated in the works [4, 5]. The solvability of multipoint problems for partial differential equations in bounded domains is frequently related to the problem of small denominators. In the scientific works [3, 7, 8, 11] metric approach have used for estimate from below of small denominators and it was proved that the conditions of solvability of such problems are satisfied for almost all (with respect to the Lebesgue measure) vectors which coordinates are the coefficients of the equations and interpolation nodes values.

The results of scientific works [3, 7, 8, 11] were generalized in the papers [2, 9, 10]. The correctness of problems with multipoint conditions holds for almost all (with respect to the Lebesgue measure) vectors which components are the interpolation nodes values (see [9, 10]). The conditions of solvability of the problem with two multiple nodes for factorized equation for almost all (with respect to the Lebesgue measure) vectors constructed by the coefficients of the equations (see [2]).

In the present work, we established the conditions of correct solvability of local two-point problem for factorized, parabolic operator (by Petrovskyi sense) in cylindrical domain which is a cartesian product of time segment and special multidimensional parallelepiped and we prove that such conditions are true for almost all (with respect to the Hausdorff measure) vectors constructed by coefficients of the equation.

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## 1 STATEMENT OF THE PROBLEM

In the domain  $Q_T^p = (0, T) \times \Pi^p$ ,  $\Pi^p = (0, \pi)^p$ , we consider the problem

$$\prod_{q=1}^2 \left( \frac{\partial}{\partial t} + \sum_{j=1}^p a_j^q L_j^b + A_q(L_1, \dots, L_p) \right) u(t, x) = 0, \quad (t, x) \in Q_T^p, \quad (1)$$

$$u(t_1, x) = \varphi_1(x), \quad u(t_2, x) = \varphi_2(x), \quad 0 \leq t_1 < t_2 \leq T, \quad x = (x_1, \dots, x_p) \in \Pi^p, \quad (2)$$

$$L_j^m u(t, x) \Big|_{x_j=0} = L_j^m u(t, x) \Big|_{x_j=\pi} = 0, \quad m \in \{0, 1, \dots, b-1\}, \quad j \in \{1, \dots, p\}, \quad (3)$$

where  $a_j^q > 0$ ,  $j \in \{1, \dots, p\}$ ,  $q \in \{1, 2\}$ ,

$$A_q(L_1, \dots, L_p) = \sum_{|s| < b} A_s^q L_1^{s_1} \dots L_p^{s_p}, \quad A_s^q \in \mathbb{C}, \quad q \in \{1, 2\}, \quad b \in \mathbb{N},$$

$L_j := -\frac{\partial}{\partial x_j} \left( p_j(x_j) \frac{\partial}{\partial x_j} \right) + q_j(x_j)$ ;  $p_j \in C^{2b-1}[0, \pi]$ ,  $q_j \in C^{2b-2}[0, \pi]$  are real-valued functions,  $p_j(x_j) \geq p_{0,j} > 0$ ,  $q_j(x_j) \geq 0$ ,  $j \in \{1, \dots, p\}$ .

We denote via  $\Lambda_j = \{\lambda_{k_j}, k_j \in \mathbb{N}\}$  and  $\{X_{k_j}(x_j), k_j \in \mathbb{N}\}$ ,  $j \in \{1, \dots, p\}$ , the set of eigenvalues and the system of responsible eigenfunctions (we suppose that  $\int_0^\pi |X_{k_j}(x_j)|^2 dx_j = 1$ ) of such problem

$$L_j X(x_j) = \lambda X(x_j), \quad X(0) = X(\pi) = 0. \quad (4)$$

It is known [6] that for each  $j$ ,  $j \in \{1, \dots, p\}$ , the eigenfunctions of the problem (4) make the total orthonormal system in the space  $L_2(0, \pi)$ . Under the set of conditions for  $p_j(x_j)$  and  $q_j(x_j)$  the next estimates

$$C_1 k_j^2 \leq \lambda_{k_j} \leq C_2 k_j^2, \quad (5)$$

$$\max_{0 \leq x_j \leq \pi} |X_{k_j}^{(r)}(x_j)| \leq N_j k_j^r, \quad r \in \{0, 1, \dots, 2b\}, \quad k_j \in \mathbb{N}, \quad j \in \{1, \dots, p\},$$

are true for all  $k_j \in \mathbb{N}$ , where  $C_1, C_2, N_1, \dots, N_p$  are positive constants; in addition to that the system of functions

$$\{X_k(x) = X_{k_1}(x_1) \dots X_{k_p}(x_p), k = (k_1, \dots, k_p) \in \mathbb{N}^p\}$$

is a total orthonormal system in the space  $L_2(\Pi^p)$ .

Denote  $\Lambda = \{\vec{\lambda}_k = (\lambda_{k_1}, \dots, \lambda_{k_p}), k \in \mathbb{N}^p\}$ ,  $|\vec{\lambda}_k^b| = \lambda_{k_1}^b + \dots + \lambda_{k_p}^b$ ,  $b \in \mathbb{N}$ ,  $\vec{\beta} = (\beta_1, \dots, \beta_p) \in \mathbb{R}^p$ ,  $(\vec{\beta}, \vec{\lambda}_k^b) = \beta_1 \lambda_{k_1}^b + \dots + \beta_p \lambda_{k_p}^b$ ;  $E_{\alpha, \vec{\beta}}^b$ ,  $\alpha \in \mathbb{R}$ ,  $\vec{\beta} \in \mathbb{R}^p$  is a space of functions  $\varphi(x) = \sum \varphi_k X_k(x)$ ,  $\varphi_k \in \mathbb{C}$ ,  $k \in \mathbb{N}^p$ , with finite norm

$$\|\varphi; E_{\alpha, \vec{\beta}}^b\| = \sqrt{\sum_{k \in \mathbb{N}^p} |\varphi_k|^2 w_k^2(\alpha; \vec{\beta}; b)}, \quad w_k(\alpha; \vec{\beta}; b) = |\vec{\lambda}_k^b|^\alpha \exp(\vec{\beta}, \vec{\lambda}_k^b);$$

$C^n([0, T]; E_{\alpha, \vec{\beta}}^b)$  is space of functions  $u(t, x) = \sum u_k(t) X_k(x)$ ,  $u_k(t) \in C^n[0, T]$ ,  $k \in \mathbb{N}^p$ , with norm

$$\|u; C^n([0, T]; E_{\alpha, \vec{\beta}}^b)\| = \sum_{j=0}^n \max_{t \in [0, T]} \|\partial^j u(t, \cdot) / \partial t^j; E_{\alpha, \vec{\beta}}^b\| < \infty.$$

2 UNIQUENESS OF A SOLUTION OF THE PROBLEM

The solution of the problem (1)–(3) in the space  $C^2([0, T]; E_{\alpha, \vec{\beta}}^b)$  has the form of series

$$u(t, x) = \sum_{k \in \mathbb{N}^p} u_k(t) X_k(x). \tag{6}$$

The coefficient  $u_k(t), k \in \mathbb{N}^p$ , is a solution of the two-point problem for ordinary differential equation

$$\prod_{q=1}^2 \left( \frac{d}{dt} + \sum_{j=1}^p a_j^q \lambda_{k_j}^b + A_q(\lambda_{k_1}, \dots, \lambda_{k_p}) \right) u_k(t) = 0, \tag{7}$$

$$u_k(t_1) = \varphi_{1k}, \quad u_k(t_2) = \varphi_{2k}, \tag{8}$$

where  $\varphi_{1k}, \varphi_{2k}, k \in \mathbb{N}^p$  are the Fourier coefficients (according to the system  $X_k(x), k \in \mathbb{N}^p$ ) of functions  $\varphi_1(x), \varphi_2(x)$  respectively. Let  $\mathcal{L}$  by a set  $\{k \in \mathbb{N}^p : \mu_1(\vec{\lambda}_k) = \mu_2(\vec{\lambda}_k)\}$ , where

$$\mu_q(\vec{\lambda}_k) = - \sum_{j=1}^p a_j^q \lambda_{k_j}^b - A_q(\lambda_{k_1}, \dots, \lambda_{k_p}), \quad q \in \{1, 2\}, \quad k \in \mathbb{N}^p. \tag{9}$$

The solution of the problem (7), (8) is defined by the formulas

$$u_k(t) = \begin{cases} D_1(\vec{\lambda}_k) e^{\mu_1(\vec{\lambda}_k)t} + D_2(\vec{\lambda}_k) e^{\mu_2(\vec{\lambda}_k)t}, & \text{if } k \in \mathbb{N}^p \setminus \mathcal{L}, \\ D_3(\vec{\lambda}_k) e^{\mu_1(\vec{\lambda}_k)t} + D_4(\vec{\lambda}_k) t e^{\mu_1(\vec{\lambda}_k)t}, & \text{if } k \in \mathcal{L}, \end{cases}$$

where  $D_j(\vec{\lambda}_k), j \in \{1, \dots, 4\}$ , is a solution of the following system of linear equations

$$\begin{cases} D_1(\vec{\lambda}_k) e^{\mu_1(\vec{\lambda}_k)t_1} + D_2(\vec{\lambda}_k) e^{\mu_2(\vec{\lambda}_k)t_1} = \varphi_{1k}, & \text{if } k \in \mathbb{N}^p \setminus \mathcal{L}, \\ D_1(\vec{\lambda}_k) e^{\mu_1(\vec{\lambda}_k)t_2} + D_2(\vec{\lambda}_k) e^{\mu_2(\vec{\lambda}_k)t_2} = \varphi_{2k}, & \\ D_3(\vec{\lambda}_k) e^{\mu_1(\vec{\lambda}_k)t_1} + D_4(\vec{\lambda}_k) t_1 e^{\mu_1(\vec{\lambda}_k)t_1} = \varphi_{1k}, & \text{if } k \in \mathcal{L}. \\ D_3(\vec{\lambda}_k) e^{\mu_1(\vec{\lambda}_k)t_2} + D_4(\vec{\lambda}_k) t_2 e^{\mu_1(\vec{\lambda}_k)t_2} = \varphi_{2k}, & \end{cases}$$

Let's denote

$$\Delta(\vec{\lambda}_k) = \begin{cases} e^{\mu_1(\vec{\lambda}_k)t_2 + \mu_2(\vec{\lambda}_k)t_1} \left[ e^{(\mu_2(\vec{\lambda}_k) - \mu_1(\vec{\lambda}_k))(t_2 - t_1)} - 1 \right], & \text{if } k \in \mathbb{N}^p \setminus \mathcal{L}, \\ (t_2 - t_1) e^{\mu_1(\vec{\lambda}_k)(t_1 + t_2)}, & \text{if } k \in \mathcal{L}. \end{cases} \tag{10}$$

**Theorem 1.** *In order that problem (1)–(3) have at most one solution in the space  $C^2([0, T]; E_{\alpha, \vec{\beta}}^b), \alpha \in \mathbb{R}, \vec{\beta} \in \mathbb{R}^p$ , it is necessary and sufficiently that the following condition be satisfied*

$$\forall k \in \mathbb{N}^p \setminus \mathcal{L} \quad \forall \ell \in \mathbb{Z} \quad (\mu_2(\vec{\lambda}_k) - \mu_1(\vec{\lambda}_k))(t_2 - t_1) \neq 2\pi i \ell. \tag{11}$$

*Proof.* The proof is carried out by the scheme used to prove theorem 5.3 in [7]. □

We get next result comes from Theorem 1 and formulas (9).

**Corollary 1.** *In order that problem (1)–(3) have the most one solution in the space  $C^2([0, T]; E_{\alpha, \vec{\beta}}^b), \alpha \in \mathbb{R}, \vec{\beta} \in \mathbb{R}^p$ , it is necessary and sufficient that for each  $(k_1, \dots, k_p) \in \mathbb{N}^p \setminus \mathcal{L}$  and each  $\ell \in \mathbb{Z}$  at least one of the equations*

$$\sum_{j=1}^p (a_j^1 - a_j^2) \lambda_{k_j}^b + \sum_{|s| < b} \operatorname{Re}(A_s^1 - A_s^2) \lambda_{k_1}^{s_1} \dots \lambda_{k_p}^{s_p} = 0, \quad \sum_{|s| < b} \operatorname{Im}(A_s^1 - A_s^2) \lambda_{k_1}^{s_1} \dots \lambda_{k_p}^{s_p} = \frac{2\pi \ell}{(t_2 - t_1)}$$

*doesn't hold.*

**Example 1.** For the problem

$$\left(\frac{\partial}{\partial t} + a\frac{\partial^4}{\partial x^4} + ia_1\frac{\partial^2}{\partial x^2}\right)\left(\frac{\partial}{\partial t} + a\frac{\partial^4}{\partial x^4} + ia_2\frac{\partial^2}{\partial x^2}\right)u(t, x) = 0, \quad (t, x) \in Q_T^1, \quad (12)$$

$$u(0, x) = 0, \quad u(T, x) = 0, \quad x \in (0, \pi), \quad (13)$$

$$\frac{\partial^{2m}u(t, x)}{\partial x^{2m}}\Big|_{x=0} = \frac{\partial^{2m}u(t, x)}{\partial x^{2m}}\Big|_{x=\pi} = 0, \quad m \in \{0, 1\}, \quad (14)$$

where  $a > 0, a_1, a_2 \in \mathbb{R}, a_1 \neq a_2, i^2 = -1$ , the determinant  $\Delta(\lambda_k), k \in \mathbb{N}$ , is calculated by the formula

$$\Delta(\lambda_k) = \begin{cases} e^{(-ak^4 + ia_1k^2)T}(e^{i(a_2 - a_1)k^2T} - 1), & \text{if } k \neq 0, \\ T, & \text{if } k = 0. \end{cases}$$

So far as  $|\Delta(\lambda_k)| = 2e^{-ak^4T}|\sin(a_2 - a_1)k^2T/2|, k \neq 0$ , then the problem (12)–(14) has in space  $C^2([0, T]; E_{\alpha, \vec{\beta}}^b)$  only trivial solution, if number  $(a_2 - a_1)T/\pi$  is irrational. If number  $(a_2 - a_1)T/\pi$  is rational, then the problem (12)–(14) has in space  $C^2([0, T]; E_{\alpha, \vec{\beta}}^b)$  countable number of linear independent solutions

$$u_r(t, x) = e^{-16ar^4n^4t} \left( e^{4ia_1r^2n^2t} - e^{4ia_2r^2n^2t} \right) \sin(2rnx), \quad r \in \mathbb{Z} \setminus \{0\}.$$

### 3 EXISTENCE OF A SOLUTION OF THE PROBLEM

In what follows, we consider that the condition (11) is satisfied. Then for every  $k \in \mathbb{N}^p$  there exists the unique solution  $u_k(t)$  of the problem (7), (8) such that

$$u_k(t) = \begin{cases} \frac{1}{\Delta(\vec{\lambda}_k)} \left[ (e^{\mu_2(\vec{\lambda}_k)t_2 + \mu_1(\vec{\lambda}_k)t} - e^{\mu_1(\vec{\lambda}_k)t_2 + \mu_2(\vec{\lambda}_k)t})\varphi_{1k} + (e^{\mu_1(\vec{\lambda}_k)t_1 + \mu_2(\vec{\lambda}_k)t} - e^{\mu_2(\vec{\lambda}_k)t_1 + \mu_1(\vec{\lambda}_k)t})\varphi_{2k} \right], & \text{if } k \in \mathbb{N}^p \setminus \mathcal{L}, \\ \frac{1}{\Delta(\vec{\lambda}_k)} \left[ (t_2 - t)e^{\mu_1(\vec{\lambda}_k)(t_2+t)}\varphi_{1k} + (t - t_1)e^{\mu_1(\vec{\lambda}_k)(t_1+t)}\varphi_{2k} \right], & \text{if } k \in \mathcal{L}. \end{cases} \quad (15)$$

We get from equations (6), (15) that the solution of the problem (1)–(3) can be represented by the Fourier series

$$u(t, x) = \sum_{k \in \mathcal{L}} u_k(t)X_k(x) + \sum_{k \in \mathbb{N}^p \setminus \mathcal{L}} u_k(t)X_k(x). \quad (16)$$

The series (16) is, generally speaking, divergent, since the nonzero quantity  $\Delta(\vec{\lambda}_k)$  can take very small for the infinite number of vectors  $\vec{\lambda}_k \in \Lambda$ . The following statement is true.

**Theorem 2.** Suppose that condition (11) is satisfied and there exist  $\omega \in \mathbb{R}$  and  $\vec{v} \in \mathbb{R}^p$  such that for all (except a finite number) vectors  $\vec{\lambda}_k \in \Lambda$  the following inequality holds

$$|\Delta(\vec{\lambda}_k)| \geq w_k(-\omega; -\vec{v}; b). \quad (17)$$

If  $\varphi_1, \varphi_2 \in E_{\alpha_0, \vec{\beta}_0}^b$ , where  $\alpha_0 = \alpha + \omega + 2, \vec{\beta}_0 = \vec{\beta} + \vec{v} - \vec{\delta}t_1, \vec{\delta} = (\delta_1, \dots, \delta_p), 0 < \delta_j < \min\{a_j^1, a_j^2\}, j \in \{1, \dots, p\}$ , then there exists the unique solution of the problem (1)–(3) from the space  $C^2([0, T]; E_{\alpha, \vec{\beta}}^b)$ , which depends continuously on the functions  $\varphi_1, \varphi_2$ .

*Proof.* It follows from equations (9) that estimates

$$-(\vec{\xi}, \vec{\lambda}_k^b) \leq \operatorname{Re} \mu_q(\vec{\lambda}_k) \leq -(\vec{\delta}, \vec{\lambda}_k^b), \quad q \in \{1, 2\}, \quad (18)$$

where  $\vec{\xi} = (\xi_1, \dots, \xi_p)$ ,  $\xi_j > \max\{a_j^1, a_j^2\}$ ,  $j \in \{1, \dots, p\}$ , are true for all (except a finite number) vectors  $k \in \mathbb{N}^p$ . So far as

$$|\mu_q(\vec{\lambda}_k)| \leq C_3 |\vec{\lambda}_k^b|, \quad q \in \{1, 2\}, \quad C_3 > \max\{a_j^1, a_j^2 : j \in \{1, \dots, p\}\}, \quad (19)$$

then we'll get from estimates (18), (19) that

$$\forall t \geq 0 \quad |(t^j e^{\mu_q(\vec{\lambda}_k)t})^{(r)}| \leq C_4 w_k(r; -\vec{\delta}t; b), \quad j \in \{0, 1\}, \quad q \in \{1, 2\}, \quad r \in \{0, 1, 2\}. \quad (20)$$

Based on estimates (17), (20) we get from the formulas (10), (15) that

$$\max_{t \in [0, T]} |u_k^{(r)}(t)| \leq C_5 \sum_{q=1}^2 |\varphi_{qk}| w_k(2 + \omega; \vec{v} - \vec{\delta}t_1; b), \quad k \in \mathbb{N}^p.$$

So

$$\begin{aligned} \|u; C^2([0, T]; E_{\alpha, \vec{\beta}}^b)\| &\leq \sum_{r=0}^2 \left( \sum_{k \in \mathbb{N}^p} \max_{t \in [0, T]} |u_k^{(r)}(t)|^2 w_k^2(\alpha; \vec{\beta}; b) \right)^{1/2} \\ &\leq C_6 \sum_{q=1}^2 \left( \sum_{k \in \mathbb{N}^p} |\varphi_{qk}|^2 w_k^2(\alpha + \omega + 2; \vec{\beta} + \vec{v} - \vec{\delta}t_1; b) \right)^{1/2} \\ &= C_6 \sum_{q=1}^2 \|\varphi_q; E_{\alpha_0, \vec{\beta}_0}^{2b}\|. \end{aligned} \quad (21)$$

The proof of the theorem implies from the inequality (21). □

**Remark 1.** *If the conditions of Theorem 2 are satisfied then for each fixed  $t_0 \in [0, T]$  the function  $u(t_0, x)$  belongs to the space  $E_{\alpha, \vec{\beta} + \vec{\delta}t_0}^b$ .*

The next statement describes the equations (1), for which estimate (17) is true with properly chosen indices  $\omega \in \mathbb{R}$  and  $\vec{v} = (v_1, \dots, v_p) \in \mathbb{R}^p$ .

**Theorem 3.** *Suppose that for each  $j \in \{1, \dots, p\}$  the following inequality holds*

$$a_j^1 > a_j^2. \quad (22)$$

*If  $\omega = 0$ ,  $\vec{v} = \vec{\xi}(t_1 + t_2) + \vec{\eta}(t_1 - t_2)$ , where  $\vec{\eta} = (\eta_1, \dots, \eta_p)$ ,  $0 < \eta_j < a_j^1 - a_j^2$ ,  $j \in \{1, \dots, p\}$ , then the estimate (17) holds for all (except for a finite number) vectors  $\vec{\lambda}_k \in \Lambda$ .*

*Proof.* We get from inequalities (22) that for all (except for a finite number) vectors  $\vec{\lambda}_k \in \Lambda$  the inequality

$$\operatorname{Re} \left( \mu_2(\vec{\lambda}_k) - \mu_1(\vec{\lambda}_k) \right) \geq (\vec{\eta}, \vec{\lambda}_k^b) \quad (23)$$

is true. It follows from the estimates (23) that the set  $\mathcal{L}$  is not over finite. Let

$$N = \begin{cases} \max_{k \in \mathcal{L}} |\vec{\lambda}_k^1|, & \text{if } \mathcal{L} \neq \emptyset, \\ 0, & \text{if } \mathcal{L} = \emptyset. \end{cases}$$

Then, for all  $k \in \mathbb{N}^p$  such that  $|\vec{\lambda}_k^1| > N$ , the determinant of  $\Delta(\vec{\lambda}_k)$  is calculated by the formula

$$\Delta(\vec{\lambda}_k) = e^{\mu_1(\vec{\lambda}_k)t_2 + \mu_2(\vec{\lambda}_k)t_1} \left( e^{(\mu_2(\vec{\lambda}_k) - \mu_1(\vec{\lambda}_k))(t_2 - t_1)} - 1 \right). \tag{24}$$

Since for any  $z \in \mathbb{C}$  such that  $\text{Re } z \geq \zeta > 0$ , the inequality  $|e^z - 1| \geq e^\zeta - 1$  is true, then based on estimates (23) we obtain from equation (24) that

$$|\Delta(\vec{\lambda}_k)| \geq e^{\text{Re}(\mu_1(\vec{\lambda}_k)t_2 + \mu_2(\vec{\lambda}_k)t_1)} \left| e^{(\vec{\eta}, \vec{\lambda}_k^b)(t_2 - t_1)} - 1 \right|,$$

for  $|\vec{\lambda}_k^1| > N$ . Considering that  $e^\zeta - 1 \geq \frac{1}{2}e^\zeta$  for all  $\zeta \geq 1$ , and the fact that for all (except for a finite number) the inequalities (18) are satisfied, we obtain that the inequality

$$|\Delta(\vec{\lambda}_k)| \geq e^{-(\vec{\zeta}(t_1 + t_2) + \vec{\eta}(t_1 - t_2), \vec{\lambda}_k^b)}$$

holds for all (except for a finite number of) vectors  $K \in \mathbb{N}^p$ . Theorem is proved. □

#### 4 METRIC ESTIMATES OF SMALL DENOMINATORS

Let's study the question of possibility for inequality (17). Let us provide some concepts related to  $\rho$ -Hausdorff measure and Hausdorff dimension of the set  $M \subset \mathbb{R}^p$ , for the ease of presentation.

**Definition 1.** A limit (finite or infinite)

$$\dim_\rho M = \liminf_{\delta \rightarrow 0} \sum_{j=1}^{\infty} (\text{diam } S_j)^\rho,$$

where the infimum is taken over all coverings of the set  $M$  by the balls  $S_j, j = 1, 2, \dots$ , such that  $M \subset \bigcup_{j=1}^{\infty} S_j$  and diameter of each ball  $S_j$  is not greater than  $\delta$ ,  $\text{diam } S_j \leq \delta$ , is called  $\rho$ -Hausdorff measure of the set  $M \subset \mathbb{R}^p$  (this limit we denote by  $\dim_\rho M$ ).

**Definition 2.** A real number  $\beta$  such that

- 1)  $\forall \rho \ \beta \leq \rho \leq p \ \dim_\rho M = 0$ ,
- 2)  $\forall \rho \ 0 < \rho < \beta \ \dim_\rho M = \infty$ ,

is called the Hausdorff dimension of the set  $M \subset \mathbb{R}^p$ .

We will use statements, proof of which is contained in [1].

**Theorem 4.** The set  $M \subset \mathbb{R}^p$  has zero  $\rho$ -Hausdorff measure if and only if when there exists a covering by balls  $\{S_j\}_{j=1}^{\infty}$  of the set  $M$  such that  $\sum_{j=1}^{\infty} (\text{diam } S_j)^\rho < \infty$ , and that every point of the set  $M$  belongs to an infinite number of balls  $S_j$ .

We denote  $s_{j,q} = \underbrace{(0, \dots, 0, q, 0, \dots, 0)}_j, j \in \{1, \dots, p\}, q \in \{1, \dots, b - 1\}$ , the multiindex of

the length  $p$  which  $j$ -th place is  $q$  and the rest places are zero;

$$y_j^q = \text{Im}(A_{s_{j,q}}^1 - A_{s_{j,q}}^2), q \in \{1, \dots, b - 1\}, j \in \{1, \dots, p\},$$

$$\vec{y}^q = (y_1^q, \dots, y_p^q), q \in \{1, \dots, b - 1\};$$

$$G = [c_1, d_1] \times \dots \times [c_p, d_p], c_j, d_j \in \mathbb{R}, c_j < d_j, j \in \{1, \dots, p\}.$$

**Theorem 5.** Let  $\rho \in (p - 1; p]$  and  $q \in \{1, \dots, b - 1\}$ . The inequality (17) holds for almost all (respectively  $\rho$ -Hausdorff measure) vectors  $\vec{y}^q \in G$  and for all (except for a finite number) vectors  $k \in \mathbb{N}^p$  if  $\omega > \omega_1(q)$ ,  $\vec{v} = \vec{\zeta}(t_1 + t_2)$ , where  $\vec{\zeta} = (\zeta_1, \dots, \zeta_p)$ ,  $\zeta_j > \max\{a_j^1, a_j^2\}$ ,  $j \in \{1, \dots, p\}$ ,

$$\omega_1(q) = \frac{p/(2b) + 1 - 1/b}{\rho - p + 1} - \frac{q}{2b}, \quad q \in \{1, \dots, b - 1\}.$$

*Proof.* Fix  $q \in \{1, \dots, b - 1\}$ . Let

$$F_q(\vec{\lambda}_k) = \sum_{|s| < b} \text{Im}(A_s^1 - A_s^2) \lambda_{k_1}^{s_1} \dots \lambda_{k_p}^{s_p} - \sum_{j=1}^p y_j^q \lambda_{k_j}^q.$$

Let's denote by  $V^\omega(\vec{\lambda}_k, m)$  a set of vectors  $\vec{y}^q \in G$  for which the inequality

$$\left| \sum_{j=1}^p y_j^q \tau \lambda_{k_j}^q + F_q(\vec{\lambda}_k) \tau - m \right| < |\vec{\lambda}_k^b|^{-\omega}, \quad \tau = (t_2 - t_1) / \pi,$$

is true for a fixed  $\vec{\lambda}_k \in \Lambda$  and  $m \in \mathbb{Z}$  and by  $V^\omega$  the set of vectors  $\vec{y}^q \in G$ , which belong to an infinite number of sets  $V_q^\omega(\vec{\lambda}_k, m)$ ,  $\vec{\lambda}_k \in \Lambda$ ,  $m \in \mathbb{Z}$ . Obviously there exists the number  $C_7 = C_7(p, b, c_1, \dots, c_p, d_1, \dots, d_p) > 0$  such that for all  $m \in \mathbb{Z}$ ,  $|m| > C_7 |\vec{\lambda}_k^{b-1}|$ , the set  $V^\omega(\vec{\lambda}_k, m)$  is empty.

We now consider the case when  $|m| \leq C_7 |\vec{\lambda}_k^{b-1}| = M(\lambda_k)$ . Let  $\lambda_{k_{j_0}} = \max_{j \in \{1, \dots, p\}} \{\lambda_{k_j}\}$ , and  $V^\omega(\vec{\lambda}_k, m, y_1^q, \dots, y_{j_0-1}^q, y_{j_0+1}^q, \dots, y_p^q) = \{y_{j_0}^q \in \mathbb{R} : (y_1^q, \dots, y_p^q) \in V^\omega(\vec{\lambda}_k, m)\}$ . If  $V^\omega(\vec{\lambda}_k, m) \neq \emptyset$ , then there exist  $y_1^q, \dots, y_{j_0-1}^q, y_{j_0+1}^q, \dots, y_p^q$  such that  $V^\omega(\vec{\lambda}_k, m, y_1^q, \dots, y_{j_0-1}^q, y_{j_0+1}^q, \dots, y_p^q)$  is not empty interval  $(\tau |\vec{\lambda}_k^b|^\omega \lambda_{k_{j_0}}^q)^{-1}$ . Then the set  $V^\omega(\vec{\lambda}_k, m)$  can be covered by the balls  $S_r(\vec{\lambda}_k, m)$ ,  $r \in \{1, \dots, J(\vec{\lambda}_k)\}$ , of the radius  $(\tau |\vec{\lambda}_k^b|^\omega \lambda_{k_{j_0}}^q)^{-1}$ , amount  $J(\vec{\lambda}_k)$  of which does not exceed  $C_8 (|\vec{\lambda}_k^b|^\omega \lambda_{k_{j_0}}^q)^{p-1}$ . Note that for  $\omega > \omega_1(q)$  the inclusion

$$V^\omega = \bigcap_{K=0}^\infty \bigcup_{|\vec{\lambda}_k| \geq K} \bigcup_{0 \leq |m| \leq M(\vec{\lambda}_k)} V^\omega(\vec{\lambda}_k, m) \subset \bigcap_{K=0}^\infty \bigcup_{|\vec{\lambda}_k| \geq K} \bigcup_{0 \leq |m| \leq M(\lambda_k)} \bigcup_{r=1}^{J(\vec{\lambda}_k)} S_r(\vec{\lambda}_k, m) \quad (25)$$

is correct. Therefore, each point of the set  $V^\omega$  belongs to an infinite number of the balls  $S_r(\vec{\lambda}_k, m)$ ,  $r \in \{1, \dots, J(\vec{\lambda}_k)\}$ ,  $0 \leq |m| \leq M(\vec{\lambda}_k)$ ,  $\vec{\lambda}_k \in \Lambda$ . On the basis of estimates (5) we obtain from (25) that

$$\begin{aligned} \sum_{k \in \mathbb{N}^p} \sum_{0 \leq |m| \leq M(\vec{\lambda}_k)} \sum_{r=1}^{J(\vec{\lambda}_k)} (\text{diam } S_r(\vec{\lambda}_k, m))^\rho &= \sum_{k \in \mathbb{N}^p} \sum_{0 \leq |m| \leq M(\vec{\lambda}_k)} \sum_{r=1}^{J(\vec{\lambda}_k)} \left( \frac{1}{\tau |\vec{\lambda}_k^b|^\omega \lambda_{k_{j_0}}^q} \right)^\rho \\ &\leq C_9 \sum_{k \in \mathbb{N}^p} \frac{1}{|\vec{\lambda}_k^1|^{(\omega b + q)(\rho - p + 1) - b + 1}} \leq C_{10} \sum_{k \in \mathbb{N}^p} \frac{1}{|k|^{2((\omega b + q)(\rho - p + 1) - b + 1)}}. \end{aligned} \quad (26)$$

For  $\omega > \frac{p/(2b) + 1 - 1/b}{\rho - p + 1} - \frac{q}{b}$  the series (26) is converges, then by Theorem 4 the  $\rho$ -Hausdorff measure of the set  $V^\omega$  is equal to zero. To complete the proof of the theorem it is given that

$$|\Delta(\vec{\lambda}_k)| \geq e^{\text{Re } \mu_1(\vec{\lambda}_k) t_2 + \text{Re } \mu_2(\vec{\lambda}_k) t_1} \left| \sin \left( \text{Im}(\mu_2(\vec{\lambda}_k) - \mu_1(\vec{\lambda}_k))(t_2 - t_1) \right) \right|, \quad k \in \mathbb{N}^p \setminus \mathcal{L}, \quad (27)$$

and that

$$\begin{aligned} \left| \sin \left( \operatorname{Im}(\mu_2(\vec{\lambda}_k) - \mu_1(\vec{\lambda}_k))(t_2 - t_1) \right) \right| &\geq \frac{2}{\pi} \left| \operatorname{Im}(\mu_2(\vec{\lambda}_k) - \mu_1(\vec{\lambda}_k))(t_2 - t_1) - m\pi \right| \\ &= 2 \left| \sum_{|s|<b} \operatorname{Im}(A_s^1 - A_s^2) \tau \lambda_{k_1}^{s_1} \dots \lambda_{k_p}^{s_p} - m \right|, \end{aligned} \quad (28)$$

where  $\tau = (t_2 - t_1)/\pi$  and an integer  $m$  is such that

$$-1/2 \leq \sum_{|s|<b} \operatorname{Im}(A_s^1 - A_s^2) \tau \lambda_{k_1}^{s_1} \dots \lambda_{k_p}^{s_p} - m < 1/2.$$

Based on the estimates (18), (27) and (28) we get that for almost all (respectively to the  $\rho$ -Hausdorff measure) vectors  $\vec{y}^q \in G$  the inequality

$$|\Delta(\vec{\lambda}_k)| \geq |\vec{\lambda}_k^b|^{-(p/(2b)+1-1/b)/(\rho-p+1)+q/b} e^{-(\vec{\zeta}(t_1+t_2), \vec{\lambda}_k^b)}$$

is true for all (except of a finite number) vectors  $\vec{\lambda}_k \in \Lambda$ . Theorem is proved.  $\square$

Let  $H_q^{\omega, \vec{v}}$ ,  $\omega \in \mathbb{R}$ ,  $\vec{v} \in \mathbb{R}^p$ , be a set of vectors  $\vec{y}^q \in G$ , for which the estimate (17) is true. From Theorem 5 the next corollary about the Hausdorff dimension of the set  $G \setminus H_q^{\omega, \vec{v}}$  follows.

**Corollary 2.** For each  $q \in \{1, \dots, b-1\}$  and arbitrary  $\omega > \frac{p}{2b} + 1 - \frac{q+1}{b}$  the Hausdorff dimension of the set  $G \setminus H_q^{\omega, \vec{v}}$  is less than  $p - 1 + \frac{p/(2b)+1-1/b}{\omega+q/b}$ , if  $\vec{v} = \vec{\zeta}(t_1 + t_2)$ .

**Remark 2.** Theorem 5 complements the results of [11].

## 5 CONCLUSIONS

The theorems of existence and uniqueness of the solution of the problem (1)–(3) in the space of exponential type are established. The lower bound estimates of small denominators for almost all (respectively to  $\rho$ -Hausdorff measure) vectors  $\vec{y}^q \in G$  are established. A class of problems with conditions (2), (3) for equations (1) for which there is no problem of small denominators, is subscribed.

The results can be extended to the next problem

$$\begin{aligned} \prod_{q=1}^n \left( \frac{\partial}{\partial t} + \sum_{j=1}^p a_j^q L_j^b + A_q(L_1, \dots, L_p) \right) u(t, x) &= 0, \\ u(t_j, x) &= \varphi_j(x), \quad t_j = (j-1)t_0, \quad j \in \{1, \dots, n\}, \quad t_0 = T/(n-1), \end{aligned}$$

where  $a_j^q > 0$ ,  $A_q(L_1, \dots, L_p) = \sum_{|s|<b} A_s^q L_1^{s_1} \dots L_p^{s_p}$ ,  $A_s^q \in \mathbb{C}$ ,  $q \in \{1, \dots, n\}$ .

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Симотюк М.М., Тимків І.Р. *Задача з двоточковими умовами для параболічного рівняння другого порядку за часом* // Карпатські матем. публ. — 2014. — Т.6, №2. — С. 351–359.

Встановлено умови коректної розв'язності задачі з двоточковими умовами за часовою змінною та умовами типу Діріхле за просторовими координатами для лінійного параболічного рівняння. Для доведення оцінок знизу малих знаменників, які виникли при побудові розв'язку задачі, використано метричний.

*Ключові слова і фрази:* параболічне рівняння, двоточкова задача, ряд Фур'є, малі знаменники, міра Гаусдорфа.

Сымотюк М.М., Тымкив И.Р. *Задача с двухточечными условиями для параболического уравнения второго порядка по времени* // Карпатские матем. публ. — 2014. — Т.6, №2. — С. 351–359.

Найдены условия корректности задачи с двухточечными условиями по временной переменной и условиями типа Дирихле по пространственным координатам для линейного параболического уравнения. Для доказательства оценок снизу малых знаменателей, возникших при построении решения задачи, использован метрический подход.

*Ключевые слова и фразы:* параболическое уравнение, двухточечная задача, ряд Фурье, малые знаменатели, мера Хаусдорфа.